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EQUIVALENCE RELATIONS IN ALGEBRAIC GEOMETRY

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1. **The cycle groups C_s.** An algebraic variety V in n-dimensional complex projective space $P^{(n)}$ is obtained by equating to zero a finite number of forms $F_1(x_0, \cdots, x_n), \cdots, F_m(x_0, \cdots, x_n)$ with complex coefficients; V is assumed to be nonempty. If V is irreducible, that is, if V is not the union of a finite number of proper subvarieties, it is possible to associate with V in several ways a complex dimension d. For example, just as $P^{(1)}$ is topologically equivalent to a real 2-dimensional sphere, so can every $P^{(n)}$ be represented topologically by a $2n$-dimensional real complex in the sense of combinatorial topology. (See [1]; numbers in brackets refer to the references.) In this representation, V goes over into an even-dimensional, connected, orientable, closed complex whose dimension is defined as $2d$. This complex is denoted by $K^{(2d)}$ and V itself by $V^{(d)}$.

Consider the set T_s of irreducible, s-dimensional subvarieties of $V^{(d)}$ for some fixed s, where $0 \leqq s \leqq d$. A function on T_s is called integral if its value for every element of T_s is a rational integer, and if the function is zero except for at most a finite number of elements of T_s; these functions constitute of course an additive group, denoted by C_s. We identify the integral function which at the elements $W_1^{(s)}, \cdots, W_k^{(s)}$ of T_s assumes the values $n_1, \cdots, n_k$ and which is zero everywhere else on T_s with the linear combination $n_1 W_1^{(s)} + \cdots + n_k W_k^{(s)}$. Since every $W_i^{(s)}$ gives rise to a $2s$-dimensional, connected, closed, orientable subcomplex of $K^{(2d)}$, the above linear combination can be interpreted as a $2s$-dimensional cycle of $K^{(2d)}$ in the sense of topology. This fact is the reason why we call the elements of C_s the s-dimensional cycles of $V^{(d)}$ and often consider C_s as a subgroup of the $2s$-dimensional cycle group of $K^{(2d)}$. A cycle is called *effective* if, considered as a function, it never assumes a negative value; otherwise the cycle is called *virtual*. The effective cycles are clearly closed under addition but not under subtraction, and every cycle is the difference of two effective cycles.

The group C_s is completely determined by the cardinal number of T_s, and hence its structure is of no interest. The importance of C_s lies in the fact that the different aspects of the geometry of $V^{(d)}$ are most conveniently studied by means of the equivalence relations which

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they impose upon C_s. Let us first study equivalence relations abstractly.

2. **Abstract equivalence relations.** Let G be an additive group, and let E be a nonempty subset of G which is closed under addition but not necessarily under subtraction. We furthermore assume that each element of G is the difference of two elements of E. In our applications, G will be C_s and E the set of effective cycles of C_s.

Suppose that a binary relation $\approx$ has been defined for just the elements of E which is reflexive, symmetric, and additive. Additivity means of course that if four elements E_1, E_2, E_3, E_4 of E are such that $E_1 \approx E_2$ and $E_3 \approx E_4$, then $E_1+E_3 \approx E_2+E_4$. In terms of $\approx$ we define a binary relation $*$ for all of G by postulating that for two elements A and B of G, $A * B$ when there exists an $X \in G$ such that $A+X$ and $B+X$ belong to E and $A+X \approx B+X$. (It is clear that X can always be chosen in E.)

STATEMENT 2.1. *The binary relation* $*$ *for all of* G *is reflexive, symmetric, transitive, and additive.*

PROOF. If $A \in G$, then $A = E_1 - E_2$ where $E_1, E_2 \in E$; $A * A$ follows from using E_2 as X. The symmetry and additivity of $*$ follow immediately from the corresponding properties of $\approx$. If $A * B$ and $B * C$, the additivity of $*$ implies that $A+B * B+C$. Hence there exists an $X^1 \in G$ such that $A+B+X^1$ and $B+C+X^1$ belong to E and $A+B+X^1 \approx B+C+X^1$. Using then $B+X^1$ as X, we conclude that $A * C$.

According to (2.1), the relation $*$ divides the elements of G into mutually exclusive classes, which can be considered as the elements of a homomorphic image of G with respect to the subgroup F which consists of all elements $A * O$ as kernel. (O is the zero element of G.) Since $A * O$ means that there exists an $X \in E$ such that $A+X \approx X$, we can also say that F consists of all elements of G which can be written as $E_1 - E_2$, where $E_1, E_2 \in E$ and $E_1 \approx E_2$.

It is clear that if $E_1, E_2 \in E$ and $E_1 \approx E_2$, then $E_1 * E_2$ while the converse may not hold. (Algebraic equivalence gives an example of this.) However, if $\approx$ is also subtractive on E, we prove immediately that $\approx$ and $*$ coincide on E. ($\approx$ is of course called subtractive if $E_1, E_2, E_3, E_4 \in E$ and $E_1 \approx E_2$, $E_3 \approx E_4$, and $E_1 - E_3 \in E$ and $E_2 - E_4 \in E$ imply that $E_1 - E_3 \approx E_2 - E_4$.)

In algebraic geometry it may happen that a binary relation $\sim$ is defined for the elements of E which is only reflexive and symmetric but not additive. (Rational equivalence gives an example of this.) We then first go over to a new binary relation $\approx$ for E by defining that, for two elements E_1 and E_2 of E, $E_1 \approx E_2$ if there exist elements

$E_{11}, \cdots, E_{1k} \in E$ and $E_{21}, \cdots, E_{2k} \in E$ such that $E_1 = E_{11} + E_{12} + \cdots + E_{1k}$, $E_2 = E_{21} + E_{22} + \cdots + E_{2k}$, and $E_{1i} \sim E_{2i}$ for $i = 1, \cdots, k$; clearly $\approx$ is reflexive, symmetric, and additive on E. The relation $\sim$ is usually given by means of a covering U of E, where $E_1 \sim E_2$ then means that E_1 and E_2 belong to at least one element of U. The above additive relation $\approx$ is then determined by the covering of E which consists of all elements of U together with those subsets of E which are sums of a finite number of elements of U. (If $U_1, \cdots, U_k \in U$, $U_1 + \cdots + U_k$ is the subset of E which consists of all sums $E_1 + \cdots + E_k$ where $E_i \in U_i$ for $i = 1, \cdots, k$.)

Every covering U of any set whatsoever is a partially ordered system, obtained by ordering the elements of U by inclusion. We call U an ***inductive covering*** if the partially ordered system U is inductive, that is, if every totally ordered subsystem of U has an upper bound in U. According to Zorn's lemma, every element of U is then contained in a maximal element of U, and hence the maximal elements of U then constitute a new covering M of the set which clearly defines the same binary relation $\sim$ as U. The following remarks concerning U and M are useful, as all coverings which occur in algebraic geometry are inductive.

STATEMENT 2.2. *The elements of M are mutually exclusive subsets if and only if any two elements of U which have a nonempty intersection are both contained in the same element of U. Clearly, the relation $\sim$ is then transitive.*

PROOF. Let the elements of M be mutually exclusive and let $U_1, U_2 \in U$ where $U_1 \cap U_2 \neq \emptyset$. U_1 and U_2 are contained in respectively the elements M_1 and M_2 of M, and since clearly $M_1 = M_2$, the desired result follows. Conversely, if the condition on U is satisfied and $M_1, M_2 \in M$, where $M_1 \neq M_2$, then $M_1 \cap M_2$ must be empty since distinct maximal elements of U can not be contained in the same element of U.

In algebraic geometry, the elements of U are always certain irreducible algebraic systems of effective cycles. Before discussing algebraic systems, let us indicate by means of an example how the above notions may be used in topology.

EXAMPLE 2.1. Let K be a complex in the sense of topology, and let G be the additive group of the s-dimensional cycles of K. Hence G consists of the finite, linear combinations with integral coefficients of the connected, closed, oriented s-dimensional subcomplexes of K. Choosing $E = G$, we say that $A \sim B$, where $A, B \in G$, if both A and B can be deformed continuously into the same s-dimensional cycle. It is trivial that $\sim$ is reflexive and symmetric and hence $\approx$ defines a homomorphic image J of G which is obviously a topological invariant of K. Clearly, if $A * B$, these two cycles are homologous in the sense of topology and consequently the s-dimensional homology group of K is a homomorphic image of J. Whenever for a complex the topological relation of homology coincides with the relation $*$, we are in the advantageous position of being able to define homology solely by notions which refer to continuous deformations. In this connection, it is interesting to observe that the well known one-dimensional cycle A of the double torus, which is homologous to zero but can not be deformed continuously into a point, nevertheless satisfies $A * O$. (O is the zero-cycle.)

3. **Algebraic systems.** Returning to §1, let $A = n_1 W_1^{(s)} + \cdots + n_k W_k^{(s)}$ be a d-dimensional cycle of an irreducible algebraic variety $V^{(d)}$. We denote by ∂_i the degree of the irreducible variety $W_i^{(s)}$, that is, the maximum number of points of intersection which can occur when $W_i^{(s)}$ is intersected with a linear, $(n-s)$-dimensional subspace $L^{(n-s)}$ of the projective space $P^{(n)}$, where neither $W_i^{(s)} \subset L^{(n-s)}$ nor $L^{(n-s)} \subset W_i^{(s)}$. The rational integer $\partial = n_1\partial_1 + \cdots + n_k\partial_k$ is called *the degree of A*. It was proved in [2] that, *for fixed s and* ∂, there exists an algebraic correspondence N from an algebraic variety Π onto $V^{(d)}$ which establishes a (1-1)-correspondence between the points of Π and all the *effective* cycles of dimension s and degree ∂ of $V^{(d)}$; Π and N depend of course on s and ∂. Precisely, Π is a subvariety of some projective space $P^{(t)}$ described by coordinates $y_0, \cdots, y_t$; that is, the forms which determine Π are forms in the variables $y_0, \cdots, y_t$; t depends also on the choice of s and ∂ and Π may be reducible. N is determined by a finite set of forms $k_1(x; y), \cdots, k_a(x; y)$ (with complex coefficients) which are homogeneous separately in $x_0, \cdots, x_n$ and $y_0, \cdots, y_t$. These forms are such that if $(\eta_0, \cdots, \eta_t)$ is a point of Π and $N(\eta_0, \cdots, \eta_t) = n_1 W_1^{(s)} + \cdots + n_k W_k^{(s)}$, the algebraic variety $W_1^{(s)} \cup \cdots \cup W_k^{(s)}$ is determined by the equations $k_1(x; \eta) = 0, \cdots, k_a(x; \eta) = 0$. The positive multiplicities $n_1, \cdots, n_k$ can be defined conveniently by means of the "associated forms" of [2] and hence their definition does not require any theory of "specialization."

The collection of effective cycles which corresponds under N to an algebraic subvariety Ω of Π is called an *algebraic system*. Dimension, irreducibility, and components of an algebraic system S are defined in terms of its "parameter variety Ω."

In general, an algebraic correspondence J from a variety V_0 onto $V^{(d)}$ is a correspondence defined by means of a finite set of doubly homogeneous forms, just as N was defined by the forms $k_i(x; y)$.

The points of V_0 are transformed by J into effective cycles of $V^{(d)}$; and, if all these cycles have the same dimension and degree, they constitute an algebraic system. This is usually the easiest way to check whether a given collection of cycles is an algebraic system. If V_0 is also irreducible, J is called a birational transformation if the correspondence is a (1-1)-point transformation except at most on proper subvarieties of V_0 and $V^{(d)}$. We say then that V_0 and $V^{(d)}$ are birational models of one another. A large part of algebraic geometry consists of searching for birational invariants, that is, invariants of the variety under consideration which do not change when the variety is submitted to birational transformations. Dimension is a birational invariant. If Ω is the parameter variety of an algebraic system S and J is an algebraic correspondence from some other algebraic variety Ω^* onto Ω which is a (1-1)-point transformation without exception, we can clearly also consider Ω^* as a parameter variety of S with N replaced by NJ.

EXAMPLES 3.1. The points of $V^{(d)}$, that is, $s=0$, $\partial=1$, constitute an irreducible algebraic system of dimension d with $V^{(d)}$ as parameter variety. The point groups of arbitrary degree ∂ (those for which $s=0$ and ∂ is arbitrary) constitute a (∂d)-dimensional, irreducible algebraic system with the product $V^{(d)} \times \cdots \times V^{(d)}$ of $V^{(d)}$ with itself ∂ times as parameter variety. The straight lines on an irreducible quadratic surface $V^{(2)}$ in 3-space (here $s=\partial=1$) constitute a reducible algebraic system with two irreducible 1-dimensional components, namely the two pencils of lines on $V^{(2)}$; observe that these pencils have no element in common. Choose $V^{(3)}=P^{(3)}$, that is, all of 3-space. When $s=\partial=1$, we obtain again the straight lines, and it is well known that they constitute an irreducible 4-dimensional algebraic system whose parameter variety $\Omega^{(4)}$ arises from the quadratic relations among the Plücker coordinates. Let us consider the 2nd degree curves of $P^{(3)}$; then $s=1$, $\partial=2$. We have in the first place the pairs of lines, and they constitute an irreducible 8-dimensional system with $\Omega^{(4)} \times \Omega^{(4)}$ as parameter variety. Secondly, we have all conics. The conics which lie in a fixed plane of $P^{(3)}$ form an irreducible 5-dimensional system and hence the conics of $P^{(3)}$ constitute an 8-dimensional, irreducible system. Consequently for $s=1$, $\partial=2$, we obtain a reducible algebraic system with two irreducible 8-dimensional components; observe that these components have a nonempty intersection, namely the 7-dimensional algebraic system consisting of the pairs of *intersecting* lines. It may happen that Π consists of only a finite number of points, even when $s<d$. For example, when $V^{(2)}$ is a nonsingular cubic surface in $P^{(3)}$, Π for $s=\partial=1$ consists of 27 points.

Warning. Let Ω be the irreducible parameter variety of an irreducible algebraic system S. If $A_1, A_2 \in S$ and Q_1, Q_2 are the points of Ω which correspond respectively to A_1 and A_2, we can join Q_1 and Q_2 by an arc which lies completely on Ω. We should not conclude from this that necessarily A_1 can be continuously deformed into A_2. For example, if S is the irreducible 8-dimensional system consisting of all pairs of lines of $P^{(3)}$ and A_1 is a pair of intersecting lines and A_2 is a pair of not intersecting lines, A_1 can not be deformed continuously into A_2; of course, in this case it is still possible to deform A_2 continuously into A_1.

Finally, we observe that irreducible algebraic systems satisfy both the ascending and the descending chain conditions. This result follows from the fact that the irreducible systems, belonging to either an increasing or a descending chain, can all be considered as irreducible subvarieties of the fixed variety Π which comprises all effective cycles of the dimension and the degree under consideration.

4. **Algebraic equivalence.** Consider the cycle group C_s for some fixed dimension $0 \leqq s \leqq d$ of our d-dimensional irreducible algebraic variety $V^{(d)}$. For two effective cycles $E_1, E_2 \in C_s$ we define that $E_1 \approx E_2$ if they belong to the same *irreducible* algebraic system; E_1 and E_2 then have the same degree. The relation $\approx$ is trivially reflexive and symmetric, and it is easily seen to be additive. For, if E_1 and E_2 belong to an irreducible algebraic system with parameter variety Ω_1, and if E_3 and E_4 belong to an irreducible algebraic system with parameter variety Ω_2, then E_1+E_3 and E_2+E_4 belong to the necessarily irreducible algebraic system whose parameter variety is the product $\Omega_1 \times \Omega_2$. We conclude from §2 that we obtain a homomorphic image of C_s by defining that $A \,||||\, B$ if A and B are any two elements of C_s for which there exists an $X \in C_s$ such that $A+X$ and $B+X$ are effective and $A+X \approx B+X$; clearly A and B must then have the same degree. The equivalence relation $||||$ is called *algebraic equivalence*. The relation $\approx$ was obtained from the covering U of the set of effective cycles of C_s whose elements are the irreducible algebraic systems. Since irreducible algebraic systems satisfy the ascending chain condition, U is inductive; M (see §2) consists of the irreducible components of the different algebraic systems which comprise all effective s-dimensional cycles of fixed degree ∂. (See §3. We have one such system for each $\partial = 1, 2, \cdots$.) The sets of M may not be mutually exclusive, as the 2nd degree curves of $P^{(3)}$ demonstrate (see examples 3.1), and consequently $\approx$ and $||||$ may not coincide for the effective cycles of C_s. For example, choosing $V^{(d)} = P^{(3)}$, a nonsingular conic A and a pair of not intersecting lines B are in the rela-

tion $||||$ but not in the relation $\approx$. (Again, A can not be deformed continuously into B, but now A and B can both be deformed continuously into the same pair of intersecting lines. Considerations of this sort gave rise to example 2.1.) In spite of the warning of §3, when studying algebraic equivalence in C_s, we can expect to be dealing with questions closely tied up with the homology relation of the $2s$-dimensional cycles of $K^{(2d)}$. When $d=2$, this is exactly the case as we will now explain.

Let $V^{(2)}$ be an irreducible algebraic surface in 3-dimensional space whose complex $K^{(4)}$ can be considered as homogeneous; this means that every point of $K^{(4)}$ is the interior of a 4-cell. Every surface has a birational model of this type. This follows from the classical result that every surface has a birational model without singularities in 5-dimensional projective space and hence one with at most so-called "ordinary singularities" in 3-dimensional space. For our $K^{(4)}$, the Poincaré duality relations are now valid and hence, if b_0, b_1, b_2, b_3, b_4 are its Betti numbers and $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ its torsion coefficients, $b_0=b_4=1$, $b_1=b_3$, $\sigma_0=\sigma_3=0$, $\sigma_1=\sigma_2$. In order to investigate b_2 and σ_2, we study C_1 under algebraic equivalence. It was proved by Lefschetz (see [3]) that if $A, B\in C_1$, then $A\ ||||\ B$ if and only if their 2-dimensional images in $K^{(4)}$ are homologous. Hence the homomorphic image H_1 of C_1 which $||||$ defines is isomorphic to a subgroup of the 2-dimensional homology group of $K^{(4)}$. Since this homology group has a finite number of generators with respect to the integers, so does H_1; and consequently H_1 is a direct sum $H_1=T\oplus B$, where T consists of all the elements of H_1 which have a nonzero order. We should expect that T is isomorphic with a subgroup of the 2-dimensional torsion group of $K^{(4)}$ but, as is proved in [3], T actually is isomorphic with that torsion group itself. Hence, the minimum number of generators of T is the algebro-geometric definition of $\sigma_1=\sigma_2$. B is indeed isomorphic with only a subgroup of the 2-dimensional Betti group of $K^{(4)}$ and hence, if ρ denotes the maximum number of linearly independent elements of B, $\rho\leqq b_2$. A birational transformation which either produces or loses exceptional curves is not a topological transformation, and hence topological invariants may not be birational invariants. Indeed, neither ρ nor b_2 are birational invariants, but both numbers change by the amount e_1-e_2 if the transformation produces e_1 exceptional curves and loses e_2 such curves. It follows that $b_2-\rho=\rho_0$ is a birational invariant of $V^{(2)}$. In the transcendental theory of algebraic surfaces, ρ occurs in the study of the logarithmic singularities of the simple integrals of the third kind of $V^{(2)}$, while the birational invariance of ρ_0 also follows from the fact that $V^{(2)}$

possesses exactly ρ_0 proper double integrals of the second kind which are linearly independent modulo the improper integrals. Every birational transformation can be decomposed into a finite number of birational transformations, each one of which produces or loses only exceptional curves of the first kind. Since such curves can not belong to T (the "virtual degree" of an exceptional curve of the first kind is -1 and of a curve of T is 0), $\sigma_2 = \sigma_1$ is also a birational invariant of $V^{(2)}$.

Examples 4.1. If $V^{(2)}$ is a plane, all effective, one-dimensional cycles of degree ∂ form an irreducible algebraic system of dimension $\partial(\partial+3)/2$, parametrized by the projective space whose coordinates are the coefficients of the forms $k(x_0, x_1, x_2)$ of degree ∂. Hence if A is a straight line of $V^{(2)}$ all these cycles are algebraically equivalent to ∂A. Since every cycle is the difference of two effective cycles, every cycle is algebraically equivalent to nA for some whole number n. It follows that $\rho = 1$ and that $\sigma_1 = \sigma_2 = 0$. A surface is called *rational* if it can be transformed birationally into a plane. Hence, the torsion coefficients of rational surfaces are always zero. Let $V^{(2)}$ be a nonsingular quadric. If we project $V^{(2)}$ on a plane from a point Q on $V^{(2)}$ as projection center, in the resulting birational transformation the two straight lines of $V^{(2)}$ which pass through Q go over into points and Q itself goes over in a straight line. Hence, $V^{(2)}$ is a rational surface, and $e_1 = 1$, $e_2 = 2$; consequently $\rho + 1 - 2 = 1$, that is, $\rho = 2$. Two straight lines A, B of $V^{(2)}$ which intersect are not algebraically equivalent and hence every one-dimensional cycle of $V^{(2)}$ is algebraically equivalent to $n_1A + n_2B$. Surfaces with nonzero torsion are the sixth degree surfaces whose ordinary singularities form the six edges of a tetrahedron. These surfaces, which consequently are never rational, are due to Enriques.

It is believed that not only the C_1 of $V^{(2)}$ has a finite number of generators with respect to $||||$, but that this is true for every C_s of every $V^{(d)}$ with respect to any groundfield of characteristic zero. This conjecture has been proved for $s = d-1$ and the complex numbers as groundfield by means of transcendental and topological methods. Beyond this case, the conjecture has been proved only for special types of varieties. Even when $d = 2$, we can not prove the conjecture when the groundfield is not the field of complex numbers.[1]

5. **Linear series.** In the theory of linear equivalence, arbitrary ir-

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[1] A proof for the case $s = d-1$ and arbitrary groundfield is sketched by A. Néron in *La théorie de la base pour les diviseurs sur les variétés algébriques*, "Deuxième Colloque de Géométrie Algébrique," Liège, 1952, pp. 119–126.

reducible algebraic systems are replaced by the special irreducible algebraic systems which are called linear series. So, let us study them first.

Consider a form $k(x_0, \cdots, x_n)$ of positive degree which is such that the hypersurface G defined by $k(x)=0$ does not contain our $V^{(d)}$. The intersection $V^{(d)} \cap G$ is then the union of a finite number of irreducible, $(d-1)$-dimensional subvarieties of $V^{(d)}$, say $V^{(d)} \cap G = W_1^{(d-1)} \cup \cdots \cup W_h^{(d-1)}$. It is proved in [4] that the intuitive intersection multiplicities which a geometer wants to associate with the intersection components $W_1^{(d-1)}, \cdots, W_h^{(d-1)}$ can be defined ideal-theoretically as follows. Let $\mathfrak{p}$ be the prime ideal which consists of all forms which vanish everywhere on $V^{(d)}$. In a decomposition of the ideal $(\mathfrak{p}, k(x))$ into largest primary ideals, exactly h isolated primary components occur, say $\mathfrak{q}_1, \cdots, \mathfrak{q}_h$. The associated prime ideal $\mathfrak{p}_i$ of $\mathfrak{q}_i$ defines a variety $W_i^{(d-1)}$, and the intersection multiplicity n_i of $W_i^{(d-1)}$ as a component of $V^{(d)} \cap G$ is the length of the primary ideal $\mathfrak{q}_i$, $i=1, \cdots, h$. It is well known that lengths of primary ideals do not generally give the correct multiplicities in intersection theory, but in our simple case, where a variety is intersected with a hypersurface, the length can always be used. From now on we consider $V^{(d)} \cap G$ as the effective $(d-1)$-dimensional cycle $n_1 W_1^{(d-1)} + \cdots + n_h W_h^{(d-1)}$. If the degrees of $V^{(d)}$ and G are respectively α and β (it is easy to show that β is the degree of the form $k(x_0, \cdots, x_n)$), the degree of the cycle $V^{(d)} \cap G$ is $\alpha\beta$. For this reason, when $k(x_0, \cdots, x_n)$ is a nonzero constant and hence G the empty set, we consider $V^{(d)} \cap G$ as the zero cycle.

EXAMPLE 5.1. Let $V^{(2)}$ be an irreducible quadratic cone in 3-space and A and B two distinct straight lines on $V^{(2)}$. The plane containing A and B is denoted by G_1, and the tangent plane of the cone which contains A by G_2. Then $V^{(2)} \cap G_1 = 1A + 1B$ and $V^{(2)} \cap G_2 = 2A$; the multiplicity 2 arises from the fact that if we deform G_2 into another plane through the vertex of the cone, $V^{(2)} \cap G_2$ splits up into two distinct straight lines.

Now let $k_0(x_0, \cdots, x_n), \cdots, k_r(x_0, \cdots, x_n)$ be r forms which all have the same degree β and which are linearly independent modulo $V^{(d)}$. If $\lambda_0, \cdots, \lambda_r$ are complex numbers, not all zero, $\lambda_0 k_0(x) + \cdots + \lambda_r k_r(x)$ is also a form of degree β and the condition of linear independence states exactly that the hypersurface $G(\lambda_0, \cdots, \lambda_r)$ with equation $\lambda_0 k_0(x) + \cdots + \lambda_r k_r(x) = 0$ does not contain $V^{(d)}$. Hence, the intersection cycle $V^{(d)} \cap G(\lambda_0, \cdots, \lambda_r)$ is a well defined effective cycle of degree $\alpha\beta$ of C_{d-1} (α is again the degree of $V^{(d)}$). The effective cycles, which in this way correspond to the points of the r-dimensional projective space $\Lambda^{(r)}$ of $(r+1)$-tuples $(\lambda_0, \cdots, \lambda_r)$, are called a *linear series* g^r of dimension r and degree $\alpha\beta$. If it is possible to subtract from all the cycles of $g^r_{\alpha\beta}$ a fixed *effective* $(d-1)$-dimensional cycle of degree β_0 so that all the remaining cycles are still effective, we consider also these remaining cycles as a linear series $g^r_{\alpha\beta-\beta_0}$ of dimension r and degree $\alpha\beta-\beta_0$, parametrized by the same projective space $\Lambda^{(r)}$ as above.

It is clear that an r-dimensional linear series g^r_t is an irreducible, r-dimensional algebraic system consisting of $(d-1)$-dimensional effective cycles of degree t for whose parameter variety we can choose the above projective space $\Lambda^{(r)}$. It follows that r can not exceed the dimension of the algebraic variety Π which parametrizes *all* the effective $(d-1)$-dimensional cycles of degree t. For example, if $d=1$, g^r_t consists of pointgroups of degree t, and we have seen in examples 3.1 that Π then has dimension t. Hence, $r \leqq t$, if g^r_t is a linear series on a curve.

Example 5.2. Let $V^{(2)}$ again be an irreducible quadratic cone in 3-space $P^{(3)}$ and A and B two distinct lines on $V^{(2)}$. All quadrics of $P^{(3)}$ which contain both A and B constitute a linear system L of dimension 4 in which occur three quadrics which are linearly independent modulo our $V^{(2)}$. Hence the quadrics of L cut out on the cone a linear series g^3_4, consisting of $A+B+D_t$, where D_t runs through all the nonsingular and singular conics on the cone. This g^3_4 has $A+B$ as fixed cycle. Consequently the sets of effective cycles $A+D_t$, $B+D_t$, D_t are all considered as 3-dimensional linear series of degrees respectively 3, 3, 2; only the last linear series has no fixed cycle.

Let $p(x_0, \cdots, x_n)$, $q(x_0, \cdots, x_n)$ be two forms of the same degree, where the hypersurface $G(q)$ with equation $q(x)=0$ does not contain $V^{(d)}$. The quotient $p(x)/q(x)$ then determines a function on $V^{(d)}$ which is well defined there except for the points of the cycle $V^{(d)} \cap G(q)$. If we add and multiply these functions *as functions on* $V^{(d)}$, we obtain a field F which is called *the field of rational functions of* $V^{(d)}$. If also the hypersurface $G(p)$ with equation $p(x)=0$ does not contain $V^{(d)}$, the function is a nonzero element of F and the well defined $(d-1)$-dimensional cycle $(V^{(d)} \cap G(p)) - (V^{(d)} \cap G(q))$ of degree zero is then called the cycle $D(\phi)$ of the rational function ϕ determined by $p(x)/q(x)$. It is easy to see that the mapping $\phi \rightarrow D(\phi)$ is a homomorphism from the multiplicative group of F into C_{d-1}, and hence these cycles $D(\phi)$ form a subgroup R of C_{d-1}. Two irreducible varieties can be mapped onto one another by means of a birational transformation if and only if their fields of rational functions are isomorphic under an isomorphism which leaves the constant functions pointwise fixed. Small wonder then that if we study C_{d-1} under the equivalence relation imposed upon it by R, and this relation will turn out to be linear equivalence, we shall discover further birational invariants.

Let E_1 and E_2 be two effective cycles which belong to the same linear series, cut out on $V^{(d)}$ by the linearly independent forms $k_0(x), \cdots, k_r(x)$; say E_1 and E_2 arise respectively from the hypersurfaces G_1, G_2 with equations $\lambda_0 k_0(x) + \cdots + \lambda_r k_r(x) = 0$ and $\mu_0 k_0(x) + \cdots + \mu_r k_r(x) = 0$. Denoting by E_0 the fixed divisor which may have been subtracted from the total intersections, $E_1 = (V^{(d)} \cap G_1) - E_0$ and $E_2 = (V^{(d)} \cap G_2) - E_0$. Hence, $E_1 - E_2 = D(\phi)$, where ϕ is the nonzero rational function of $V^{(d)}$ determined by the quotient $\lambda_0 k_0(x) + \cdots + \lambda_r k_r(x) / \mu_0 k_0(x) + \cdots + \mu_r k_r(x)$. We prove just as easily that all cycles of R can be obtained in this way; that is, *a cycle belongs to R if and only if it is the difference of two effective cycles which belong to the same linear series.* This statement establishes the connection between linear series and rational functions.

6. **Linear equivalence.** This type of equivalence is restricted to the group C_{d-1} of our $V^{(d)}$. For two effective cycles $E_1, E_2 \in C_{d-1}$ we define that $E_1 \approx E_2$ if they belong to the same linear series, that is, if $E_1 - E_2$ belongs to the subgroup R of C_{d-1}. (See the previous section.) We conclude immediately that the relation $\approx$ is reflexive, symmetric, additive, subtractive, and transitive. It follows from §2 that we obtain a homomorphic image of C_{d-1} with kernel R by defining that $A \equiv B$ if A and B are any two elements of C_{d-1} for which there exists an $X \in C_{d-1}$ such that $A + X$ and $B + X$ are effective and $A + X \approx B + X$. Clearly, if $A \equiv B$, these cycles have the same degree and the relations $\equiv$ and $\approx$ coincide on the set of effective cycles. Observe that if $A \equiv B$, then certainly $A \,||||\, B$.

The relation $\approx$ was obtained from the covering U of the set of effective cycles of C_{d-1} whose elements are the linear series of $V^{(d)}$. Since linear series are irreducible algebraic systems, U is again inductive. The elements of M (see §2) are now called *complete linear series.* In order to show that two distinct complete linear series are mutually exclusive, we must prove that when two linear series have a cycle in common, they are contained in the same linear series. (See statement 2.2.) This fact, whose analogue we have seen to be false for algebraic equivalence, does not follow trivially from anything which has been said in this exposition; a simple proof of it can be found in §17 of [5]. Accepting this fact as established, we see that every effective, $(d-1)$-dimensional cycle E is contained in a unique complete linear series $|E|$ which consists of all effective cycles, linearly equivalent to E; exactly the same remark can be made concerning any linear series g^r and the unique complete linear series $|g^r|$ to which all cycles of g^r belong. If $|g^r|$ has dimension r^1, the *deficiency* δ of g^r is defined as $\delta = r^1 - r$; clearly $\delta \geqq 0$, and $\delta = 0$ if and only if g^r is complete.

Let $W^{(d-1)}$ be an irreducible, $(d-1)$-dimensional subvariety of $V^{(d)}$ and let us choose $E = W^{(d-1)}$. Suppose that $|E|$ has dimension r and is cut out on $V^{(d)}$ by the hypersurfaces of the system $\lambda_0 k_0(x) + \cdots + \lambda_r k_r(x) = 0$, where of course an appropriate fixed, effective cycle E_0 may have been subtracted from the total intersection cycles. Since $W^{(d-1)}$ is irreducible, our $|E|$ has no fixed cycle, unless $r = 0$ in which case $|E|$ consists of $W^{(d-1)}$ alone. (This happens for example when $W^{(1)}$ is one of the 27 straight lines on a cubic surface without singularities.) We make the special assumption that there does not exist a $(d-2)$-dimensional subvariety of $V^{(d)}$ which belongs to all cycles of $|E|$; this clearly is stronger than assuming that $r > 0$. Under this assumption, the linear series g^{r-1}, cut out on E by the cycles of $|E|$, can be defined as follows, avoiding all questions of intersection theory we have not dealt with already. Among the above forms $k_0(x), \cdots, k_r(x)$ there are precisely r forms, say $k_0(x), \cdots, k_{r-1}(x)$, which are linearly independent modulo $W^{(d-1)}$. The g^{r-1} cut out on E by $|E|$ is exactly the linear series *without fixed cycle* cut out on $W^{(d-1)}$ by the hypersurfaces $\lambda_0 k_0(x) + \cdots + \lambda_{r-1} k_{r-1}(x) = 0$. (This may of course necessitate subtracting a fixed, effective, $(d-2)$-dimensional cycle of $W^{(d-1)}$ from the total intersection divisors of $W^{(d-1)}$ with the hypersurfaces $\lambda_0 k_0(x) + \cdots + \lambda_{r-1} k_{r-1}(x) = 0$.) The fact that $|E|$ is complete does not imply that g^{r-1} is complete; on the contrary, the deficiency of g^{r-1} may very well be positive.

EXAMPLES 6.1. $V^{(3)}$ is all of 3-space and $W^{(2)}$ is one of its planes. $|E|$ now consists of all planes of 3-space and hence $r = 3$. The g^2 consists of all straight lines of $W^{(2)}$ and hence is complete. Now let $V^{(2)}$ be all of 2-space and $W^{(1)}$ an irreducible, third degree curve in $V^{(2)}$ with a double point. $|E|$ consists of all third degree curves of $V^{(2)}$ and hence $r = 9$. The g^8 clearly has degree 9; that is, it is a g_9^8. Since our $W^{(1)}$ is a rational curve, this g_9^8 is not complete, but has deficiency $\delta = 1$. Observe that, if we had chosen for our $W^{(1)}$ an irreducible third degree curve without singularities, the corresponding g_9^8 would have been complete.

REMARK. Whenever $W^{(s)}$ is an irreducible, s-dimensional subvariety of $V^{(d)}$ and g a linear series of $V^{(d)}$, such that there does not exist an $(s-1)$-dimensional subvariety of $W^{(s)}$ which belongs to all cycles of g, the linear series of $W^{(s)}$ which results from intersecting $W^{(s)}$ with the cyles of g can be defined as above.

Let us now return to the case $d=2$. We assume again that $V^{(2)}$ is an irreducible surface of 3-space with only ordinary singularities whose complex $K^{(4)}$ is homogeneous. We shall call an irreducible curve $W^{(1)}=E$ of $V^{(2)}$ "general" if:

(1) There exists on $V^{(2)}$ no point which belongs to all cycles of $|E|$.

(2) A tangent plane of $V^{(2)}$ contains at most one simple tangent of $W^{(1)}$.

The first condition guarantees that the linear series g_t^{r-1} of $W^{(1)}$, obtained by intersecting $W^{(1)}$ with the cycles of $|E|$, can be defined as above. This g_t^{r-1} is called the *characteristic series of* $W^{(1)}$ and its degree t the *virtual degree of* $W^{(1)}$. The second condition states that on every two-dimensional branch of $V^{(2)}$ our curve $W^{(1)}$ behaves as a nonsingular curve. $V^{(2)}$ has general curves galore. For example, the intersection of $V^{(2)}$ with a hypersurface will almost always be a general curve. We are using here the term "general" as is customary in algebraic geometry. Precisely, if ∂ is large enough, all one-dimensional cycles of degree ∂ are general curves, except for perhaps those which correspond to the points of a subvariety Ω of the variety Π which parametrizes all one-dimensional cycles of degree ∂; the important thing is that the dimension of Π exceeds the dimension of Ω. (The dimension of a reducible variety is defined as the largest dimension which occurs among its irreducible components.)

Castelnuovo has proved in [6] that the deficiencies of the characteristic series of the general curves of $V^{(2)}$ reach a finite maximum on $V^{(2)}$. This maximum is usually denoted by q and is called *the irregularity of the surface*; when $q=0$, the surface is called *regular*, otherwise *irregular*. The Betti number $b_1=b_3$ of $K^{(4)}$ is equal to $2q$ (see [3]), and hence we have now found an algebro-geometric definition for this Betti number. We see that the connected, orientable, closed, homogeneous, four-dimensional complexes, which arise from algebraic surfaces, are not the most general complexes of this type; a restriction on their topology is that b_1 is even. Since q is the maximum number of simple integrals of the first kind of $V^{(2)}$ which are linearly independent modulo the constants, q is a birational invariant of $V^{(2)}$. Observe from examples 6.1 that the second condition in the definition of a general curve can not be left out, when quoting the Castelnuovo theorem; namely, for the plane, $q=0$ but, for the cubic curve with double point, the deficiency of its characteristic series is 1.

The following birational invariants also belong in the theory of linear equivalence. The singular points of $V^{(2)}$ fill up an algebraic curve. If $D_1^{(1)}, \cdots, D_k^{(1)}$ are the irreducible components of this curve, we denote by D the one-dimensional cycle $D_1^{(1)}+\cdots+D_k^{(1)}$; if $V^{(2)}$ has no singularities, we consider D as the zero cycle. The surfaces of 3-space which contain D, that is, which contain $D_1^{(1)}, \cdots, D_k^{(1)}$, are called the *adjoints of* $V^{(2)}$. If m is the degree of $V^{(2)}$, the maximum number of linearly independent adjoints of degree $m-4$ is a birational invariant of $V^{(2)}$, called *the geometric genus* p_g *of* $V^{(2)}$. Hence, the adjoints of degree $m-4$, if there are any, cut out on $V^{(2)}$ a linear series of dimension p_g-1 which contains at least D as a fixed cycle; the (p_g-1)-dimensional linear series which is obtained by subtracting D from all the total intersection cycles is called *the impure canonical system* K *of* $V^{(2)}$. Every linear series of $V^{(2)}$, obtained by intersecting $V^{(2)}$ with all its adjoints of a fixed degree and subtracting D from the total intersection cycles, is a complete linear series; in particular, K is complete. The birational invariance of p_g follows from the fact that the maximum number of double integrals of the first kind of $V^{(2)}$, which are linearly independent modulo the constants, is exactly p_g. The birational invariant p_g-q is called the *arithmetic genus* p_a *of* $V^{(2)}$; p_a may very well be negative.

It is usually not difficult to compute p_g directly from its definition. We now show how p_a, and hence q, can be computed. The planes of 3-space constitute an irreducible algebraic system of dimension 3. Except for at most a subvariety of planes (that is, an algebraic system of planes) of dimension 2, a plane intersects $V^{(2)}$ in an irreducible curve. The genus of such an irreducible plane curve of degree m is of course at most $(m-1)(m-2)/2$. We denote by p the maximum genus of the irreducible plane curves of $V^{(2)}$ and by α the maximum number of linearly independent adjoints of degree $m-3$ of $V^{(2)}$. It follows from a theorem of Picard (see [7, vol. 2, p. 437]) that $p_a=\alpha-p$.

EXAMPLES 6.2. Let $V^{(2)}$ be a plane. Then $m=1$ and hence there exist no adjoints of degree $m-4$ or $m-3$; hence $p_g=\alpha=0$. Intersection of $V^{(2)}$ with a plane gives rise to a straight line; hence $p=0$, which shows that $p_a=q=0$. Hence, the three birational invariants q, p_g, p_a are also zero for all rational surfaces. An example of a nonrational surface with $q=p_g=p_a=0$ is furnished by the sixth degree surfaces of Enriques. (See examples 4.1.) In that case, $m-4=2$; and, since there exist no quadrics which contain all six edges of a tetrahedron, $p_g=0$. A general plane intersects $V^{(2)}$ in a 6th degree curve with 6 ordinary double points; hence, $p=4$. The four faces of the tetrahedron, taken three at a time, furnish four linearly independent adjoints of degree 3, from which it follows that $\alpha=4$ and hence that $p_a=q=0$. All surfaces $V^{(2)}$ without singularities are regular. Thus, the number of linearly independent surfaces of degree $m-4$ is $(m-1)(m-2)(m-3)/6=p_g$; in the same way, $\alpha=1m(m-1)(m-2)/6$. A general plane intersects $V^{(2)}$ in a nonsingular curve of degree m; hence $p=(m-1)(m-2)/2$. It follows that $p_a=\alpha-p=(m-1)(m-2)\cdot(m-3)/6$ and hence that $q=0$. Ruled surfaces never have adjoints of degree $m-4$ or $m-3$, while p can be arbitrarily large. Hence, these surfaces furnish examples of irregular surfaces, as now $p_g=0$, $p_a=-p$, $q=p$.

The only topological invariant of $K^{(4)}$ for which we have not yet given an algebro-geometric definition is the second Betti number b_2. We have seen that the straight lines of 3-space form a 4-dimensional irreducible algebraic system. The number of tangent planes of $V^{(2)}$ which pass through a straight line of 3-space is always the same, except for at most a 3-dimensional algebraic system of lines. This number is called *the class c of $V^{(2)}$*. The Euler characteristic $b_0-b_1+b_2-b_3+b_4$ of $K^{(4)}$ turns out to be $c-4p-m+4$, where p has the same meaning as above; this follows from a formula of Alexander (see Rendiconti dei Lincei, August 1914, and [5, p. 96]). Hence, $2-4q+b_2=c-4p-m+4$, which shows that $b_2=c-4p-m+4q+2$. This formula can be considered as an algebro-geometric definition of b_2 and enables us to compute b_2 easily.

EXAMPLES 6.3. If $V^{(2)}$ is a plane, $c=p=q=0$ and $m=1$; hence, $b_2=1$. Consequently, $\rho_0=0$ and since ρ_0 is a birational invariant (see §4), $\rho_0=0$ for all rational surfaces. We conclude for example that for the nonsingular quadrics, $b_2=2$.

We have not attempted to define all known birational invariants of surfaces. Equivalence relations have further usage, besides discovering and studying birational invariants. For example, the Riemann-Roch theorem for surfaces belongs properly in the theory of linear equivalence. Let $W^{(1)}$ be a general curve of genus p, and with characteristic series g_t^{r-1}. Let i be the number of linearly independent cycles of the impure canonical system K which contain $W^{(1)}$ as a component. Then the Riemann-Roch theorem asserts that $r\geqq t-p+p_a-i+1$.

7. **Rational equivalence.** It is true that, using only the part of intersection theory which is based on the notion of length of a primary ideal, we have been able to define several of the invariants of algebraic surfaces and state some of their properties. This should not mislead the reader into thinking that the algebrogeometric theory of

surfaces can be developed without some thorough, general intersection theory. *It is inconceivable that the assertions of the previous sections can be proved, using only intersection multiplicities which are based on the notion of length.*

Let us return to our irreducible variety $V^{(d)}$. If $E_1, \cdots, E_h$ are h effective, $(d-1)$-dimensional cycles of $V^{(d)}$ where $1 \leqq h \leqq d$, it is part of the task of intersection theory to define the $(d-h)$-dimensional intersection cycle $E_1 \cdot \cdots \cdot E_h$ of $E_1, \cdots, E_h$. If $W_1^{(d-h)}, \cdots, W_k^{(d-h)}$ are the irreducible, $(d-h)$-dimensional components of $E_1 \cdot \cdots \cdot E_h$ and $n_1, \cdots, n_k$ their respective intersection multiplicities, the cycle $E_1 \cdot \cdots \cdot E_h$ is exactly $n_1W_1^{(d-h)} + \cdots + n_kW_k^{(d-h)}$. It is usually intuitively evident what $W_1^{(d-h)}, \cdots, W_k^{(d-h)}, n_1, \cdots, n_k$ are, but intersection theory tries to give definitions for them which are acceptable both from the point of view of mathematical rigor and mathematical simplicity. Let us suppose that this has been done, that is, that the intersection cycle $E_1 \cdot \cdots \cdot E_h$ has been well defined.

Consider h linear series $L_1, \cdots, L_h$ of $V^{(d)}$, $1 \leqq h \leqq d$. Each cycle E_i of L_i corresponds to a point of the parametrizing projective space Ω_i of L_i, and hence to each point of the product space $\Omega_1 \times \cdots \times \Omega_h$ we can associate the intersection cycle $E_1 \cdot \cdots \cdot E_h$. The effective, $(d-h)$-dimensional cycles, obtained in this way, form an irreducible algebraic system, parametrized by $\Omega_1 \times \cdots \times \Omega_h$ and called an *elementary system of rational equivalence*. When $h=1$, we obtain again the linear series and, as in the case of linear series, we allow subtraction of fixed cycles if they occur. For two effective, $(d-h)$-dimensional cycles E_1, E_2 we define that $E_1 \sim E_2$ if they belong to the same elementary system of rational equivalence. The binary relation $\sim$ is reflexive and symmetric, but may not be additive. If it is not, we first go over to an additive relation $\approx$ for the effective, $(d-h)$-dimensional cycles by the procedure described in §2; the covering U now has as elements the elementary systems of rational equivalence. Then, by our standard method, we introduce an equivalence relation for the group C_{d-h} of $(d-h)$-dimensional cycles which gives rise to a homomorphic image of C_{d-h}. The relation for the cycles of C_{d-h}, obtained in this way, is called *rational equivalence*. Hence, two arbitrary cycles A, $B \in C_{d-h}$ are rationally equivalent if there exists an $X \in C_{d-h}$ such that $A+X=E_{11}+\cdots+E_{1k}$, $B+X=E_{21}+E_{22}+\cdots+E_{2k}$, where the effective cycles $E_{1i} \sim E_{2i}$ for $i=1, \cdots, k$. Rationally equivalent cycles clearly have the same degree and are necessarily algebraically equivalent; if $h=1$, rational equivalence coincides with linear equivalence.

The rational equivalence of pointgroups, that is, of 0-dimensional cycles, is used in the proofs of the theorems we stated for surfaces. Efforts to extend these theorems to varieties of arbitrary dimension lead to outstanding problems of algebraic geometry and in this work the rational equivalence for arbitrary C_{d-k} is used.

8. **How to get started in algebraic geometry.** This section is meant for readers who want to learn algebraic geometry but have to do this studying by themselves.

For most people, the easiest way to get into this field is probably through the so-called algebro-geometric method. This is the method whose main instruments come from projective geometry and modern algebra and which uses analysis and topology only as secondary tools; it is the method used in this exposition.

First, then, some knowledge of projective geometry and modern algebra has to be acquired. When selecting one of the many books on projective geometry, choose one which treats this subject from the technical rather than from the axiomatic point of view. A book of about 150 to 200 pages which *defines* n-dimensional projective space in terms of $(n+1)$-tuples, discusses general properties of n-dimensional space rather than special properties of the straight line, the plane, and 3-space, *uses* projective coordinates whenever they are convenient, and so on, is the sort of book best suited for the present purpose. As far as modern algebra is concerned, some knowledge of field extensions, valuation theory, and commutative ring theory should be obtained. It is not necessary to study any of these subjects extensively to get started in algebraic geometry. For example, §§6 through 41, 50 through 52, and chapters 8, 10, 11, 12, 13 of the English edition of van der Waerden's *Modern algebra* should be more than enough.

As a first book on algebraic geometry, try *Algebraic geometry* by S. Lefschetz which is announced as No. 18 of the Princeton Mathematical Series, to be published in June, 1953. This book makes tough, but rewarding, reading and is particularly strong in giving the reader an insight into the transcendental methods of curve and surface theory, although only the algebro-geometric method is used. When the reading gets too tough, the very clearly written *Algebraic curves* by R. J. Walker may be of help; this book is more elementary than the first and treats many of its subjects. Next, it is advisable to study volume 2 of *Methods of algebraic geometry* by W. V. D. Hodge and D. Pedoe. Here we find a good treatment of algebraic systems, algebraic equivalence, intersection theory, and so on. After these two books have been studied, the student ought to be willing to relax his mathematical rigor somewhat, so that he can get an insight into the over-all accomplishments of classical algebraic geometry and acquaint himself with the outstanding problems of present day algebraic geometry. There is no doubt that as yet this can not be done if we insist that every statement be proved with complete mathematical rigor, as we understand that term today. Excellent books for this purpose are reference [5] of this exposition and further literature given there.

Once the above material has been absorbed, the student should be ready to specialize in whatever direction he chooses. Besides the algebro-geometric direction, there are the algebraic direction and the transcendental direction.

If the student wants to go on in the algebro-geometric direction, he should read the papers by O. Zariski in order to get an insight into how valuation theory is used in the study of varieties of arbitrary dimension.

The algebraic direction is the one where as little projective geometry as possible is used, where the groundfield is made as arbitrary as possible, and where one tries to use only algebra as his tool. Although the distinction between the algebro-geometric method and the algebraic method is necessarily vague, it is safe to say that the latter method is the one which is most important for algebraic number theory. The student who wants to pursue the strictly algebraic direction should begin by learning the abstract theory of algebraic function fields of one variable. An excellent account of this theory can be found in *Algebraic numbers and algebraic functions* by E. Artin; this is a set of lecture notes, mimeographed in 1950–1951 by New York University. The student should then study *Foundations of algebraic geometry* by A. Weil, published as vol. 29 of the American Mathematical Society Colloquium Publications.

The transcendental direction is the one which is based on analysis and topology. There one can get a start by studying in references [3] and [7] of this exposition. As a preparation for [7], the *Traité d'Analyse* by É. Picard is helpful. Except for a 1934–1935 mimeographed set of Princeton lectures, given by S. Lefschetz and written by N. E. Steenrod and H. Wallman, the author is not able to mention an efficient preparation for [3].

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THE OCTOBER MEETING IN NEW YORK

The four hundred ninety-fifth meeting of the American Mathematical Society was held on Thursday through Saturday, October 22–24, 1953, at Columbia University in New York City, in conjunction with a Conference on Training in Applied Mathematics sponsored jointly by the National Research Council and the Society. About 350 persons attended including the following 309 members of Society:

J. C. Abbott, Milton Abramowitz, C. R. Adams, M. I. Aissen, A. A. Albert, Linda Allegri, B. A. Amira, R. D. Anderson, R. L. Anderson, M. M. Andrew, Joseph Andrushkiw, R. G. Archibald, P. N. Armstrong, N. J. Artin, W. A. Asprey, Silvio Aurora, W. G. Bade, Frederick Bagemihl, F. E. Baker, K. Y. Bal, R. P. Bambah, R. B. Barrar, J. H. Barrett, L. K. Barrett, Walter Bartky, E. G. Begle, A. A. Bennett, R. R. Bernard, Lipman Bers, M. A. Biot, I. E. Block, H. W. Bode, H. F. Bohnenblust, S. G. Bourne, J. W. Bower, J. R. Bowman, E. H. Boyle, J. W. Brace, C. C. Bramble, Eleazer Bromberg, F. E. Browder, A. A. Brown, A. B. Brown, J. L. Brown, Jr., F. H. Brownell, R. S. Burington, C. W. Burrill, S. S. Cairns, Lorenzo Calabi, F. P. Callahan, Jr., E. W. Cannon, E. D. Carey, Abraham Charnes, K. L. Chung, S. I. Ciolkowski, Paul Civin, F. E. Clark, E. J. Cogan, L. W. Cohen, R. M. Cohn, T. F. Cope, Richard Courant, J. H. Curtiss, M. D. Darkow, M. D. Davis, Philip Davis, R. B. Davis, Allen Devinatz, R. J. DeVogelaere, A. H. Diamond, D. J. Dickinson, V. J. Doberly, H. L. Dorwart, Avron Douglis, L. A. Dragonette, D. M. Dribin, Nelson Dunford, Jacques Dutka, Aryeh Dvoretzky, H. A. Dye, B. E. Dyer, H. P. Edmundson, Bernard Epstein, D. I. Epstein, Paul Erdös, G. C. Evans, H. P. Evans, W. H. Fagerstrom, Michael Fekete, J. M. Feld, W. E. Ferguson, N. J. Fine, C. D. Firestone, Irwin Fischer, D. A. Flanders, William Forman, R. M. Foster, Phyllis Fox, R. H. Fox, Gerald Freilich, Bernard Friedman, H. D. Friedman, Orrin Frink, T. C. Fry, H. K. Fulmer, I. N. Gál, I. S. Gál, J. J. Gergen, Murray Gerstenhaber, H. A. Giddings, Wallace Givens, Sidney Glusman, J. K. Goldhaber, Saul Gorn, Arthur Grad, Harold Grad, H. M. Griffin, Emil Grosswald, Laura Guggenbuhl, Felix Haas, V. B. Haas, P. R. Halmos, M. E. Hamstrom, G. H. Handelman, G. A. Hedlund, I. N. Herstein, T. H. Hildebrandt, Abraham Hillman, W. M. Hirsch, S. B. Hobbs, P. G. Hodge, Jr., S. P. Hoffman, Banesh Hoffmann, F. E. Hohn, S. S. Holland, Jr., H. G. Hopkins, Gabriel Horvay, D. B. Houghton, B. K. Hovey, C. C. Hsiung, A. O. Huber, Witold Hurewicz, Eugene Isaacson, J. R. Isbell, W. S. Jardetzky, R. V. Kadison, Shizuo Kakutani, Aida Kalish, E. L. Kaplan, Irving Kaplansky, D. E. Kibbey, S. G. Kneale, George Klein, Morris Kline, Daniel Kocan, I. I. Kolodner, B. O. Koopman, A. G. Kostenbauder, Saul Kravetz, J. B. Kruskal, Jr., M. D. Kruskal, Harold W. Kuhn, O. E. Lancaster, R. E. Langer, C. W. Langley, P. D. Lax, Solomon Leader, E. H. Lee, Solomon Lefschetz, R. A. Leibler, W. W. Leutert, Howard Levene, M. E. Levenson, H. M. Lieberstein, S. C. Lowell, G. S. S. Ludford, Eugene Lukacs, Harold Luxenberg, B. H. McCandless, S. W. McCuskey, Brockway McMillan, E. J. McShane, L. A. MacColl, D. B. MacMillan, H. M. MacNeille, Wilhelm Magnus, A. J. Maria, M. H. Maria, Lawrence Markus, M. H. Martin, F. J. Massey, C. W. Marshall, F. H. Miller, K. S. Miller, H. J. Miser, Deane Montgomery, Morris Morduchow, B. J. Morse, David Moskovitz, C. A.

Nelson, D. J. Newman, Louis Nirenberg, A. B. Novikoff, C. O. Oakley, G. G. O'Brien, Alex Orden, J. C. Oxtoby, J. S. Oxtoby, L. J. Paige, Emanuel Parzen, L. E. Payne, E. J. Pellicciaro, Anna Pell-Wheeler, A. S. Peters, W. A. Pierce, H. O. Pollak, William Prager, Hans Rademacher, M. S. Rees, Haim Reingold, Russell Remage, Helene Reschovsky, Moses Richardson, P. D. Ritger, E. K. Ritter, I. F. Ritter, Herbert Robbins, Mark Robinson, Robin Robinson, Selby Robinson, I. H. Rose, M. E. Rose, P. C. Rosenbloom, J. E. Rosenthal, M. F. Rosskopf, J. P. Russell, E. A. Saibel, H. E. Salzer, J. E. Sammet, W. C. Sangren, Arthur Sard, Herbert Scarf, H. M. Schaerf, A. T. Schafer, R. D. Schafer, Miriam Schapiro, S. A. Schelkunoff, E. V. Schenkman, M. M. Schiffer, Albert Schild, K. C. Schraut, Seymour Schuster, Abraham Schwartz, C. H. W. Sedgewick, I. E. Segal, Wladimir Seidel, H. N. Shapiro, V. L. Shapiro, I. M. Sheffer, I. M. Singer, James Singer, M. L. Slater, David Slepian, O. K. Smith, J. J. Sopka, J. J. Stoker, Walter Strodt, Sister M. Carmel Sullivan, M. M. Sullivan, R. L. Swain, Alexander Tartler, J. T. Tate, A. H. Taub, Olga Taussky, T. Y. Thomas, D. L. Thomson, Jr., John Todd, L. F. Tolle, M. L. Tomber, C. B. Tompkins, A. W. Tucker, J. W. Tukey, H. E. Vansant, H. V. Waldinger, Jack Warga, J. H. Weiner, Alexander Weinstein, Louis Weisner, Bernard Weitzer, David Wellinger, John Wermer, F. J. Weyl, A. L. Whiteman, Hassler Whitney, Albert Wilansky, A. B. Willcox, Sister Teresa Dolores Wolf, Jacob Wolfowitz, K. G. Wolfson, E. S. Wolk, V. M. Wolontis, Arthur Wouk, F. B. Wright, Jr., Michael Yanowitch, D. M. Young, L. A. Zadeh, Arthur Zeichner, Leo Zippin, O. J. Zobel.

By invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings, Professor Paul Erdös of the University of Notre Dame addressed a general session Saturday afternoon on *Combinatorial problems in set theory*. Professor Kakutani presided.

Sessions for contributed papers were held Saturday morning, Professor Kaplansky presiding, and afternoon, Professors Oxtoby and Schiffer presiding.

The program of the conference on Training in Applied Mathematics consisted of papers invited by the National Research Council Committee on Applied Mathematics and distributed in sessions Thursday afternoon, Friday morning and afternoon, Saturday morning as follows:

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Sessions on Applied Mathematics in the Traditional Departmental Structure: Professors G. C. Evans, University of California; T. H. Hildebrandt, University of Michigan; R. E. Langer, University of Wisconsin.

Session on The Role of Mathematics in the Integrated School of Applied Science: Professors J. B. Wiesner, Massachusetts Institute of Technology; H. W. Emmons, Harvard University; A. H. Taub, University of Illinois.

Session on The Graduate Institute for Applied Mathematics: Professor Richard Courant, Institute for Mathematics and Mechanics, New York University; Professor William Prager, Brown University; Professor M. M. Schiffer, Stanford University; Professor M. H. Martin, University of Maryland.

Session on Applied Mathematics Training in Europe: Dr. F. J. Weyl, Office of Naval Research, Chairman; Professor G. C. McVittie, University of Illinois; Professor P. Germain, Université de Poitiers; Professor A. van Wijngaarden, Mathematisch Centrum, Amsterdam.

Session on The Mathematician in Government Establishments: Dean Mina S. Rees, Hunter College; Dr. C. C. Bramble, Naval Proving Ground, Dahlgren; Dr. W. W. Leutert, Aberdeen Proving Ground; Dr. H. J. Miser, USAF Operations Analysis Division; Dr. C. B. Tompkins, Institute for Numerical Analysis, Los Angeles.

Session on The Mathematician in Industrial Organizations: Dean Walter Bartky, University of Chicago; Dr. E. C. Nelson, Hughes Aircraft Company; Dr. A. A. Brown, A. D. Little, Inc.; Dr. H. Gerschinowitz, Shell Development Company; Dr. H. W. Bode, Bell Telephone Laboratories.

General Session: Summary Address, Dr. T. C. Fry, Bell Telephone Laboratories; Professor Leon W. Cohen, National Science Foundation and Queens College, presiding.

The abstracts of contributed papers presented at the meeting follow. Those with "*t*" after the abstract number were presented by title. Where a paper with joint authorship was presented in person, (p) follows the name of the author presenting it. Dr. Schneider was introduced by Dr. I.T.A.C. Adamson.

Algebra and Theory of Numbers

1*t*. Leonard Carlitz: *A problem involving quadratic forms in a finite field.*

Let $q=p^n$, $p>2$, and let A denote a nonsingular symmetric matrix of order m with elements in $GF(q)$. We determine the number of $m\times t$ matrices such that $\begin{vmatrix} A & X \\ X' & 0 \end{vmatrix}=\beta$, where $\beta\in GF(q)$ is fixed. Application is made to the evaluation of a generalized Gauss sum. (Received August 24, 1953.)

2*t*. Leonard Carlitz: *Certain special equations in a finite field.*

The equation $a_1x_1^2+\cdots+a_rx_r^2=bx_1x_2\cdots x_r+c$, where a_i, x_i, b, $c\in GF(q)$, is discussed for $r=3$, 4. In the case $r=3$ a simple explicit formula is found for the number of solutions. In the case $r=4$ the number of solutions is expressed in terms of a Jacobsthal sum; in certain special cases explicit results are obtained. (Received August 24, 1953.)

3*t*. Leonard Carlitz: *Representations by skew forms in a finite field.*

Let $q = p^n$, $p > 2$, and let A and B denote skew-symmetric matrices of order m and t, respectively, with elements in $GF(q)$. Explicit formulas are found for the number of $m \times t$ matrices X such that $X'AX = B$. As an application the number of solutions $X = X(m, t)$ of the equation $\left|\begin{smallmatrix} A & X \\ X' & 0 \end{smallmatrix}\right| = \beta$ is determined, where A is skew-symmetric and nonsingular. (Received August 24, 1953.)

4t. Leonard Carlitz: *Sets of primitive roots.*

The following result is typical. Let $f_1(x), \cdots, f_r(x)$ denote quadratfrei polynomials with coefficients in $GF(q)$ that are relatively prime in pairs and of degree $\geqq 1$; let N_r denote the number of α's in $GF(q)$ such that $f_i(\alpha)$ is a primitive root of $GF(q)$ for $i = 1, \cdots, r$. Then $N_r \sim \phi^r(q-1)/q^{r-1}$, $q \to \infty$. (Received August 24, 1953.)

5. V. J. Doberly (Dobroliuboff): *Doberly's logarithmic series.*

For preparation of mathematical tables and solution of problems involving direct calculation of natural, denary, and other logarithms, Doberly's logarithmic series provide a larger scope of usefulness and practicability, because, unlike any other known logarithmic series, they permit the use of unlimited number of different infinite progressions—all of different convergibility—through which to calculate the logarithmic number. Of many series of the kind those applicable to calculation of natural logarithms are the simplest, Of them the series $\log_{\text{nat}} x = m + (x/e^m - 1) - (1/2)(x/e^m - 1)^2 + (1/3)(x/e^m - 1)^3 - (1/4)(x/e^m - 1)^4 + \cdots$, where m is any number satisfying the condition $m > \log_{\text{nat}} x/2$, and $\log_{\text{nat}} x = m + ((x - e^m)/x) + (1/2)((x - e^m)/x)^2 + (1/3)((x - e^m/x)^3 + (1/4)((x - e^m/x)^4 + \cdots$, where $m < \log_{\text{nat}} 2x$, are especially valuable. When the logarithm sought is approximately known, the substitution of the approximate value of $\log x$ for m in the Doberly's series makes them strongly convergent, thus virtually permitting one to obtain a sufficiently precise figure of the logarithm by discarding in the computation all but the few first terms of the series. (Received August 4, 1953.)

6. I. N. Herstein: *On the Lie and Jordan rings of a simple associative ring.*

Given any associative ring A, to it one can attach two other rings, possibly no longer associative, A^J the Jordan ring, and A^L the Lie ring of A by defining new products for the elements of A, $a \cdot b = ab + ba$ to yield A^J, and $[a, b] = ab - ba$ to yield A^L. The following results are proved: (1) If A is a simple ring of characteristic $\neq 2$, then A^J is a simple Jordan ring. (2) If A is a simple, non-nil ring of characteristic $\neq 2$, then any Lie ideal U of A^L is either contained in the center of A or must contain $[A, A] = \{ab - ba \mid a, b \in A\}$. (Received September 11, 1953.)

7. J. R. Isbell: *More on the continuity of the real roots of an algebraic equation.*

Hewitt announced in *Rings of real-valued continuous functions.* I. Trans. Amer. Math. Soc. vol. 64, the following theorem: Let C (X, R) be the ring of all continuous functions on completely regular X to the real line R; let M be a maximal ideal in $C(X, R)$. The residue field $C(X, R)/M$ is real closed. However, Hewitt's proof that a polynomial of odd degree has a root is defective. Henriksen and Isbell, *On the continuity of the real roots of an algebraic equation*, Proc. Amer. Math. Soc. vol. 4, restored the proof with use of the Tietze extension theorem and thus only in case X is normal. A new proof is given here which avoids extension and thus establishes the theorem as Hewitt stated it. (Received September 21, 1953.)

8*t*. Bjarni Jónsson: *Representations of lattices.* I.

Whitman showed that every lattice L is isomorphic to a lattice L' of equivalence relations. If we regard binary relations as sets of ordered pairs, then multiplication and inclusion in the lattice L' coincide with set-theoretic intersection and inclusion while the lattice sum of two relations R and S in L' is the set-theoretic union of the nondecreasing infinite sequence of relative products $R; S, R; S; R, R; S; R; S, \cdots$. If for every R and S in L' this sequence is constant from the nth term on, then we speak of a representation of L of type n. *Results*: Every lattice L has a representation of type 3. In order for L to have a representation of type 2, it is necessary and sufficient that L be modular. Previously we have shown that there exist modular lattices which do not have a representation of type 1 (a representation by commuting equivalence relations), cf. Bull. Amer. Math. Soc. Abstract 59-4-341. (Received September 8, 1953.)

9*t*. Bjarni Jónsson: *Representations of lattices.* II. Preliminary report.

Consider the four classes (i) $\mathcal{E}$, (ii) $\mathcal{N}$, (iii) $\mathcal{A}$, (iv) $\mathcal{L}$, consisting, respectively, of all modular lattices which can be represented isomorphically by means of (i) commuting equivalence relations, (ii) normal subgroups of a group, (iii) subgroups of an Abelian group, (iv) subspaces of a (possibly degenerate) Desarguesian projective space. Let $\mathcal{C}$ be the class of all complemented modular lattices and $\mathcal{D}_n$ the class of all n-dimensional modular lattices. *Results*: 1. $\mathcal{E}\cap\mathcal{C}=\mathcal{N}\cap\mathcal{C}=\mathcal{A}\cap\mathcal{C}=\mathcal{L}\cap\mathcal{C}$. 2. $\mathcal{E}\cap\mathcal{D}_n=\mathcal{N}\cap\mathcal{D}_n=\mathcal{A}\cap\mathcal{D}_n=\mathcal{L}\cap\mathcal{D}_n$ for $n\leqq 4$. 3. $\mathcal{E}\cap\mathcal{D}_5\neq\mathcal{N}\cap\mathcal{D}_5$. 4. If $L\in\mathcal{E}$, then L satisfies the following condition (α): For any $a_0, a_1, a_2, b_0, b_1, b_2\in L$, if $y=(a_0+a_1)\cdot(b_0+b_1)[(a_0+a_2)(b_0+b_2)+(a_1+a_2)(b_1+b_2)]$, then $(a_0+b_0)(a_1+b_1)(a_2+b_2)\leqq a_0(a_1+y)+b_0(b_1+y)$. 5. If $L\in\mathcal{C}$ or $L\in\mathcal{D}_n$ with $n\leqq 4$, and if the condition (α) holds, then $L\in\mathcal{E}$. 6. Suppose $L\in\mathcal{C}$ is of order $k\geqq 3$ (in the sense of von Neumann). Then L is isomorphic to the lattice of all principal right ideals of a regular ring if and only if (α) holds. (Received September 8, 1953.)

10*t*. Jakob Levitzki: *Contributions to the theory of nilrings.*

We introduce four new types of nilrings which are encountered in the theory of iterated representations (C. J. Everett, Jr., Duke Math. J. vol. 5 (1939)) in case the iteration is transfinite, and also in the theory of transfinitely iterated socles (R. Baer, *Radical ideals*, Amer. J. Math. vol. 65 (1943)). The method by which these rings are defined is extended to enable a useful classification of the various hitherto known types of nilrings. A ring S is right (left) vanishing, in short, a v_r-ring (v_l-ring), if for any infinite sequence (a_i) in S there are an n and an m such that $a_n a_{n+1}\cdots a_{n+m}=0$ (resp. $a_{n+m}a_{n+m-1}\cdots a_n=0$), and it is a $v_{r,l}$-ring if it is both a v_r-ring and a v_l-ring. If above condition holds only for sequences of type $^*\omega+\omega$, the ring is called a v^*-ring. One recalls that a ring S is called an L-ring if it coincides with its lower radical. One has the following implications: Nilpotent rings→$v_{r,l}$-rings→v_r-ring (v_l-rings)→v^*-rings→L-rings, and all these types of nilrings are distinct. A v_l-ring is consumed by a transfinite ascending chain of iterated right annihilators culminating in the ultimate r. annihilator $M_r^{(\nu)}$. The ring $S-M_r^{(\nu)}$ is l. definite (i.e. its left representation is an isomorphism) and $M_r^{(\nu)}=$ intersection of all A such that $S-A$ is l. definite. $S=M_r^{(\nu)}$ if and only if S is a $\mathfrak{v}_I$-ring, S is transfinitely right nilpotent and $\vec{s}^{\tau+1}=0$, while for a $\mathfrak{v}^*$-ring the ultimate kernel is equal to 0. (Compare J. Levitzki, *Powers with transfinite exponents*. II, Riveon Lematematika vol. 2 (1947)). (Received September 14, 1953.)

11*t*. F. R. Olson: *Some determinants involving Bernoulli and Euler numbers of higher order.*

Determinants whose elements are Bernoulli, Euler, and related numbers of higher order are evaluated. Using the notation of Nörlund, *Differenzrechnung*, Chap. 6, it is shown, for example, that $|B_i^{(a+jd)}| = \prod_{k=0}^{m} (-d/2)^k k!$ for a, d constants; $i, j=0, 1, \cdots, m$. Like results are obtained for the other numbers. (Received August 24, 1953.)

12. R. D. Schafer (p) and M. L. Tomber: *A new simple Lie algebra of characteristic* 2.

Let $\mathfrak{C}$ be any Cayley algebra over a field Φ of characteristic 2, and $\mathfrak{J}$ be the 27-dimensional space of all those 3×3 hermitian matrices X with elements in $\mathfrak{C}$ which have scalar elements in the principal diagonal. Then $\mathfrak{J}$ is proved to be a (restricted) Lie algebra of characteristic 2 relative to the composition $X \circ Y = XY + YX$. The derived algebra $\mathfrak{J}_0$ (the set of all X of trace 0) is a new simple (restricted) Lie algebra, being central simple and of dimension 26 over Φ. It is shown that the derivation algebra $\mathfrak{D}$ of $\mathfrak{J}$ is 52-dimensional, the outer derivation algebra $\mathfrak{D}/\text{ad } \mathfrak{J}_0$ being simple. If $\mathfrak{C}$ is assumed to have divisors of zero, as would be the case if Φ were assumed algebraically closed, it is shown that $\mathfrak{D}/\text{ad } \mathfrak{J}_0 \cong \mathfrak{J}_0$. (Received July 30, 1953.)

13*t*. Hans Schneider: *The null ideal belonging to a commutative ring of linear transformations.*

Let V be a finite-dimensional vector space over a commutative field k. Let $a_1, \cdots, a_m$ be commutative linear transformations on V. Let $R=k[x_1, \cdots, x_m]$, where $x_1, \cdots, x_m$ are indeterminates, and let $A=k[a_1, \cdots, a_m]$. The null ideal $\mathfrak{n}$ belonging to A is defined to be the kernel of the natural homomorphism ϕ of R onto A. Let $\mathfrak{n}=\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ be the short representation of $\mathfrak{n}$ as the intersection of primary ideals. It is proved that the prime ideals belonging to $\mathfrak{q}_1, \cdots, \mathfrak{q}_s$ are maximal. Let Q_i be the image of $\mathfrak{q}_i$ under ϕ. It is pointed out that V is also an A-space and that $V=V_1 \oplus \cdots \oplus V_s$, where V_i is the A-subspace of V annihilated by Q_i. The ideals $\mathfrak{q}_1, \cdots, \mathfrak{q}_s$ are prime if and only if V is completely A-reducible. The space V is the direct sum of A-subspaces, which are one-dimensional over k, if and only if A is isomorphic to a ring of diagonal matrices with coefficients in k. Thus if A is isomorphic to such a ring, then V is completely A-reducible. The reverse implication holds when k is algebraically closed. (Received July 30, 1953.)

14*t*. Hans Zassenhaus: *Invariant bilinear forms on Lie algebras.*

Any representation Δ of a Lie algebra L over a field F determines uniquely the symmetric bilinear form $(a, b)=\text{tr }(\Delta a \cdot \Delta b)$ on L satisfying the invariance condition $(a \circ b, c)=(a, b \circ c)$. For an ideal A of L the set κA of all solutions x of $(x, A)=0$ is an ideal of L. Also the set $L^{-1} \circ A$ of all solutions x of $L \circ x \subseteq A$ is an ideal of L. Theorem 1: $\kappa(L^{-1} \circ (A+\kappa L))=L \circ A+\kappa L$, $\kappa(L \circ A)=L^{-1} \circ (A+\kappa L)$. Theorem 2: For a nilpotent subalgebra H coinciding with its normalizer in L, $H \circ H \leq \kappa L$. Theorem

3: Let F be algebraically closed, $zL=0$, and let $D^2L=DL$ if char. $F=2$. Then L is the algebraic sum of r mutually orthogonal Lie algebras L_i which are either simple or, only if char. $F\neq 0$, $L_i \circ L_i/z(L_i)$ is simple nonabelian and dim $z(L_i)=$ dim $L_i/L_i \circ L_i$ $=1$. If Δ is irreducible then $r=1$. The proofs make use of the representation theory of nilpotent Lie algebras developed by the author earlier. Application is made to simplicity proofs for classical Lie algebras. Symmetric bilinear forms on an F-submodule of L invariant under a set of derivations of L are set in 1-1 correspondence with certain elements of degree 2 in the Birkhoff-Witt algebra of L, if the form is nondegenerate and char. $F\neq 2$. (Received August 24, 1953.)

ANALYSIS

15. Joseph Andrushkiw: *A note on polynomials all of whose zeros are real and of the same sign.*

Let all the zeros of the polynomial $f(z)=f(z;1)=1+a_1z+\cdots+a_nz^n$ be negative. Consider the polynomial $f(z;x)=1+a_1^xz+a_2^xz^2+\cdots+a_n^xz^n$, x real. Its discriminant expressed in terms of coefficients is a continuous function of x. From theorem of Schurmalo (e.g. J. Schur, *Zwei Sätze über alg. Gleichungen mit lauter reellen Wurzeln*, J. Reine Angew. Math. vol. 144 (1914)) and the properties of a polynomial with real zeros only (author's paper, *On the zeros of integral functions*, Bull. Amer. Math. Soc. Abstract 58-3-289) it follows that the above discriminant has a greatest positive zero X_0. For $x\geqq X_0$ all the zeros of $f(z;x)$ are negative. On the other hand, if $0<x_0$ $=\max\left[\log((r+1)/(r+2))/\log(a_{r+1}^2/a_ra_{r+2})\right]$, $r=0,1,\cdots,n-2$, the polynomial $f(z;x)$ has at least one pair of imaginary zeros for $x\leqq x_0$. (Received September 8, 1953.)

16*t*. W. G. Bade: *Bounded Boolean algebras of projections.*

Let X be a reflexive Banach space and $\mathfrak{B}$ be a bounded Boolean algebra (B.A.) of projections in X (for some M, $\|E\|\leqq M$, $E\in\mathfrak{B}$). $\mathfrak{B}$ is complete if for every subset $\{E_\alpha\}\subseteq\mathfrak{B}$, $X=M\oplus N$ where $M=\text{cls}\,\{E_\alpha X\}$, $N=\bigcap_\alpha(I-E_\alpha)X$, and $\mathfrak{B}$ contains the projection $E=\bigvee_\alpha E_\alpha$ defined by this decomposition. A complete bounded B.A. of projections in X contains every projection in its weak closure. If X is separable, a σ-complete bounded B.A. of projections is complete. The algebra generated by a complete bounded B.A. of projections is weakly closed. A new proof is given of a theorem of N. Dunford (Bol. Soc. Mat. Mexicana vol. 3 (1946)) that the closure of $\mathfrak{B}$ in the strong operator topology is the least complete B.A. containing $\mathfrak{B}$. (Received September 9, 1953.)

17. W. G. Bade: *Weakly closed algebras of spectral operators and strong limits.*

Let $\mathfrak{A}$ be a commutative algebra of scalar type spectral operators in a reflexive Banach space (see N. Dunford, Bull. Amer. Math. Soc. Abstract 59-2-185 for definitions). If the Boolean algebra $\mathfrak{B}$ of projections generated by the resolutions of the identity for elements of $\mathfrak{A}$ is bounded, then every element of $\overline{\mathfrak{A}}^w$ (weak closure) is a scalar type spectral operator. If $\mathfrak{A}$ is generated (uniform operator topology) by $\mathfrak{B}$, then $\overline{\mathfrak{A}}^w$ is generated by $\overline{\mathfrak{B}}^s$ (strong closure). If X is separable, then $\overline{\mathfrak{A}}^w$ is generated by the resolution of the identity of a single element. This result generalizes a theorem of von Neumann for commutative W^*-algebras in Hilbert space. With a suitable restriction on the spectra of the limiting operators, a limit in the strong operator topology of scalar type spectral operators is of this type without commutativity. For spectral operators not of scalar type a strong limit need not be spectral. (Received September 9, 1953.)

18t. Frederick Bagemihl and Wladimir Seidel: *Spiral and other asymptotic paths, and paths of complete indetermination, of analytic and meromorphic functions.*

Let λ, μ, and ν be non-negative integers with $1 \leq \nu = \lambda + \mu$, and in the complex plane let G be a region not containing the point ∞ whose boundary consists of ν mutually exclusive circles $K_1, K_2, \cdots, K_\nu$, some or all of which may degenerate to single points. If $\lambda > 0$, let $\{S_n\}$ be an arbitrary enumerable set of mutually exclusive paths (which may be spirals) in G, each of which converges monotonically to one of the K_j, and $\{w_n\}$ be a sequence of complex numbers, some or all of which may be infinite. Then there exists a single-valued nonconstant function $f(z)$ in G, regular if $\lambda = \nu = 1$, or if $\mu = 0$ and no K_j is degenerate, but meromorphic otherwise, such that, if $\lambda > 0$, $f(z)$ possesses the asymptotic value w_n along S_n for every n, and, if $\mu > 0$, every path in G converging to any one of the circles $K_{\lambda+1}, \cdots, K_{\lambda+\mu}$ is a path of complete indetermination of $f(z)$ (i.e., is mapped by $f(z)$ on a set which is everywhere dense in the complex plane). This sharpens or generalizes results of Gross [Monatshefte für Mathematik und Physik vol. 29 (1918) p. 14] and Valiron [C.R. Acad. Sci. Paris vol. 198 (1934) p. 2065]. If $f(z)$ is nonconstant and meromorphic in $|z| < 1$, and $\sum(1 - |z_n|) = \infty$ for the zeros z_n of $f(z)$ in every sector of $|z| < 1$, then $f(z)$ has a residual set of radii of complete indetermination. Other sufficient conditions are obtained for paths of complete indetermination. (Received September 11, 1953.)

19. Lipman Bers: *An application of function theory to quasi-linear elliptic equations.*

The following lemma is established with the aid of the uniformization theorem, Privaloff's theorem on conjugate functions, and the Ahlfors-Lavrentyeff theorem on quasi-conformal homeomorphisms: Let $w(z)$ be a quasi-conformal function defined in a domain D with sufficiently smooth boundary. Assume that the component of w in a given Hölder-continuously varying direction satisfies on the boundary a Hölder condition. Then w satisfies a Hölder condition on the closure of D. Since the gradient $w = \phi_x - i\phi_y$ of a function $\phi(x, y)$ satisfying an elliptic equation (1) $A\phi_{xx} + 2B\phi_{xy} + C\phi_{yy} = 0$ is a quasi-conformal function, the lemma leads to a priori estimates for the modulus and Hölder continuity of w under the assumption that ϕ satisfies appropriate boundary conditions. This method, which is independent of the so-called "three point condition," leads to a variety of existence theorems for nonlinear equations of the form (1), A, B, C being functions of x, y, ϕ, ϕ_x, ϕ_y. In particular, multiply-connected and unbounded domains, and boundary conditions other than the first, can be treated. In the present paper only the main idea of the method is presented; detailed applications will be given in subsequent publications. (Received September 9, 1953.)

20t. Lipman Bers: *Local theory of pseudo-analytic functions.*

In this paper the local theory of pseudo-analytic functions (L. Bers, Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 130–136, vol. 37 (1951) pp. 42–47 and *Theory of pseudoanalytic functions*, mimeographed notes, New York University, 1953) is extended to the case in which the generators may be nondifferentiable but are merely assumed to satisfy a Hölder condition. The theory now applies to any linear homogeneous elliptic system of two first-order equations for two unknown functions of two variables, with Holder-continuous coefficients. The extension of the global theory to this case will be considered elsewhere. (Received September 9, 1953.)

21*t*. R. P. Boas: *Interference phenomena for entire functions.*

Let $f(z)$ be an entire function of exponential type π which is $o(|x|)$ on the real axis and which is bounded at the positive and negative integers. S. Bernstein, [Izvestiya Akad. Nauk SSSR. Ser. Mat. vol. 12 (1948) pp. 421–444] proved that the combination $f(x+1/2)+f(x-1/2)$ is bounded on the real axis, and A. F. Timan [Doklady Akad. Nauk SSSR. vol. 89 (1953) pp. 17–20] showed more generally that $o(|x|)$ in the hypothesis can be replaced by $o(|x|^2)$ and that the more general transform $\int_{-\infty}^{\infty} f(x+t)d\rho(t)$ is always bounded under the condition $\int_{-\infty}^{\infty} e^{\sigma|x|}|d\rho(x)| < \infty$ for some $\sigma > \pi$, if and only if $\int_{-\infty}^{\infty} e^{\pm i\pi t}d\rho(t) = 0$. Timan also gave a more general result when $f(x)=o(|x|^q)$, $q>2$. In this paper it is shown that (i) if $f(x)=o(|x|)$ and $\{f(n)\}$ is bounded, and if $\lambda(z)=\sum\lambda_n z^n$ is regular for $|z|\leqq\pi$, then $\lambda(D)f(z)=\sum\lambda_n f^{(n)}(z)$ is always bounded on the real axis if and only if $\lambda(i\pi)=\lambda(-i\pi)=0$; (ii) if $\{f(n)\}$ is bounded and $f(x)=o(|x|^q)$, $q>1$, then $f(z)$ is of the form $g(z)+P(z)\sin \pi z$, where $g(x)=o(|x|)$ and $P(z)$ is a polynomial. Timan's results follow from these facts. (Received August 10, 1953.)

22*t*. D. G. Bourgin: *Some mapping theorems.*

Denote by X a point symmetric set in R^n, the Euclidean n-space, which carries a symmetric m cycle. Require that this m cycle, on identification of antipodal points, be nonbounding in Q, the product of a projective n-space and the unit segment. Let f map X into R^j. As generalizations of theorems of Borsuk and of Dyson it is stated: (A) The symmetric set, each of whose points is assigned the same point of R^j as its antipode, carries an $n-j$ cycle which on identification of antipodal points does not bound in Q. (B) There are $m-j+1$ symmetric orthogonal segments whose vertices lie in X and map into some $j-1$ sphere about the origin in R^j. Continuity of the map may be weakened to upper semi-continuity of the functions $f_i(x)-f_i(-x)$, where $f_i(x)$ is the ith coordinate of $f(x)$. (Received September 4, 1954.)

23*t*. F. E. Browder: *The asymptotic distribution of the eigenvalues of the non-self-adjoint elliptic differential operator.*

If D is a bounded domain in E^n, let K be a suitably differentiable linear elliptic differential operator on D, B a suitably differentiable linear differential operator of a single sign on D. Generalizing a result due to Carleman for second-order operators in three independent variables, the asymptotic distribution of the eigenvalues of K with respect to B on D is determined in terms of the number of independent variables as well as the orders and coefficients of K and B. K is not assumed to be self-adjoint. (Received September 15, 1953.)

24. F. H. Brownell: *An extension of Weyl's asymptotic law for eigenvalues.*

It has long been known for a bounded domain D in the plane that the eigenvalues λ_n, $0<\lambda_n\leqq\lambda_{n+1}$, of the membrane problem are distributed according to Weyl's law $N(t)=\sum_{\lambda_n\leqq t}1=(\mu_2(D)/4\pi)t+o(t)$ as $t\to+\infty$. In a recent paper (Comm. Sem. Math. Univ. Lund, suppl. (Riesz) (1952) p. 177) Pleijel gets over $\omega\geqq 1$ the estimate

$\sum_{n=1}^{\infty} 1/\lambda_n(\lambda_n+\omega) = (\mu_2(D)/4\pi)(\ln \omega/\omega)+(C/\omega)+(l(B)/8)(1/\omega^{3/2})+O(1/\omega^2)$ for an infinitely differentiable boundary B, $l(B)$ being the total length of B. This estimate is much more detailed than the usual one from which Weyl's law follows by Tauberian theorems, and Pleijel suggests that by using his methods to extend his estimate over a cone of complex ω it should be possible to get a more precise form of Weyl's law. We show that from Pleijel's estimate alone follows $N(t) = (\mu_2(D)/4\pi)t-(l(B)/4\pi)t^{1/2} +\tilde{O}(\ln t)$ as $t \to +\infty$, where precisely the averaged $\tilde{O}$ estimate says that $F(t) = N(t) - \{(\mu_2(D)/4\pi)t-(l(B)/4\pi)t^{1/2}\}$ has $\left|\int_{\lambda_1}^{\infty} \exp\ (-(\rho^2/2)(\ln u/t)^2)dF(t)\right| \leq M_\rho \ln u$ over $u \geq e$ for every $\rho > 0$. Furthermore, any ordinary asymptotic series for $N(t)$ must agree with this $\tilde{O}$ type one term for term as far as the ordinary series goes. This consistency result proves false a conjecture of Minakshisundaram (*Symposium on spectral theory*, Stillwater, Okla., 1951, p. 331, no. 2). (Received October 19, 1953.)

25*t*. R. B. Davis: *A special case of the normal derivative problem for a composite third-order linear partial differential equation.*

The normal derivative problem for a third-order composite partial differential equation consists in prescribing u around the boundary of a region, and $\partial u/\partial n$ around part of this boundary. This problem is solved for the equation $\partial \Delta u/\partial x = \Delta(\partial u/\partial x) = 0$, where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, for a special type of convex region, bounded in part by a straight line $x =$ constant. Both existence and uniqueness are proved. Smoothness of bounding arcs, and of boundary values, is assumed in a certain sense, although the region contains corners. The method involves solution first of a Neumann problem for $\Delta u = 0$, by which the given problem is made symmetric and is reduced to a different type of boundary value problem considered in O. Sjöstrand, Arkiv för Matematik, Astronomi och Fysik vol. 26A (1939) pp. 1–10. The hypothesis includes certain restrictions which are necessary to the process of symmetrization. (Received September 8, 1953.)

26. D. J. Dickinson: *On Bessel and Lommel polynomials.*

The modified Lommel polynomials $h_{n,\nu}(x)$ are defined in terms of the Lommel polynomials (see Watson's *Bessel functions*, §9.6) by $h_{n,\nu}(x) = R_{n,\nu}(1/x)$. The $h_{n,\nu}(x)$, for $\nu > 0$, are orthogonal polynomials with respect to the complex weight function $J_\nu(1/x)/J_{\nu-1}(1/x)$ where the integration is taken over a circle in the complex plane that includes the finite singularities of the weight function. Also, let $j_{\nu-1,n}$ be the nth positive zero of the Bessel function $J_{\nu-1}(x)$, $\nu > 0$, and for $n < 1$, let $j_{\nu-1,n} = -j_{\nu-1,-n+1}$. Then for $\nu > 0$, $\int_a^b h_{n,\nu}(x)h_{m,\nu}(x)d\alpha_\nu(x) = \delta_{m,n}/2(\nu-n)$ where the real interval $[a, b]$ is $[-j_{\nu-1,1}^{-1}, +j_{\nu-1,1}^{-1}]$. The $d\alpha_\nu(x)$ is a Stieltjes distribution where $\alpha_\nu(x)$ is a nondecreasing step function with a jump of $j_{\nu-1,n}^{-2}$ at the point $x = j_{\nu-1,n}^{-1}$ for all integral n, positive, negative, and zero. One relation between the Bessel polynomials $y_n(x)$ (see Krall and Frink, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 100–115) and the modified Lommel polynomials is $i^n y_n(-ix) = h_{n,1/2}(x) + ih_{n-1,3/2}(x)$. From this relation and a known property of the Lommel polynomials, it is deduced that the absolute value of the zeros of $y_n(x)$ have an upper bound that approaches zero as n increases. From similar relations between the Bessel polynomials and the modified Lommel polynomials, one may deduce new facts about one polynomial set from known facts about the other. (Received September 15, 1953.)

27. Michael Fekete: *Asymptotic behavior of certain polynomials of least deviation on point-sets of a given transfinite diameter.*

CALCUTTA CENTRAL

Given an infinite compact point set of the z-plane, associate to every polynomial $A_n(z)=z^n+\sum_{\nu=1}^{n}a_\nu z^{n-\nu}$ of degree $n\geqq 1$ with leading coefficient 1, in an arbitrary but definite way, a non-negative finite number $D=D(A_n(z), S)$ as its *deviation* on S (from 0). Call D a *monotone* deviation if D decreases when $A_n(z)$ is replaced by its *under-polynomials* $B_n(z)=z^n+\sum_{\nu=1}^{n}b_\nu z^{n-\nu}$ on S (if such there be) of the same degree and with the same leading coefficient which satisfy $|B_n(z)|<|A_n(z)|$ whenever $z\in S$ except the common roots $z\in S$ of $A_n(z)=0$, $B_n(z)=0$ if there are such. Call D a *continuous* deviation if $|D(A_n(z), S)-D(B_n(z), S)|\leqq\epsilon$ whenever $|A_n(z)-B_n(z)|\leqq\delta(\epsilon)$ for all $z\in S$. Monotonity and continuity of D ensure the existence of *a polynomial of least deviation* $A_n^*(z; S, D)=z^n+\sum_{\nu=1}^{n}a_\nu^* z^{n-\nu}$ on S of degree n with respect to D. The *Chebyshev deviation* $D=\operatorname{Max}_{z\in S}|A_n(z)|$ and the *Bessel-deviation* $D=\int_S|A_n(z)|^2ds$ (where S is a rectifiable Jordan arc or Jordan contour) have both aforesaid properties thus possess polynomials $A_n^*(z)$ of least deviation. The author (Math. Zeit. (1923)) and G. Szegö (Math. Zeit. (1921)) have respectively shown: the pertaining *least deviations* $\delta_n=D(A_n^*(z), S)$, in both cases, satisfy $\lim_{n\to\infty}\delta_n^{1/n}=\tau(S)=$ the transfinite diameter of S; S being, however, in the second case, *analytic*. Recent allied investigations of P. Davis and H. Pollak (Proc. Amer. Math. Soc. (1952)) induced author to include all mentioned results in one general proposition, that is: Let $D=D(A_n(z), S)$ be a monotone and continuous deviation on S and $A_n^*(z)$ a polynomial of least deviation on S w. r. to D. Put $\delta_n=D(A_n^*(z), S)$, $\mu_n=\operatorname{Max}_{z\in S}A_n^*(z)$. Then $\lim_{n\to\infty}\delta_n^{1/n}=\lim_{n\to\infty}\mu_n^{1/n}=\tau(S)$ provided there is a *supermultiplicative Hausdorff function* $h(r)$ [continuous and strictly increasing with r, subject to $h(0)=0$, $h(1)=1$], such that the ratio $h(D(A_n(z), S)):h(\operatorname{Max}_{z\in S}|A_n(z)|)$ admits, for all $n\geqq 1$, an upper bound $U(S)$ and, for $n\geqq n_0(\epsilon)$, a lower bound $L_n(S, \epsilon)$ as well, satisfying $\lim_{n\to\infty}(L_n(S, \epsilon))^{1/n}=1-\epsilon$. Inclusion of the particular results of Davis and Pollak alluded to follows from the *semicontinuity* of $\tau(S)$ *from below* in some noteworthy special cases (to be published later). (Received November 2, 1953.)

28t. A. O. Huber: *The reflection principle for polyharmonic functions.*

Let G denote a region of R_n, the boundary of which contains an open subset S of $x_1=0$. If the function w is p-harmonic in G and if w/x_1^{p-1} assumes the boundary value 0 on S, then w can be continued analytically beyond S into the reflected domain by putting $w(-x_1, x_2, \cdots, x_n)=(-1)^p\sum_{k=0}^{p-1}A_k x_1^{p+k}\Delta^k[w(x_1, x_2, \cdots, x_n)/x_1^{p-k}]$. The A_k are independent of p and defined by: $A_k=[(-1)^k-\sum_{\nu=0}^{k-1}A_\nu(k+\nu)!/(k-\nu)!]/(2k)!$ $(k=1, 2, 3, \cdots)$, $A_0=1$. This result is known for $p=1$ and $p=2$ (for the latter case see R. J. Duffin, Bull. Amer. Math. Soc. Abstract 59-4-358). This work was supported by the United States Air Force through the Office of Scientific Research. (Received September 4, 1953.)

29t. Raphael M. Robinson: *Extremal problems for star mappings.*

The maximum possible value of $\Re\{B\log f'(z_0)+C\log[f(z_0)/z_0]\}$, where B and C are complex numbers (not both zero) and $0<|z_0|<1$, for the class of functions $f(z)=z+\cdots$ which map $|z|<1$ conformally onto star-shaped regions in a one-to-one way, can be attained only by a function $w=f(z)$ which maps $|z|<1$ onto the w-plane with one or two radial slits. Examples are given of both alternatives. In particular, it is shown that the set of possible values of $f'(z_0)$ is the map of $|z|\leqq|z_0|$ by $w=(1+z)/(1-z)^3$, at least if $|z_0|\leqq 0.6$. (Received September 11, 1953.)

30. Arthur Sard: *Approximation and projection.*

Let H_Y be the usual space of (classes of) functions $u=u(y)$, $y\in Y$, absolute square integrable relative to a measure m on a space Y. Similarly let H_Ω be the space of functions $s=s(\omega)$, $\omega\in\Omega$, absolute square integrable relative to a probability p on Ω. Assume that H_Y and H_Ω are separable. Let $H_{Y\Omega}$ be the space of functions $v=v(y, \omega)$, $y\in Y$, $\omega\in\Omega$, absolute square integrable mp on $Y\times\Omega$. Then $H_{Y\Omega}$ is the Kronecker product of H_Y into H_Ω. Let H_X be any separable Hilbert space; let $H_{X\Omega}$ be the Kronecker product of H_X into H_Ω. Suppose given an element g of $H_{X\Omega}$, an element k of $H_{Y\Omega}$, and a linear set $\mathfrak{T}$ of bounded linear operators on H_X to H_Y. Efficient and strongly efficient operators T (relative to g, k, $\mathfrak{T}$) are defined in a natural way with a view to the approximation of k by Tg, $T\in\mathfrak{T}$. Operators arbitrarily close to efficiency always exist; they are characterized. Necessary and sufficient conditions for strong efficiency and nearly strong efficiency are given. In the case in which the measure m is atomic, a necessary and sufficient condition that efficiency imply strong efficiency is given. The possibility of an advantageous change of topology in H_X is discussed. (Received August 18, 1953.)

31. I. E. Segal: *Direct limits of probability spaces.*

A theorem of Kolmogoroff's, that there exist random variables having preassigned joint distributions provided these satisfy the obviously necessary consistency conditions, is extended from the case of real-valued to that of general-valued random variables. The required variables are defined as set transformations and exhibited essentially as functions on a space that is a direct limit of probability spaces associated with the given distributions, and such limits are examined in general. The noncommutative analogue of such direct limits and also of conditional probability are treated at the same time, and noncommutative extensions obtained for some known martingale convergence theorems. The results are pertinent to the algebraic isomorphism problem for finite rings of operators. (Received September 4, 1953.)

32. V. L. Shapiro: *Subharmonic functions of order r.*

$F(x, y)$ is said to be subharmonic of order r in a domain G if $F(x, y)$ is in class $C^{2(r-1)}$ and $\Delta^{r-1}F(x, y)$ is subharmonic in G (Δ^r standing for the Laplacian operator iterated r times and coinciding with the identity operator when $r=0$). $F(x, y)$ is said to have a generalized rth Laplacian of the second kind at (x_0, y_0) equal to a_r if $\pi^{-1}t^{-2}\int_0^t u\,du\int_0^{2\pi}F(x_0+u\cos\theta, y_0+u\sin\theta)d\theta=a_0+a_1t^2/2(2!)^2+\cdots+a_rt^{2r}/(r+1)(2^rr!)^2+o(t^{2r})$ where $t>0$ and a_i $(i=0, \cdots, r)$ are constants. Designating this generalized rth Laplacian by $\Delta_2^rF(x_0, y_0)$, the following theorem is proved: Let $F(x, y)$ be in class $C^{2(r-1)}$ in a domain G and let E be a bounded closed set of capacity zero contained in G. Suppose $\Delta_2^rF(x, y)$ is non-negative for (x, y) in $G-E$. Then $F(x, y)$ is subharmonic of order r in G. (Received August 10, 1953.)

33t. Milton Sobel: *On a generalisation of an inequality of Hardy, Littlewood, and Pólya.*

Let $A=\{a_1\leq a_2\leq\cdots\leq a_n\}$, $B=\{b_1\leq b_2\leq\cdots\leq b_n\}$ denote sets of real numbers. Let (B_1, B_2) denote any of the $C^n_{n_1}$ partitions of B with n_1 elements in B_1 and $n_2=n-n_1$ elements in B_2. Let $A_1^*=\{a_\alpha\}$, $B_1^*=\{b_\alpha\}$, $A_2^*=\{a_\beta\}$, $B_2^*=\{b_\beta\}$ $(\alpha=1, 2, \cdots, n_1;\ \beta=n_1+1, \cdots, n)$. For each partition (B_1, B_2) consider the $n_1!\,n_2!$ crossproducts $v_{pq}=\sum_{i=1}^n a_ib_{j_i}$ $(p=1, 2, \cdots, C^n_{n_1};\ q=1, 2, \cdots, n_1!\,n_2!)$ formed by associating B_i with A_i^* $(i=1, 2)$. Let v_q^* denote the $n_1!\,n_2!$ crossproducts formed by associating B_i^* with A_i^* $(i=1, 2)$. Then for each partition p there is a one-to-one pairing of the

v_q^* and v_{pq} such that $v_q^* \geq v_{pq}$ $(q=1, 2, \cdots, n_1! \, n_2!)$. A generalization of this result in which B is partitioned into $(B_1, B_2, \cdots, B_k)$ with n_i elements in B_i also holds. The case $n_i=1$ $(i=1, 2, \cdots, k)$ reduces to the result: For each p $(p=1, 2, \cdots, n!)$, $\sum_{i=1}^n a_i b_i = v^* \geq v_p = \sum_{i=1}^n a_i b_{i_p}$ given in *Inequalities*, p. 261, by Hardy, Littlewood, and Pólya. Research sponsored by ARDC. (Received August 24, 1933.)

34t. J. S. Thale: *Univalence of continued fractions and Stieltjes transforms.*

Using only the definition and some elementary properties of univalent functions, the author obtains first some theorems on the univalence of functions of the form $\int d\phi(t)/(1+xt)$, $\int x d\phi(t)/(1+xt)$ with suitable upper and lower limits for the Stieltjes integrals. These theorems also apply to certain classes of continued fractions associated with such Stieltjes transforms. Further results on univalence and starlike character of additional classes of continued fractions are obtained by means of an iterative process based on value region results for continued fractions. A typical result is: If $|a_p| \leq 1/4$, then the function $f(x)=x/1+a_1x/1+a_2x/1+\cdots$ is univalent for $|x| < 4(2^{1/2}/(3+2(2^{1/2}))$ and is starlike with respect to the origin for $|x| \leq 8/9$. (Received September 8, 1953.)

35t. Louis Weisner: *Group-theoretic origin of certain generating functions.*

Let $L(x, d/dx, n)v=0$ be a linear ordinary differential equation containing a parameter n, and suppose the coefficients, as functions of n, are rational integral functions. Substituting $y\partial/\partial y$ for n, the equation is converted into a linear partial differential equation $L(x, \partial/\partial x, y\partial/\partial y)u=0$. If $v=v_n(x)$ is a solution of the ordinary differential equation, then $u=y^n v_n(x)$ is a solution of the corresponding partial differential equation; and conversely. There are four types of linear partial differential equations in two variables, of the second order, that admit a three-parameter Lie group and one that admits a five-parameter group. The group is used to obtain solutions of the partial differential equation. These solutions provide generating functions for solutions of the corresponding ordinary differential equations. In this way a large number of generating functions are obtained systematically for certain classes of hypergeometric, and related, functions. (Received September 1, 1953.)

APPLIED MATHEMATICS

36t. Lipman Bers: *Existence and uniqueness of a subsonic gas flow past a given profile.* II.

A new, much simpler, proof is given for the gas-dynamical existence theorem announced in the first abstract with the same title (Bull. Amer. Math. Soc. Abstract 59-4-402). The new proof does not use the Schauder-Leray degree; it is an application of a general method of estimating solutions of boundary value problems for quasi-linear elliptic equations in the plane (see abstract of paper *An application of function theory of quasi-linear elliptic equations.*) (Received September 9, 1953.)

37. Gabriel Horvay (p) and F. N. Spiess: *Orthogonal edge polynomials in the solution of boundary value problems.*

The paper, now being published in the Quarterly of Applied Mathematics, generalizes the variational method of Kantorovitch-Poritsky by using in the expansion

$\phi(x, y) = \sum c_n g_n(x) f_n(x, y)$ of the unknown function ϕ, an orthonormal set of boundary polynomials $f_n(x, y)$ which are complete with respect to the prescribed boundary conditions. The factors $g_n(x)$ are determined from Euler-Lagrange equations. (Received August 27, 1953.)

38. Martin Krakowski and Abraham Charnes (p): *Stokes' paradox and biharmonic flows.*

A generalized form of Stokes' paradox is exhibited as a consequence of a new uniqueness theorem for biharmonic *flows* in the complement of compact, possibly multiply connected, plane regions allowing $o(\log r)$ velocity growth at infinity. This answers a question of G. Birkhoff (*Hydrodynamics*, 1950). The single-valuedness of pressure and vorticity required by the flow interpretation precludes an additional possibility suggested by S. G. Mihlin (pp. 209–210 of *Integral equations and their applications*, OGIZ, 1949) from an elasticity interpretation. (Received September 24, 1953.)

39. I. F. Ritter: *Accelerated steepest descent methods.* I.

For real matrix A the *least squares solution* of the equation $Ax = y$ is found by *explicit* use of the system of *orthogonal vectors* forming the columns of a matrix $B = AT$. The triangular matrix T is built up simultaneously with B. Because $B'B$ is diagonal the solution $x = T(B'B)^{-1}(B'y)$ is easily computed. If A is square and nonsingular, then $A^{-1} = T(B'B)B'$. For any matrix A the residual vector $y - Ax$ has the minimum possible norm. Nontrivial solutions of $Ax = 0$, if any, appear as columns of T corresponding to zero columns in B. These columns of T are multiplied by arbitrary parameters in forming x. Besides unrestricted applicability the procedure has the advantages of skipping the computation of the normal equations $A'Ax = A'y$ and of permitting a rapid correction of the error in x caused by rounding off or small mistakes. The approximate solution x^1 is improved by $x = x^1 + T(B'B)^{-1}(B'r)$ with $r = y - Ax^1$ and this is iterated until the norm of $y - Ax$ assumes its minimum value. (Received November 2, 1953.)

40*t*. I. F. Ritter: *Accelerated steepest descent methods.* II.

Methods of steepest descent for solving the eigenvalue problem $Ay = \lambda By$, with matrices A and B, for eigenvalues λ and eigenvectors y [Hestenes and Karush, Journal of Research, National Bureau of Standards vol. 47 (1951) or Cooper, Quarterly of Applied Mathematics vol. 6 (1948)] maximize or minimize the Rayleigh quotient while minimizing the residual vector. The consequent restriction to Hermitian matrices can be removed if the Rayleigh quotient is replaced by the quotient $g(x) = (Ax \cdot \overline{Bx})/(Bx \cdot \overline{Bx})$ of two Hermitian scalar products for the approximating vector x. With a suitable choice $x^{(0)}$ the sequence of approximations appears as $x^{(i+1)} = x^{(i)} + a_i r^{(i)}$, with $r^{(i)} = Ax^{(i)} - g(x^{(i)})Bx^{(i)}$, $i = 0, 1, 2, \cdots$. The scalars a_i and $h_i = g(x^{(i+1)}) - g(x^{(i)})$ can be treated as independent variables in minimizing the residual norm $(r^{(i+1)} \cdot \bar{r}^{(i+1)})$, the a_i with smallest $|h_i|$ being readily found approximately from the resulting equations. With this a_i the sequence $x^{(i)}$ converges to an eigenvector y belonging to the eigenvalue $\lambda = g(y)$ nearest to $g(x^{(0)})$, without requiring repreated orthogonalizations. To accelerate the procedure the sub-routine of finding a_i is replaced by a simpler-one using for a_i one root of the quadratic equation which eliminates h_i from those two equations of the system $r^{(i+1)} = 0$ in which the numerically largest components of $r^{(i)}$ appear. As soon as h_i is sufficiently small the method given in part I leads from a rough approximate eigenvalue g_i to a greatly improved eigenvector y as solution of $Ay = g_iBy$. (Received November 2, 1953.)

GEOMETRY

41. A. P. Dempster and Seymour Schuster (p): *Constructions of poles and polar primes.*

Von Staudt, in 1847, introduced the idea of handling a symmetric polarity in projective space by means of a self-polar simplex and another pair of corresponding elements. Neither he nor anyone in the school of projective geometers which followed gave any explicit construction for the prime of an arbitrary point, or the pole of an arbitrary prime relative to a given polarity. H. S. M. Coxeter exhibited such constructions for two-space. In the present paper we give two methods for such constructions in n-space: one, which calls heavily upon the use of Hesse's Theorem (as does Coxeter's construction); and a second, which reduces the problem in n-space to 2^r constructions in $(n-r)$-flats $(1 \leq r \leq (n-1))$. (Received September 10, 1953.)

42. A. O. Huber: *An isoperimetric inequality for surfaces of variables Gaussian curvature.*

Let C denote a rectifiable Jordan curve of length L enclosing on a surface of piecewise continuous curvature K a simply-connected piece of surface S of area A. Then the following inequality holds: $L^2 \geqq 2A(2\pi - I)$, where $I = \iint_S K^+ dS$ with $K^+ = \max [K, 0]$. Equality holds if and only if $K = 0$ on S and C is a geodesic circle. For $K \leqq 0$ we thus obtain a theorem of E. F. Beckenbach and T. Radó (Trans. Amer. Math. Soc. vol. 35 (1933) pp. 662–674), and for $K \geqq 0$ a result of F. Fiala (Comment. Math. Helv. vol. 13 (1940–41) pp. 293–346). The proof makes use of an inequality of T. Carleman (Math. Zeit. vol. 9 (1921) pp. 154–160), Hölder's inequality, and some properties of conformal mapping. This work was supported by the United States Air Force through the Office of Scientific Research. (Received September 9, 1953.)

LOGIC AND FOUNDATIONS

43t. Theodore Hailperin: *Identity and description in first-order axiom systems.*

In first-order axiom systems quantifiers apply only to individual variables and not to predicates, which are assumed fixed for the system. In such a system the predicate identity can be defined in terms of the subject-matter predicates and need not be taken as a primitive undefined notion. Having identity in a system it then becomes possible to introduce descriptions ("the x such that Fx"), either as explainable in context or axiomatically. This paper gives, for an arbitrary first order system, complete sets of axioms for the essentially different versions of descriptions found in Hilbert-Bernays' *Grundlagen der Mathematik* and in Rosser's *Logic for mathematicians*. (Received August 25, 1953.)

STATISTICS AND PROBABILITY

44t. R. E. Bechhofer and Milton Sobel: *On a sequential ranking procedure.* Preliminary report.

Let X_{ij} be normally, independently distributed from population π_i with mean

μ_i^0 and unit variance ($i=1, 2, \cdots, k$; $j=1, 2, \cdots$). The k ranked means $\nu_1^0 < \cdots < \nu_k^0$ are known, it being unknown which population has mean ν_r^0 ($r=1, 2, \cdots, k$). Problem: Which population has mean ν_k^0? Let $X_i^m = \sum_{j=1}^m X_{ij}$; denote the ranked X_i^m by $Y_1^m < \cdots < Y_k^m$. Let $S_m = [\sum \exp\{\sum_{i=1}^{k-1} Y_{j_i}^m \nu_i + Y_k^m \nu_k\}] \div [\sum \exp\{\sum_{i=1}^{k} Y_{j_i}^m \nu_i\}]$, $(j_1, j_2, \cdots, j_s)$ denoting a permutation of $(1, 2, \cdots, s)$, the summations in the numerator and denominator being over all permutations of $(1, 2, \cdots, k-1)$ and $(1, 2, \cdots, k)$, respectively. For $1/k < \gamma < 1$, the sequential procedure P_γ below has probability of a correct choice $\geq \gamma$. Procedure P_γ: "At the mth stage ($m=1, 2, \cdots$) observe $(x_{1m}, \cdots, x_{km})$. Stop if $S_m \geq \gamma$, and decide that the population yielding Y_k^m has mean ν_k^0; continue if $S_m < \gamma$." The probability of termination is unity. The case of unknown ranked means has also been treated. Similar procedures apply for selecting the t populations with largest parameters from k Koopman-Darmois distributions having the same form. Research sponsored by ARDC. (Received August 24, 1953.)

45t. Jack Kiefer and Jacob Wolfowitz: *On the theory of queues with many servers.*

Ergodic properties of the following stochastic process are studied: For $i \geq 1$, let $t_i \geq t_1 = 0$ be the time of arrival of the ith person at a system of $s \geq 1$ machines, where he waits his turn until a machine is available to serve him, say at time $t_i + w_i \geq t_i$. This machine is then occupied by him for time $R_i \geq 0$. Let $g_i = t_i - t_{i-1}$. $\{R_i\}$ and $\{g_i\}$ are independent sequences of identically distributed and independent chance variables. An s-dimensional random walk W_i, with w_i its first component, is studied. Let F_i (F_i^*) be the d.f. of W_i (w_i). It is shown that $F(x) = \lim_{i\to\infty} F_i(x)$ exists and satisfies a certain integral equation (I.E.); $F^*(x) = \lim_{i\to\infty} F_i^*(x)$ also exists. Put $\rho = ER_i/sEg_i$. Then F and F^* are d.f.'s if and only if $\rho < 1$, and F is then the unique d.f. solution to the I.E. Except in the trivial case where $P\{R_i = sg_i\} = 1$, if $\rho \geq 1$ then $F = 0 = F^*$, and the I.E. has no d.f. solution. Always $F^*(x) = F(x, \infty, \cdots, \infty)$. Results on the limiting length of the line are also proved. (Received August 5, 1953.)

TOPOLOGY

46. R. D. Anderson and Mary-Elizabeth Hamstron (p): *A note on continuous collections of continuous curves filling up continuous curves in the plane.*

It is proved that if G is a continuous collection of nondegenerate continuous curves filling up a compact continuous curve in the plane, then with respect to its elements regarded as points G is either an arc or a simple closed curve. Considerable use is made of some of R. D. Anderson's results (*Continuous collections of continuous curves in the plane*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 647–657) particularly Theorem VI, which states that if G is a compact continuous curve with respect to its elements, then G is a hereditary continuous curve such that the closure of its set of emanation points is totally disconnected. Use is also made of Theorem I of that paper, from which it follows that if H is a closed subcollection of G, then it is not true that both the collection of arcs and the collection of simple closed curves of H are dense in H. (Received September 8, 1953.)

47. Lorenzo Calabi: *Topologies on sets of local mappings.*

$\mathfrak{M}$ and $\mathfrak{S}$ being two families of subsets of a set E, F a uniform space with entourages V, call: $\mathcal{L}_{\mathfrak{M}}(E, F)$ the set of the pairs (u, M) where $M \in \mathfrak{M}$ and u is a mapping of M into F; $(A, V)_u$ with $A \in \mathfrak{S}$, $A \subset M$, the set of local mappings $(v, N) \in \mathcal{L}_{\mathfrak{M}}(E, F)$ verifying $N \supset A$ and $(v(x), u(x)) \in V$ for $x \in A$. The sets $(A, V)_u$ are fundamental neighborhoods of (u, M) in the topological space $\mathcal{L}_{\mathfrak{M}\mathfrak{S}}$ provided $\mathfrak{M}$ and $\mathfrak{S}$ satisfy some simple conditions. $\mathcal{L}_{\mathfrak{M}\mathfrak{S}}$ is not separated, not uniformisable; under some assumptions on $\mathfrak{M}$, $\mathcal{L}_{\mathfrak{M}\mathfrak{S}}$ is connected and has no nonnonstant continuous mappings into a separate space. Suppose E compact, metric space; $\mathfrak{M}$ set of the open sets with finite many components; $\mathfrak{S}$ set of the compact sets different from E. Put then $\mathcal{L}_{\mathfrak{M}\mathfrak{S}} = \mathcal{L}_s$. Assume $\mathcal{W} \subset \mathcal{L}_s$ to verify some of the properties of the set of analytic functions in the sense of Weierstrass. Call $\mathcal{A}_s$ the subspace of $\mathcal{L}_s$ whose elements are unions of restrictions of mappings in $\mathcal{W}$. $\mathcal{A}_s$ generalizes the space of local analytic functions of Fantappie, is closed in $\mathcal{L}_s$, connected and arcwise connected. If E and F have convenient algebraic structures, $\mathcal{A}_s$ is locally arcwise connected. The results apply in particular for complex and hypercomplex analytic functions. (Received September 3, 1953.)

48. H. D. Friedman: *Topologies on a space of continuous functions.*

Let S be the set of all real-valued continuous functions defined on the closed interval $[0, 1]$. Seven well known topologies for S are related—those resulting from: uniform, pointwise, bounded pointwise, and weak convergences; and pointwise, bounded pointwise, and weak neighborhood systems. All seven spaces are shown to be Hausdorff spaces, but in none of the seven is the unit sphere compact. The topologies are partially ordered under the relation "weaker than." The most important result follows: Let $\{f_n(x)\} \subset S$, $f(x) \in S$. We say that $f_n(x)$ converges *weakly* to $f(x)$ provided that for every function $\alpha(x)$ of bounded variation ($\alpha(x)$ not necessarily in S) $\lim_{n\to\infty} \int_0^1 f_n(x)d\alpha(x) = \int_0^1 f(x)d\alpha(x)$. Then for $f_n(x)$ to converge weakly to $f(x)$ it is necessary (and sufficient, but this is well known) that $\{f_n(x)\}$ be uniformly bounded and pointwise convergent to $f(x)$. (The above is an abstract of the writer's doctoral dissertation. The writer is indebted to Professor Orrin Frink, of the Pennsylvania State College, for his advice and encouragement.) (Received September 10, 1953.)

49. I. S. Gál: *Relations between knot invariants.*

In a previous abstract [Bull. Amer. Math. Soc. Abstract 59-6-698] we defined a knot invariant Λ in terms of an abstract cycle associated with a regular normed projection of a knot K. Now we can prove the following results: Let $\Gamma = (v_{i,2} - v_{i,1}, v_{i,4} - v_{i,3}, \cdots, v_{i,2k} - v_{i,2k-1})$ be the Seifert matrix where the v_{ij}'s denote the crossing numbers. Let $\Delta(x) = \sum a_k \cosh kx$, where $(\cdots, a_1, 2a_0, a_1, \cdots)$ are the coefficients of the Alexander polynomial of K. Then we have $\Lambda/2 = \sum (v_{2i-1,2j-1}v_{2i,2j} - v_{2i-1,2j}v_{2i,2j-1}) = \Delta''(0) = \sum k^2 a_k$. Consequently if the Alexander polynomial of K is trivial, then $\Lambda = 0$, but not conversely. (Received September 10, 1953.)

50t. R. D. Johnson, Jr.: *An antipodal point theorem for mapping the 2-sphere into the reals.*

Let $f(S^2) = R_1$ be a (continuous) mapping of the 2-sphere into the reals. Then there exists a point r contained in R_1 such that some component of $f^{-1}(r)$ contains a pair of antipodal points of S^2. The following result is obtained as a corollary to this theorem: Let A and B be two closed sets such that $A \cup B = S^2$. Then there exists a continuum K containing a pair of antipodal points of S^2 which is contained in one of the three sets $(A - B)$, $(B - A)$, and $(A \cap B)$. (Received September 15, 1953.)

51. Harold W. Kuhn: *Contractibility and convexity.*

The intersection of a set X in a euclidean space with a hyperplane of support is called a *face* of X. With this definition, the following purely geometrical result appears as a by-product of the theory of games. Theorem: If X is a contractible polyhedron with contractible faces, then X is a convex set. (Received September 9, 1953.)

52t. C. W. Saalfrank: *On the universal covering space and the fundamental group.*

Let X be an arcwise connected, Hausdorff space, and let X_p^* be the universal covering space associated with X. It is shown that if A is a retract of X, then A_p^* is homeomorphic to a retract of X_p^*. A subgroup F of a group G is called a retract of G provided there exists a homomorphism h such that $h(G)=F$ and such that $h(f)=f$ for all $f\in F$. It is shown that if A is a retract of X, then the fundamental group $\pi(A)$ is isomorphic to a retract of the fundamental group $\pi(X)$. (Received June 16, 1953.)

53. E. S. Wolk (p) and J. K. Goldhaber: *Maximal ideals in rings of bounded continuous functions.*

Let X be a topological space, A a topological ring, and $C^*(X, A)$ the ring of bounded (in the sense of Shafarevich, C. R. (Doklady) Acad. Sci. URSS. vol. 40 (1943) pp. 133–135, and Kaplansky, Amer. J. Math. vol. 69 (1947) pp. 153–183) continuous functions from X to A. On the basis of the definition of boundedness as given by Shafarevich and Kaplansky, a definition of X completely regular with respect to A is given. Now let Σ be the set of all infinite sequences of distinct points of X which have no limit point in X, and let S be a maximal sub-lattice in Σ. Define $M(S)=[f\in C^*(X, A)\,|\,\lambda\in S$ implies that $f\to 0$ on a subsequence of $\lambda]$. The following theorems are proved: (1) Let A be a topological division ring of type V (Kaplansky, Duke Math. J. vol. 14 (1947) pp. 527–541) with continuous inverse and satisfying the first axiom of countability. Let X be a noncompact Hausdorff space which is completely regular with respect to A. Then $M(S)$ is a proper maximal free ideal in $C^*(X, A)$. (2) $C^*(X, A)/M(S)$ is isomorphic with A if and only if every bounded closed set in A is compact. The following question remains open: What further conditions on A, if any, are required to insure that every proper maximal free ideal in $C^*(X, A)$ is of the above form? (Received August 24, 1953.)

L. W. COHEN,
Associate Secretary

THE NRC-AMS CONFERENCE ON TRAINING IN APPLIED MATHEMATICS

This is a summary report on a conference on training in applied mathematics which was held under the joint sponsorship of the National Research Council and the American Mathematical Society in conjunction with the latter's regular meeting at Columbia University on 22–24 October 1953. The conference was organized as part of the current Survey of Training and Research in Applied Mathematics conducted by the Mathematics Division of the National Research Council under provisions made by the National Science Foundation jointly with the Army's Office of Ordnance Research, the Office of Naval Research, and the Office of Scientific Research of the Air Force. Complete proceedings are being submitted to the National Science Foundation as a report under the Survey contract. Inquiries regarding the availability of copies should be sent to Dr. L. W. Cohen, Program Director for Mathematical Sciences, National Science Foundation, Washington 25, D. C., prior to 30 April 1954. Final arrangements for distribution will depend on the anticipated demand.

Purpose and agenda. From the beginning the plans for the Survey included as an important part the arrangement of one or several conferences on the various phases of training and research in applied mathematics. Breadth of coverage and the effective presentation of all relevant viewpoints were to be the primary aim of these meetings regardless of whether or not general agreement would ultimately be attained. Two conferences appeared to be indicated, one of them to present characteristic current research to a large audience of mathematicians, illustrating particularly active sectors of the front along which mathematics interacts today with other scientific disciplines, and the other one to subject the possible patterns of training applied mathematicians to an intensive discussion by a smaller group of persons specifically interested and concerned. The Conference on Training in Applied Mathematics was arranged to serve this latter purpose, bringing together both producers and consumers of applied mathematicians and having them compare their ideas, experiences, and expectations.

In accordance with this plan the meeting was broken down into two parts, during the first of which training programs were reviewed by outstanding faculty members from universities which have made major efforts to provide a home for applied mathematics on their campus, while during the second part leaders from governmental and industrial organizations discussed the present and the anticipated future role played by mathematicians in their establishments. These two parts were separated by a group of three invited addresses presenting characteristic aspects of the education of applied mathematicians in certain European countries, and a particularly effective summarizing address by Dr. Thornton C. Fry of the Bell Telephone Company concluded the conference.

Summary of discussions. The conference certainly achieved the aim of being comprehensive, and very little, if anything at all, was left unsaid which can reasonably be said about applied mathematics and applied mathematicians. Yet in building up from a somewhat confused and groping start to the firm final summary, gathering sense and direction as it went along, the work of the conference managed to reveal a broad, common fabric of recognized trends, concordant ideas, and established objectives. It is this which gave the conference its real impact and significance.

The American applied mathematical scene as we see it today is a product of World War II. It has since been kept from falling apart by the research supporting efforts of the Federal Government, and there is as yet little indication of a change. Whatever the appearance that current interest in applied mathematics merely reflects the desire of mathematicians to capitalize on the availability of such funds, there is a general conviction that this interest is really and honestly due to the wealth of substantial problems which confront us especially in the production of equipment and organizations capable of meeting the challenge of combat conditions. These are problems where the mathematician can make unique and effective contributions since tradition and experience are often lacking or inaccessible; moreover they are of real mathematical significance, engaging the best he has to offer. Thus, the military climate of the war and cold war years has catalyzed a situation which civilian and commercial competition would have taken yet another decade or more to produce.

The needs of the Federal Government. In keeping with this state of affairs, the Federal Government sets at present the pace for the demand of applied mathematicians. Numerous specifically mathematical positions, ranging from the lower rungs to the very top of the Civil Service ladder, have been established in Federal Departments and in the laboratories they operate. To these must be added the jobs of similar nature, existing in laboratories operated under contract by outside agencies for work on Government problems. Government spokesmen could therefore be particularly specific as to what must be required and expected of the applied mathematician. First and foremost he must be a good mathematician, sound and well-rounded of training. Mathematical statistics and probability should in every case be part of his background. Aside from this it is almost immaterial whether or not he is skilled in any one particular technique. If he needs it, he can learn it on the job, while this would not be a favorable environment for extending his fundamental mathematical background. In addition he will need substantial knowledge in some other related fields of science, again on a sufficiently basic level so that he is not only able to handle the problems of today in his field but to learn and to develop what he may need in analyzing and designing the mechanisms and systems yet to be realized in the course of his life-time. A second group of prerequisites can be gathered under the heading of technical geniality, used by one of the speakers: A ready interest in other peoples' problems, the ability to work as member of a team, effectiveness and clarity in the communication of ideas. It is inevitable that any coordinated scientific effort, mathematical or otherwise, brings with it the responsibility for other persons' work or for the management of funds or programs. Hence, a certain amount of executive ability, be it inborn or acquired,—of trustworthiness, responsibility, and common sense, are often demanded.

One of the most significant factors affecting the demand for mathematicians at first in Government agencies and now also in industry is the emergence of the high-speed automatic computer. It has enhanced the usefulness of the applied mathematician in his traditional role of carrying out preliminary analyses of proposed systems by permitting a higher level of complexity in the mathematical models which can be subjected to final numerical evaluation. Even more important is the growing need for new mathematical insight and methods to insure the effective use of such machines. This creates a demand for an "applied" sort of mathematician much deeper grounded in "pure" mathematics than his traditional predecessor ever was. The effective use, in particular, of the large-scale electronic computers includes the burgeoning science of numerical simulation in which quite abstract phases of mathematics, and therefore the mathematicians who are experts in them, are proving very useful.

While much of this is equally true of the situation in industry, at least in a qualitative way, it is characteristic for the Government that there is a sizeable number of jobs primarily concerned with the administration rather than the actual performance of mathematical research. It should be a matter of concern to the entire mathematical community that competent mathematicians, possessing the necessary additional abilities, be available to fill, in particular, the positions of this type on the policy level: the Program Director for Mathematics of the National Science Foundation, the Mathematical Sciences Divisions and Mathematics Branches in the three military Offices of Research, and the Mathematics Division of the National Bureau of Standards. Beyond these there are positions, such as the Science Director of the Office of Naval Research, the Director of the Weapons Systems Evaluation Group, and others, for which mathematicians are eligible but in competition with representatives of other scientific fields. In all of them there is added to the duties of program administration the challenging and important assignment of creating, within the latitude allowed by Civil Service regulations, an acceptable environment in which a staff of competent scientists can work.

Industrial demands. With some notable exceptions, practically all of the mathematical work in industrial laboratories is done by engineers, physicists, and chemists who command a knowledge of the theoretical aspects of their fields. The more extensive employment in such environments of people who are trained and act primarily as mathematicians is generally considered a matter of the future whose potential benefits are freely acknowledged. In the meantime, however, there exists in industry nothing like the elaborate system of well-defined mathematical positions now found in the Federal Government. Mathematics majors, hired fresh out of college, are generally not slated by their subsequent training for a career as mathematicians but they use their mathematical background to get a good start in one or the other engineering specialty. The unbridged gap of understanding between the mathematician and the practicing scientist, due to the insufficient knowledge which each has of the other's field, was principally held responsible for this continued dearth of career mathematicians in industry. As of now the theoretically inclined scientist is therefore called upon to acquire such mathematical skills as may be needed, mostly on an ad hoc basis, yet occasionally to the point where he becomes a mathematician in all but the name.

It is recognized that this approach is reaching the point of diminishing returns as it becomes profitable to bring into play mathematical modes of thought and analysis which lie beyond the specifically devised ad hoc techniques. The availability of electronic computers will bring about a truly explosive acceleration in this process, and, unless it is not too late even now, there is certainly no time to lose in making adequate preparations. The IBM Company has now installed all over the country more than 1,000 electronic punched card machines (Type 604), each requiring at least one competent mathematician to operate, and about 100 Card Programmed Calculators, each requiring a team of from three to six mathematicians, if they are to be effectively used. The type of applied mathematician, moreover, for which these installations are looking, is not the junior technician specifically drilled in coding problems; it is the same competent, well-rounded mathematician for whom the Government jobs are competing. Thus, these IBM installations alone are currently opening up roughly 1,500 new jobs for applied mathematicians with graduate training.

The most notable of the exceptions mentioned at the beginning is the Mathematical Research Department of the Bell Telephone Laboratories which can now look back on about 25 years of fruitful cooperation between mathematicians and engineers. Even with every consideration being given to the fact that the problems of communications engineering are particularly accessible to a broadly based mathematical approach, the experiences which are reflected in the operation of this group today are particularly relevant at the present time.

Training programs. It is impossible to find a natural dividing line between pure and applied mathematics. Just as the various branches of mathematics interact with each other so they interact with other scientific disciplines, deriving inspiration and often motivation from both. Has Boolean algebra, it was asked, become a different subject because it was discovered to be of use in the design of switching circuits? Hence, no such line should be drawn artificially in the curriculum of prospective mathematicians. Applied mathematics must therefore be considered primarily a matter of attitude and motivation; the designation of secular in contrast to monastic mathematics was suggested as doing more justice to the situation than the traditional pure and applied.

The discussions were mostly concerned with graduate training, driving home in varied ways the salient point that its aim must be the formation of good general mathematicians who have at the same time a broad knowledge in some other related sciences. In the considerable variety of programs which were presented as having been fashioned towards these ends, there were two points about which prevailing opinions seem to be particularly diverse. The first of these concerns the relative size and administrative position of programs training a type of mathematician who is neither urged nor desires to learn what significance his subject has in its relation to other scientific disciplines. The second one is that of establishing the amount of experience in other scientific fields which should minimally be demanded and maximally be provided for. The adopted or proposed solutions to these problems determine the spectral structure in the broad-band concept of applied mathematician: There is the mathematical engineer, described as a man of action, able to deal with practical situations without waiting indefinitely until all conceivable data are in and the theory has reached academic perfection, and who more likely than not has topped off his formal education by participating in a research project involving experimental work. There is the applied mathematical dreamer and tinkerer who constructs his ingenious models to capture the thoughts the engineers will be thinking three or five years hence. There is the mathematician who uses long unsolved problems of other sciences to give both them and mathematics a new lively turn; the mathematical consultant with his broadly integrated knowledge and research experience; and many others.

To the extent that this variety of approaches is due to the flexibility of our institutions of higher learning in adapting their applied mathematics programs to their particular environment and to their individual administrative structure, this is all to the good. A continued effort of cooperation will be needed, however, to eliminate from this picture everything which is the result of adapting to misapprehensions about the demands of the future as well as to the traditional prejudices that the pure takes precedence over the applied and that the extensive contact with other sciences weakens the mathematical fibre.

The greatest single difficulty, generally experienced, is that of attracting good students in sufficient number. Numerous reasons were given for this lack of enthusiasm on the part of students to go into applied mathematics: Pure mathematics is taught more challengingly and, on the whole, by more inspiring teachers. A longer period of education and greater maturity are needed to do substantial work in applied mathematics while in the abstract phases a talented mathematician can gain recognition at an early age; on the other hand, engineering careers lead more quickly to higher salaries. Students generally do not know enough about the careers now opening up in applied mathematics; or again, they consider them unattractive because they are afraid that, by going into government or industry, they will lose their contact with living mathematics, and they know that it will be difficult to find academic jobs.

The discussion turned to the mathematical training of undergraduates as the proper point at which to attack this problem, either by the establishment of a specific applied mathematical curriculum paralleling the traditional one, or by adopting a single modernized and broadened course of study in mathematics which would produce a graduate equally qualified to pursue advanced work either in the pure or the applied direction. Either approach would make it possible to give the student a fair amount of information about the various professional careers which are open to him.

In the past most of the leaders in the field started out either as pure mathematicians, or as engineers and physicists, turning to applied mathematics only after having done substantial research in the field of their first interest. Similarly one might hope to see in the future the ranks of applied mathematicians unexpectedly strengthened by outstanding talents who are now working successfully along highly abstract lines. In the meantime, however, there are strong faculties in the purer branches of mathematics who are experiencing difficulty in finding suitable candidates to take care of the applied aspects.

Conclusion. In revealing large areas of agreement as far as facts, trends, and objectives are concerned, and in providing for a forthright expression of opinions where obvious differences existed, the conference has significantly contributed towards a clarification of the problems faced by the various groups who have a stake in the training of applied mathematicians. It reflected, to borrow the words of Thornton Fry, a growing appreciation by the monastic of the dignity of science in the market place, and by the secular of the dignity of pure thought. The tide is clearly running toward a cofraternity of mutual respect and mutual understanding, so necessary in our complete and interdependent world.

F. JOACHIM WEYL, *Investigator,*
Committee on Training and Research
in Applied Mathematics

THE NOVEMBER MEETING IN SPARTANBURG

The four hundred ninety-sixth meeting of the American Mathematical Society was held at Wofford College, Spartanburg, South Carolina, on Friday and Saturday, November 27–28, 1953. The total attendance was over 100, including the following 86 members of the Society:

R. W. Bagley, E. G. Begle, J. W. Brace, A. T. Brauer, E. A. Cameron, C. E. Capel, Haskell Cohen, H. S. Collins, F. E. Cothran, A. C. Downing, Jr., B. M. Drucker, L. A. Dye, S. E. Dyer, Jr., J. C. Eaves, D. O. Ellis, M. H. M. Esser, W. M. Faucett, F. A. Ficken, L. R. Ford, L. R. Ford, Jr., M. K. Fort, Jr., Tomlinson Fort, H. K. Fulmer, Seymour Ginsburg, Wallace Givens, R. A. Good, W. L. Gordon, E. E. Grace, J. S. Griffin, Jr., H. C. Griffith, E. H. Hadlock, B. F. Hadnot, S. W. Hahn, Dick Wick Hall, E. A. Hedberg, Nickolas Heerema, R. T. Herbst, P. S. Herwitz, J. G. Horne, Jr., A. S. Householder, G. B. Huff, S. L. Jamison, R. J. Koch, F. W. Kokomoor, G. B. Lang, J. W. Lasley, Jr., T. H. Lee, F. A. Lewis, R. A. Lytle, C. W. McArthur, J. S. MacNerney, Nathaniel Macon, W. R. Mann, D. D. Miller, J. C. Morelock, J. A. Nohel, F. R. Olson, W. V. Parker, S. E. Pence, B. J. Pettis, C. M. Petty, E. E. Posey, J. H. Roberts, R. A. Roberts, W. C. Royster, B. M. Seelbinder, A. L. Shields, J. R. Shoenfield, W. S. Snyder, G. L. Spencer, H. D. Sprinkle, W. L. Strother, P. M. Swingle, H. E. Taylor, J. M. Thomas, H. E. Vaughan, T. L. Wade, Jr., A. D. Wallace, M. J. Walsh, H. C. Wang, A. E. Welch, G. T. Whyburn, L. S. Whyburn, W. M. Whyburn, R. A. Willoughby, C. T. Yang.

By invitation of the Committee to Select Hour Speakers for Southeastern Sectional Meetings, Professor B. J. Pettis of Tulane University and Dr. A. S. Householder of the Oak Ridge National Laboratory addressed the Society. Dr. Householder's address, entitled *Generation of errors in digital computation*, was delivered at 8:00 P.M. on Friday, with Professor Wallace Givens presiding. The address by Professor Pettis, entitled *Certain aspects of linear topological spaces*, was given at 10:00 A.M. on Saturday, with Professor G. T. Whyburn presiding.

Four sessions for the presentation of contributed papers were held on Friday afternoon and Saturday morning. Presiding officers for these sessions were Professors Tomlinson Fort, W. V. Parker, F. A. Lewis, and F. W. Kokomoor.

All sessions were held in the Whitefoord Smith Library on the campus of Wofford College. Members of the Wofford College faculty and their wives were hosts at coffee hours on Friday morning, Friday afternoon, and Saturday morning. President Pendleton Gaines of Wofford College extended a cordial welcome to the Society at its opening session on Friday. He and Mrs. Gaines entertained with a reception in the president's home on Saturday morning. A resolution expressing the appreciation of the Society to its hosts was offered by Professor G. B. Huff and unanimously adopted.

Abstracts of the papers presented follow below. Abstracts whose numbers are followed by the letter "*t*" were presented by title. In the case of joint papers, (p) following one of the authors' names indicates the one who actually read the paper. Mr. McAuley was introduced by Professor Jones, Mr. C. W. Huff by Professor G. B. Huff, Professor Ohtsuka by Professor Lohwater, Dr. Pohrer by Professor Roberts, Miss Norton by Professor Miles, Mr. Aucoin by Professor Wang, Dr. Saaty and Mr. Gass by Professor Whyburn.

ALGEBRA AND THEORY OF NUMBERS

54. A. T. Brauer and B. M. Seelbinder (p): *On a problem of partitions.* II.

Let $a_1, a_2, \cdots, a_k$ be relatively prime positive integers. Let d_r be the greatest common divisor of $a_1, a_2, \cdots, a_r$ for $r=1, 2, \cdots, k$. It was proved in part I (Amer. J. Math. vol. 64, pp. 299–312) that the Diophantine equation $a_1x_1+a_2x_2+\cdots+a_kx_k=n$ always has solutions in positive integers $x_1, x_2, \cdots, x_k$ if $n>T=T(a_1, a_2, \cdots, a_k)=a_2(d_1/d_2)+a_3(d_2/d_3)+\cdots+a_k(d_{k-1}/d_k)$. Moreover it was proved that T is the smallest possible bound for the solution of the given equation by positive integers if for $r=3, 4, \cdots, k$ the numbers a_rd_{r-1}/d_r can be written in the form $a_rd_{r-1}/d_r=a_1y_{r1}+a_2y_{r2}+\cdots+a_{r-1}y_{r,r-1}$, where all the y's are non-negative integers. For $k=3$ the condition was proved necessary. It is proved in this paper that this condition is necessary for every k. If T is not the best bound, then $T-\min(a_1, a_2, \cdots, a_k)$ is a bound for the solution of the given equation by positive integers. (Received September 21, 1953.)

55*t*. Leonard Carlitz: *An application of the multiplication formulas for the Jacobi elliptic function.*

Put $\operatorname{sn} tx/t \operatorname{sn} x=\sum_0^\infty \beta_{2m}(t)x^{2m}/(2m)!$, where t is an odd integer and $u=k^2$ is an indeterminate; also put $\operatorname{sn} x=\sum_0^\infty A_{2m+1}(u)x^{2m+1}/(2m+1)!$. It is proved that $\beta_{2m}(t)=H_m(t)-\sum A^{m/(p-1)}(u)/p$, where the summation is over all primes p such that $p\mid t$ and $p-1\mid 2m$, and $H_m(t)$ is a polynomial in u with integral coefficients. It is also proved that $t\sum_{s=0}^{r}(-1)^{r-s}C_{r,s}\beta_{m+s(p-1)}(t)A_p^{r-s}(u)\equiv 0 \pmod{p^m, p^r}$ for $p>2$, $r\geq 1$. (Received October 5, 1953.)

56*t*. Leonard Carlitz: *Congruences for the solutions of certain difference equations of the second order.*

D. H. Lehmer (Ann. of Math. vol. 33 (1932) pp. 143–150) discussed the difference equation $u_{n+1}=(an+b)u_n+cu_{n-1}$. In particular he proved that if $(a, m)=1$, then $a_{n+m}\equiv c^m a_n \pmod{m}$. 1. In the present paper, taking the equation in "normal" form $u_{n+1}=nu_n+cu_{n-1}$, with c and c^{-1} prime to m, it is shown that $\Delta^{2r-1}u_n\equiv\Delta^{2r}u_n\equiv 0 \pmod{m^r}$ for $r\geq 1$ and all n, where $\Delta^r u_n=\sum_{s=0}^{r}(-1)^{r-s}C_{r,s}c^{m(r-s)}u_{n+sm}$; for m even the modulus is $2^{-r+1}m^r$. 2. Turning next to the nonhomogeneous equation $u_{n+1}=nu_n+bcu_{n-1}+b^n+c^n$ with $u_0=0$, $u_1=1$, we assume that $b\equiv c \pmod{p}$ for each prime p dividing m and show that $\Delta^{2r}u_n\equiv\Delta^{2r+1}\equiv 0 \pmod{m^rm'}$, where $m=\prod p^e$, $m'=\prod p^{e-1}$,

m odd, and now $\Delta^r u_n = \sum(-1)^{r-s}C_{r,s}c^{m(r-s)}u_{n+ms}$. The nonhomogeneous case includes the so-called ménage polynomials. We remark that in the proof of these results it seems desirable to let n take on negative values. (Received October 5, 1953.)

57*t*. Leonard Carlitz: *Congruence properties of the ménage polynomials.*

If $U_n = U_n(t)$ denotes the ménage polynomials and $\Delta f_n = f(n+m) - (t-1)^m f(n)$, $\Delta^r f(n) = \Delta \cdot \Delta^{r-1} f(n)$, it is proved that (1) $\Delta U_n \equiv 0 \pmod{m'}$ and (2) $\Delta^{2r} U_n \equiv \Delta^{2r+1} U_n \equiv 0 \pmod{m^r m'}$, where m' is defined by $m = \prod p^e$, $m' = \prod p^{e-1}$, m odd. For previous results see J. Riordan, *The arithmetic of ménage numbers*, Duke Math. J. vol. 19 (1952) pp. 27–30 and L. Carlitz, *Congruences for the ménage polynomials*, Duke Math. J. vol. 19 (1952) pp. 549–552. (Received October 5, 1953.)

58*t*. Leonard Carlitz: *Congruence properties of the polynomials of Hermite, Laguerre and Legendre.*

For the Hermite polynomial $H_n(x)$ it is proved that (1) $H_{n+m}(x) \equiv (2x)^m H_n(x) \pmod{m}$, where m is an arbitrary integer, and (2) $\Delta^{2r} H_n(x) \equiv \Delta^{2r-1} H_n(x) \equiv 0 \pmod{m^r}$ for all $r \geqq 1$, where $\Delta^r H_n(x) = \sum_{s=0}^{r}(-1)^{r-s} C_{r,s} H_{n+ms}(x) H_{(r-s)}(x)$. Similar results are proved for the polynomial $\Lambda_n^{(\alpha)}(x) = n!\, L_n^{(\alpha)}(x)$, where $L_n^{(\alpha)}(x)$ denotes the general Laguerre polynomial. For the Legendre polynomial $P_n(x)$ it is proved that $n!\, P_{n+m}(x) \equiv P_n(x) P_m(x)$, $n!\, P_{m-n-1}(x) \equiv n!\, P_n(x) P_{m-1}(x) \pmod{m}$; like results hold for the ultraspherical polynomials. (Received October 5, 1953.)

59*t*. Leonard Carlitz and F. R. Olson: *Some theorems on Bernoulli and Euler numbers of higher order.*

Employing the notation of Nörlund's *Differenzrechnung*, Chap. 6, for the numbers $B_m^{(k)}[\omega_1, \cdots, \omega_k]$, etc., congruences are proved that generalize theorems of Glaisher and Nielsen. Similar theorems are proved for the numbers of negative order as well as for numbers of "mixed" order that include both those of positive and negative order. Finally most of the results are extended to the case of more general sequences. (Received August 24, 1953.)

60. R. A. Good, III and D. W. Hall (p): *On a class of orthogonal matrices.*

We study the class $\{Q_n\}$ of all orthogonal matrices of order n with rational elements, with determinant one, and with the additional property that the sum of the elements in each row and each column is one. The determination of all members of the class $\{Q_n\}$ may be reduced to the search for all positive rational automorphs of a certain quadratic form. Using this approach and other special techniques, we give a parametrization of the members of each class $\{Q_n\}$ for $n \leqq 5$. (Received October 14, 1953.)

61. E. H. Hadlock: *Ternary quadratic forms equivalent to forms with one term of type $2b_{ij}y_iy_j$, $j \neq i$.*

Let (c_{ij}) be a matrix of determinant one which takes the ternary quadratic form $f = \sum x_i X_i$, with determinant not equal to zero, into an equivalent form f' containing one term of type $2b_{ij}b_ib_j$, $j \neq i$. The elements of the third column of (c_{ij}) are arbitrary, except that their g.c.d. is one. All integral solutions of a linear homogeneous equation, involving as variables the elements of the first column of (c_{ij}), yield all values for which $b_{13}=0$. Similarly, values of the elements of the second column are restricted so that $b_{23}=0$. It is found that a necessary and sufficient condition that f be equivalent to f' is that f represents, primitively, for certain values of the variables, a divisor of g, the g.c.d. of the three linear functions X_i for those same values of the variables. Moreover it is necessary that this g be a divisor of the determinant of f. (Received October 14, 1953.)

62. C. W. Huff: *On pair of matrices (of order two) A, B satisfying the condition $e^Ae^B=e^{A+B}\neq e^Be^A$.*

M. Fréchet has pointed out [Rend. Circ. Mat. Palermo (1) vol. 2 (1953); see also Mathematical Reviews vol. 14 (1953) p. 237] that in his attack on pairs of matrices X, Y satisfying the equation $e^Xe^Y=e^{X+Y}$ [Rend. Circ. Mat. Palermo (1) vol. 1 (1952)], he had actually described pairs satisfying the relationship $e^Xe^Y=e^{X+Y}=e^Ye^X$. This paper completes Fréchet's work in the following way. Let (A, B) have property Q mean that $e^Ae^B=e^{A+B}\neq e^Be^A$. Let A' be the transpose of the matrix A. Then an equivalence relation, $\sim$, is defined by $(A, B)\sim(X, Y)$ means that there exists a nonsingular matrix C and scalar matrices H and K such that either $X=C(A+H)C^{-1}$ and $Y=C(B+K)C^{-1}$ or $X=C(B'+H)C^{-1}$ and $Y=C(A'+K)C^{-1}$. Two theorems are established: (1) If (X, Y) has property Q then there exists a pair of matrices A and B such that $A=\left(\begin{smallmatrix} x & 0\\ 0 & 0\end{smallmatrix}\right)$, $x\neq 0$, and $B=\left(\begin{smallmatrix} a & z\\ 0 & y\end{smallmatrix}\right)$, $z\neq 0$, and $(X, Y)\sim(A, B)$. (2) If $f(s)=(e^s-1)/s$ for $s\neq 0$ and $f(0)=1$, $A=\left(\begin{smallmatrix} x & 0\\ 0 & 0\end{smallmatrix}\right)$, $x\neq 0$, and $B=\left(\begin{smallmatrix} a & b\\ 0 & y\end{smallmatrix}\right)$, then (A, B) has property Q if, and only if, $b\neq 0$, $f(y)=f(y-x)$ and $f(x)f(y)\neq 0$. (Received October 15, 1953.)

63t. H. L. Lee: *On a certain congruence of variable degree.* Preliminary report.

The congruence $x^{x-1}\equiv 1 \pmod p$ where p is a prime is considered. A sufficient condition is obtained for the solution of this congruence, viz. $p=na^n-n+1$ with $x=a^n$. By the use of the primitive root g of p, it is found that a necessary and sufficient condition that the congruence have a solution is that for $g^h\equiv s \pmod p$, $p-1\mid h(s-1)$. (Received October 16, 1953.)

64. T. H. Lee: *Similarity of matrices with rational generalised quaternions as elements.*

Let α and β be square-free rational integers and r_0, r_1, r_2, and r_3 rational numbers. We define i and j by the equations $i^2=-\alpha$, $j^2=-\beta$, and $ij=-ji$. The elements $r=r_0+r_1i+r_2j+r_3ij$ form a rational generalized quaternion algebra $Q(\alpha, \beta)=Q$. C. G. Latimer (*Quaternion algebras*, Duke Math. J. vol. 15 (1948) pp. 357–366) has shown that there are infinitely many distinct rational generalized quaternion algebras Q, and all except one (which we shall denote by Q_{-1}) are division algebras. In this paper, matrices A whose elements belong to a rational generalized quaternion algebra Q, with $Q\neq Q_{-1}$, are studied. Invariant factors are defined and shown to be rational polynomials. A necessary and sufficient condition for similarity (with respect to rational generalized quaternions) of matrices with elements in Q is that the two matrices have the same invariant factors. This is a generalization of the work of Louise A. Wolf (*Similarity of matrices in which the elements are real quaternions*, Bull. Amer. Math. Soc. vol. 42 (1936) pp. 737–743). But the proof given here is different from that of Miss Wolf's proof in that a method is given for constructing the transforming matrix for the similarity. (Received October 8, 1953.)

65. F. R. Olson: *Arithmetic properties of Bernoulli numbers of higher order.*

L. Carlitz (Quart. J. Math. Oxford Ser. (2) vol. 4 (1953) pp. 112–116) has proved $B_{p+1}^{(p+1)}\equiv p^3/6 \pmod{p^4}$ and $B_p^{(p)}\equiv(-p^2/2)(p-1)! \pmod{p^3}$ for primes $p\geqq 5$. These results are sharpened to $B_{p+1}^{(p+1)}\equiv p^4/4-(p^3/6)(p-1)! \pmod{p^5}$, $p\geqq 5$, and $B_p^{(p)}\equiv(-p^2/2)(p-1)!+(p^3/36)B_{p-3} \pmod{p^4}$, $p\geqq 7$. Like congruences are obtained for generalized Bernoulli numbers $B_n^{(k)}[\omega_1, \omega_2, \cdots, \omega_k]$, Nörlund, *Differenzenrechnung*, Chap. 6, which apply to numbers of either positive or negative order and indeed to the numbers of "mixed" order discussed in *Some theorems on Bernoulli and Euler numbers of higher order*, Bull. Amer. Math. Soc. Abstract 60-1-59. (Received August 24, 1953.)

66. W. V. Parker: *A note on normal matrices.*

In an earlier paper [Duke Math. J. vol. 15 (1948) pp. 439–442] the author proved: If $A=(a_{ij})$ is a normal matrix and the characteristic function of A is $\prod(x-a_{ii})$, then A is a diagonal matrix. This result is extended to give the following: If a normal matrix A can be partitioned into a block matrix in such a way that the characteristic function of A is the product of the characteristic functions of the diagonal block, than A is a diagonal block matrix. (Received October 15, 1953.)

67. R. G. Pohrer: *On some types of equations in a finite field.*

The author classifies several types of equations in a finite field according to the nature of the number of solutions. By direct methods, the number of solutions of equations formed from various combinations of these types, among themselves and conjoined with more general kinds, can be determined. Special examples include equations formed from combinations of quadratic forms, as $c_1Q_1(x_1)Q_2(x_2)+\cdots+c_sQ_{2s-1}(x_{2s-1})Q_{2s}(x_{2s})=\alpha$, where α, $c_i\in GF(q)$, $q=p^n$, $Q_{2i-1}(x_{2i-1})$, $Q_{2i}(x_{2i})$ are $2s$ quadratic forms $(i=1, 2, \cdots, s)$; the results hold for q odd and even. Simple combinations of Moore determinants, considered as equations over the $GF(q)$, can also be treated, as $c_1D_1(x_1)+\cdots+c_sD_s(x_s)=\alpha$, where $\alpha\in GF(q)$, $c_i\in GF(p)$, $D_i(x_i)$ are s Moore determinants $(i=1, 2, \cdots, s)$. (Received October 13, 1953.)

68. R. A. Willoughby: *A note on functional completeness.*

Let $F=\{0, 1, 2, \cdots, n-1\}$ where $n\geqq 2$ and let $\times$ be a multiplication such that (0) if $a, b\in F$ then $a\times b\in F$, (1) $a\times b=b\times a$, (2) $(a\times b)\times c=a\times(b\times c)$, (3) $a\times 1=1\times a=a$, (4) $a\times 0=0\times a=0$. Suppose that F can be partitioned into two sets F_1 and F_2 such that (5) $1\in F_1$, (6) $0\in F_2$, (7) if $a, b\in F_1$ then $a\times b\in F_1$, (8) there exists a $c\in F_1$ and a positive integer m such that $(c\times b)^m=0$ for all $b\in F_2$, (9) either $(c\times\prod_{a\in F_1}a)^m=1$ or F_2 contains a nonzero nilpotent element or F_1 contains at least three distinct elements. Then, there exists a 0→1 transitive permutation on F, Δ, such that $(F, \times, \Delta)$ is functionally complete. Multiplication modulo n satisfies (0)–(9) above. (Received September 14, 1953.)

ANALYSIS

69t. J. H. Barrett: *Differential equations of non-integer order.* II. *Properties of solutions.*

Let α be a positive real number; n be the positive integer such that $n-1<\alpha<n$; $I(-\alpha;\ a,\ x|f)=D_x I(n-\alpha;\ a,\ x|f)$, where $I(\beta;\ a,\ x|f)=\int_a^x f(t)((x-t)^{\beta-1}/\Gamma(\beta))dt$ for

$\beta > 0$. Then there is a unique solution $Y_\alpha(x)$ of the differential equation $I(-\alpha; 0, xy) + y(x) = 0$, $x > 0$, which satisfies the boundary condition $I(1-\alpha; 0, 0^+ | y) = 1$ and, if $n > 1$, the additional conditions $I(i-\alpha; 0, 0^+ | y) = 0$, $i = 2, 3, \cdots, n$. For $0 < \alpha < 1$ and $x > 0$, $Y_\alpha(x)$ satisfies the following relationships, also satisfied by $Y_1(x) = e^{-x}$: $Y(x) > 0$; $\int_0^\infty Y(t)dt = 1$ and $\lim_{x\to\infty} I(\beta; 0, x | Y_\alpha) = 0$ for $0 < \beta < 1$. These results are applied to the complex entire function $E_\alpha(z) = \sum_{p=0}^\infty z^p/\Gamma(p\alpha+1)$ to show that it does not vanish, is less than 1, and $D_x E_\alpha(x) > 0$ on the negative real axis. For $\alpha > 1$, results of Mittag-Leffler and Wiman (Acta Math. vol. 29) are used to show that for $1 < \alpha < 2$ and $x > 0$, $Y_\alpha(x)$ has only a finite number of zeros and for $\alpha > 2$ that $Y(x)$ is oscillatory. (Received October 13, 1953.)

70. Seymour Ginsburg: *On the λ-dimension and the A-dimension of partially ordered sets.*

In this paper the λ-dimension (Komm, *On the dimension of partially ordered sets*, Amer. J. Math. vol. 70 (1948) pp. 507–520) of the partially ordered set consisting of sequences of real numbers, ordered as follows, is shown to be $\aleph_0$. $(x_1, \cdots, x_n, \cdots) \leq (y_1, \cdots, y_n, \cdots)$ if and only if $x_n \leq y_n$ for all n. A new kind of dimension is also discussed. The linear order L on an abstract set S is called an A-extension of the partial order P on S if L is an extension of P which is similar to a set having property A. By the A-dimension of a partial order P on S is meant the minimum number of A-extensions of P which realize P. A nontrivial example is given to show that the λ-dimension and the A-dimension need not be equal, even when both dimensions exist. (Received October 13, 1953.)

71. B. F. Hadnot: *Note on convergence of a class of series.*

R. D. Carmichael has established [Trans. Amer. Math. Soc. vol. 17, pp. 218–220] convergence theorems for $D(s) = \sum_{n=0}^\infty C_n \exp\,(\lambda_{1n}s + \lambda_{2n}s^2 + \cdots + \lambda_{tn}s^t)$ where the λ's are subject to severe conditions. The author has established convergence theorems for $\vartheta(s) = \sum_{n=0}^\infty C_n \exp\,(\lambda_{1n}W_1(s) + \lambda_{2n}W_2(s) + \cdots + \lambda_{tn}W_t(s))$ for more general λ's and arbitrary functions $W_j(s)$. (Received October 13, 1953.)

72. P. S. Herwitz: *On the approximate solutions of differential equations by difference equations.*

This paper considers round-off and truncation errors which arise when differential systems are approximated by difference systems in the manner described by W. M. Whyburn in Amer. J. Math. vol. 51, pp. 265–269. The Picard method of successive approximations with variations due to Moulton and others is used, and questions of stability for the numerical solutions are investigated. (Received October 14, 1953.)

73. J. S. MacNerney: *Hellinger integrals and Stieltjes integral equations.*

Each of m and n is a real, continuous, nondecreasing, not constant function on the number interval $[0, 1]$. The Hellinger integral $\int_0^x [df\,dg]/dm$ is developed as a tool for studying the system (*) $A(x) - A(0) = \int_0^x B\,dm$, $B(x) - B(0) = \int_0^x A\,dn - z\int_0^x A\,dk + C(0) - C(x)$ with various two-point boundary conditions on A and B (k continuous and increasing, C continuous and of bounded variation). The study of the integral provides an elementary proof that there are uncountably many mutually exclusive (except for the origin) linear manifolds in Hilbert space, each dense in the space.

Solutions of (*) are provided by a general existence theorem due to Wall (Rend. Circ. Mat. Palermo (2) vol. 2, (1953)). Boundary value problems are studied by introducing a resolvent pair K, L for (*) with each set of boundary conditions—K behaves as a Green's function. Expansion theorems are obtained as follows: the Hellinger integral is used to define an inner product function Q; for each set of boundary conditions there is found a Hilbert space M (of continuous functions), with inner product Q, in which the corresponding K is the reproducing kernel; orthogonal eigenfunctions of K, suitably normalized, also form an orthonormal set relative to Q and this set is complete in M. (Received October 13, 1953.)

74*t*. J. S. MacNerney: *Note on a functional Hilbert space*. Preliminary report.

Each of m and n is a continuous function from the real numbers to the real numbers which is nondecreasing and not constant; $Q[f, g] = \int_{-\infty}^{+\infty} [df dg^*]/dm + \int_{-\infty}^{+\infty} [fg^*]dn$ for all ordered pairs f, g of functions from the real to the complex numbers for which these integrals exist (see preceding abstract); M is the linear orthogonal space of all f such that $Q[f, f]$ exists, with inner product function Q. It is proved that M is a Hilbert space of continuous functions with a continuous reproducing kernel K, and there is paired with K a function L such that, if C is continuous and of bounded variation on each number interval then, in order that, for each s, $A(s) = \int_{-\infty}^{+\infty} K(s, t)dC(t)$ and $B(s) = \int_{-\infty}^{+\infty} L(s, t)dC(t)$, it is necessary and sufficient that (1) $\int_x^y B dm = A(y) - A(x)$ and $\int_x^y A dn = B(y) - B(x) + C(y) - C(x)$ for all x and y and (2) $A(x)L(x, s) - B(x)K(x, s) \rightarrow 0$ as $x \rightarrow -\infty$ and as $x \rightarrow +\infty$ for each number s. (Received October 13, 1953.)

75. J. A. Nohel: *Stability of perturbed periodic motions.*

In the nonlinear system (1) $x' = Ax + \mu f(x, t, \mu)$ $(' = d/dt)$, x is a real vector of n components, A is a real constant $n \times n$ matrix, μ is a real parameter, and f is a real vector of n components with period 2π in t (2π need not be the least period). Let the unperturbed linear system have a real periodic solution $x = p(t)$ with period 2π. Sufficient conditions for the asymptotic stability of the periodic solution of (1) $x = q(t, \mu)$ which (i) has period 2π in t, (ii) is continuous for all t and $|\mu|$ sufficiently small, and (iii) for which $q(t, 0) = p(t)$, are established for $\mu > 0$ and sufficiently small. Such a solution exists by Theorem 1 of E. A. Coddington and N. Levinson (*Perturbations of linear systems with constant coefficients*, Annals of Mathematics Studies, no. 29, pp. 19–35, Princeton, 1952). In case system (1) is autonomous, the orbital stability of the continuous periodic solution $x = q(t, \mu)$ (Ibid., Theorem 4) with period $2\pi + \tau(\mu)$, where $\tau(0) = 0$ and $q(t, 0) = p(t)$, is investigated for $\mu > 0$ and sufficiently small. (Received October 7, 1953.)

76*t*. Makoto Ohtsuka: *Note on functions bounded and analytic in the unit circle.*

W. Gross [Math. Ann. vol. 79 (1918) pp. 201–208] gave an example of an entire function $w = \phi(z)$ with the property that, for any complex number w, there exists an uncountable set of asymptotic paths along which $\phi(z)$ tends to w. By conformal mapping it is then a simple matter to give an example of a function analytic in the unit circle $|z| < 1$ with a similar property in the neighborhood of a point z_0 of $|z| = 1$. If the function is required to be bounded, however, it can have at most one asymptotic value at a point of $|z| = 1$. It is proved that there exists a function $w = f(z)$, analytic and bounded in $|z| < 1$, such that $f(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta})$ has modulus 1 for almost all $e^{i\theta}$ on $|z|=1$, and such that, for any α with $|\alpha|\leq 1$, the equation $f(e^{i\theta})=\alpha$ is satisfied on an uncountable set of $e^{i\theta}$ on $|z|=1$, i.e. that the radial limit values of $f(z)$ at a set of linear measure zero on $|z|=1$ fills the open set $|z|<1$. (Received October 7, 1953.)

77. W. C. Royster: *Note on values omitted by p-valent functions.*

Let $\mathcal{F}$ be the class of p-valent functions $w=f(z)$, p a positive integer, which are regular in $|z|<1$ and have a p-tuple zero at the origin. Hayman (*Symmetrization in the theory of functions*) states that if $f(z)\in\mathcal{F}$ then $f(z)$ takes on every value in $|w|<(1/4)^p$ exactly p times. This result is sharp. Using this result and a boundedness condition, one proves that if $f(z)\in\mathcal{F}$ omits a value α at least once for z in $|z|<1$, then a lower bound for $|\alpha|$ is found in sharp form. Also using a theorem due to Biernacki (Bull. Sci. Math. (2) vol. 70) a lower bound for $|\alpha|$ is found in sharp form if $f(z)$ omits a value α exactly p times. By constructing bounded functions from $\mathcal{F}$ and applying the above results other theorems are proven. (Received October 14, 1953.)

78. A. L. Shields: *Measure on a projective limit.*

Let A be a directed set, and for each $a\in A$ let (X_a, S_a, m_a) be a σ-finite measure space. For each pair $a, b\in A$ such that $a\leq b$ there is a function ϕ_{ab} from X_b to X_a such that, if $E\in S_a$, then $\varphi_{ab}^{-1}(E)\in S_b$, and $m_a(E)=m_b(\phi_{ab}^{-1}E)$. Let X be the projective limit of the spaces X_a under the maps ϕ_{ab}. Let $E_a\in S_a$ for some fixed a. Let $[E_a]$ be the set of all those points of X whose ath coordinate is in E_a. The sets $[E_a]$ form a ring. If we define $m([E_a])=m_a(E_a)$ this is a σ-finite measure on the ring, hence can be extended to the σ-ring generated by the ring. In case the X_a and X are locally compact groups, m_a is Haar measure, ϕ_{ab} are homomorphisms, then the domain of definition of the completion of m is the Borel field in X, and m is a Haar measure on X. (Received November 27, 1953.)

79. J. M. Thomas: *Reflex solutions of linear differential equations.*

The term "reflex functions" will describe the class of even and odd functions and "equation" will mean "ordinary linear homogeneous differential equation." An equation is reflex if its coefficients are alternately even and odd, the initial being unity. A reflex equation of order n has both an even solution with $[(n+1)/2]$ arbitrary constants and an odd solution with $[n/2]$. If each coefficient in an equation is replaced by its even or by its odd part, the choice alternating from term to term, two equations M_0, M_1 of orders n and at most $n-1$ respectively result. M_0 is reflex and M_1 can be made reflex in a region where its initial does not vanish. The system M_0, M_1 is satisfied by the reflex solutions of the original equation, the even corresponding to initial determinations with derivatives of odd order zero. The system M_0, M_1 can be applied to determine all equations with reflex solutions. Extension is made to the case where the zero of the independent variable is singular for the equation. This involves modifying the form of the initial determination and the number of arbitrary constants. (Received October 14, 1953.)

APPLIED MATHEMATICS

80. F. A. Ficken: *Reduction of a nonlinear parabolic problem with piecewise constant coefficients.* Preliminary report.

A problem consisting of $(E)\,[u_x-\psi u(1-u)-pu]_x=ku_t$ (p, ψ, k constant), an initial condition, and certain end conditions at $x=0$ and $x=N$ has been linearized by Montroll and Newell (Journal of Applied Physics vol. 23 (1952) pp. 184–194). When ψ, p, and k are step-functions with jumps at $x=j_k$ $((j_0=)0<j_1<\cdots<j_{n-1}<N(=j_n))$, standard junction conditions (u continuous, u_x jumps) do not linearize directly. Now u_k is expressed for $j_{k-1}<x<j_k$ in terms of a new unknown z_k satisfying a linear homogeneous (E_k^*) and a function $\theta_k(t)$ satisfying $(\lambda_{k-1})\,k_k\dot\theta_k=l_{k-1}z_k$, where $l_{k-1}z_k$ is a linear combination of z_k and z_{kx} at j_{k-1}. By using the jump condition at j_k $(1\leqq k\leqq n-1)$ one obtains a function $\phi_k=c_k\theta_{k+1}^{r_k}$ (c_k, r_k known constants) such that $(\mu_k)\,k_k\dot\phi_k=m_kz_k$, where m_kz_k is a linear combination of z_k and z_{kx} at j_k. The *linear* problem consisting of (E_k^*), an initial condition, and the nonhomogeneous end conditions (λ_{k-1}) and (μ_k) (at $x=N$ the transformed (homogeneous) end condition supplies μ_n) has a solution $z_k(\theta_k, \theta_{k+1}; x, t)$ (z_n involving only θ_n). By substituting these n functions z_k into the transformed end condition at $x=0$ (which involves θ_1 if the feed-concentration is variable), and into the $n-1$ continuity conditions at j_k, one obtains n nonlinear functional equations for the n functions θ_k. Vice versa, solutions θ_k of these equations, if available, would yield via z_k solutions u_k of the original problem. (Received October 15, 1953.)

81. W. R. Mann (p) and J. F. Blackburn: *A steady state nonlinear heat problem.*

The problem is to find a function, $u(x, y)$, which satisfies Laplace's equation in the semi-infinite strip, $y\geqq 0$, $0\leqq x\leqq\pi$; which vanishes along the two vertical sides, and which satisfies the nonlinear condition $u_y(x, 0)=-G[u(x, 0)]$ along the horizontal side. G is a continuous decreasing function which vanishes when $u=1$. The method of approach is to reduce the problem, by means of the finite Fourier transform, to the nonlinear integral equation $f(x)=(1/\pi)\int_0^\pi K(x, \lambda)G[f(\lambda)]d\lambda$, where $K(x, \lambda)=\log\,[1-\cos\,(x+\lambda)/1-\cos\,(x-\lambda)]$ and where $f(x)=U(x, 0)$. The main results are proofs of the existence and uniqueness of $f(x)$ and the fact that $0\leqq f(x)\leqq 1$. An iterative approximation to $f(x)$ is also discussed. (Received October 13, 1953.)

82*t*. T. L. Saaty and S. S. Gass: *Parametric objective function.*

In problems such as the linear programming problem where there is a linear function to be optimized (called the objective function), it is desirable to study the behavior of solutions when the (cost) coefficients in the objective function are allowed to change. The problem is then to find the x_j $(j=1, 2, \cdots, n)$ which minimize the linear form $\sum_{j=1}^{n}(c_j+\lambda c_j')x_j$ and satisfy the constraints $x_j\geqq 0$ and $\sum_{j=1}^{n}a_{ij}x=a_{i0}$ $(i=1, 2, \cdots, m)$, where c_j, c_j', a_{ij}, and a_{i0} are constants (with at least one $c_j\neq 0$) and λ a parameter. Using the Simplex method, a computational procedure is described which enables one, given a feasible solution, to determine the values of the parameter (if any) for which the solution minimizes the objective function; and given a minimum feasible solution how one can generate by the Simplex algorithm all minimum feasible solutions and the corresponding values of the parameter. To a minimum feasible solution there corresponds one connected set of λ called a maximal determined range. Two maximal determined ranges intersect in at most one point and the set of maximal ranges is connected. The geometry of the problem is discussed in terms of the dual problem and an illustrative example is given. (Received October 2, 1953.)

GEOMETRY

83*t*. M. K. Fort, Jr.: *A cylindrical curve with maximum length and maximum height.*

In a recent paper, *Curves encircling a cylinder*, Amer. Math. Monthly vol. 60 (1953) pp. 30–31, J. W. Green considers curves C which have diameter 2 and which encircle a right circular cylinder of unit radius. Each such curve C has a length $L(C)$ and a height $H(C)$. ($H(C)$ is defined to be the minimum distance between two planes perpendicular to the cylinder and containing C between them.) Green proves that $2\pi \leq L(C) \leq 2^{3/2}\pi$ and $0 \leq H(C) \leq 2^{1/2}$. In the present paper, a curve C is constructed for which $L(C) = 2^{3/2}\pi$ and $H(C) = 2^{1/2}$. This example answers two questions raised by Green in his paper. (Received September 4, 1953.)

84. C. M. Petty: *Centroid surfaces.*

In $E^{(n)}$, let M be given in polar coordinates ρ, u by $0 \leq \rho \leq r(u)$, $u \in \Omega_n$, where Ω_n is the surface of the n-dimensional unit sphere and $r(u)$ is a positive, even, continuous function defined on Ω_n. The centroids of the half-volumes of M bounded by hyperplanes through the origin z (center of M) constitute a convex surface C^* of class $C^{(2)}$ with center z. The surface C^* is called the centroid surface of M and determines M uniquely. If C^* is expanded by the factor $(n+1)K_n/2K_{n-1}$ $(K_r = \pi^{r/2}/\Gamma(r/2+1))$ we obtain the expanded centroid surface C of M. One has the inequality $|C| \geq |M|$ between the volumes where the equality holds only if M is an ellipsoid. The volume $|M|$ of M can be expressed by (1) $|M|^{n-1} = (1/n)(K_n^{n-2}/K_{n-1}^{n})\int_{\Omega_n} |M(u)|^{n-1}|C(u)|\,d\omega_n$, where $M(u)$ is the cross-section of M with a hyperplane H through z perpendicular to u and $C(u)$ is the expanded centroid surface of $M(u)$ in H. The relation (1) is a special case of a more general relation involving $(n-1)$ bodies of the type considered above. (Received October 21, 1953.)

TOPOLOGY

85. C. V. Aucoin: *On skew rings.*

A skew ring consists of a set R and two binary operations "$*$" and "o" such that $\{R, *\}$ forms a group and the operation "o" is two-sided distributive. If the operation "$*$" is abelian then one has a ring without the associative law. This paper proves the following three theorems: 1. If $\{R, *\}$ is a simple group, then either R is isomorphic with the ordinary field of integers mod a prime, or R is trivial (i.e. $x \circ y = e$ for all x and y of R). 2. If R is a finite skew ring and I is the set of trivial elements (i.e. $I = \{x | x \circ r = r \circ x = e$ for each $r \in R\}$), then R/I is isomorphic with the direct product $R_1 \otimes R_2 \otimes \cdots \otimes R_n$ of rings R_i whose basic groups are abelian p-groups. 3. A compact connected topological skew ring satisfying the second countability axiom is trivial. (Received October 19, 1953.)

86. C. E. Capel: *Inverse limits of arcs and simple closed curves.*

Let $\{X_\lambda, \Pi_{\lambda\mu}, \Lambda\}$ be an inverse system of compact Hausdorff spaces in which the bounding maps, $\Pi_{\lambda\mu}$, are monotone; let $X = \lim_{\leftarrow}\{X_\lambda, \Pi_{\lambda\mu}, \Lambda\}$ be the inverse limit space. If each X_λ is locally connected, then X is locally connected. If each X_λ is an arc (a simple closed curve) and Λ is countable, then X is an arc (a simple closed curve). (Received October 13, 1953.)

87. H. S. Collins: *A new class of complete bornologique spaces.*

Recent extensions of the closed graph theorem have aroused considerable interest in a certain class of locally convex linear topological spaces; namely, the complete bornologique spaces. It is the purpose of this note to show that certain additional function spaces belong to this class. Let S be any locally compact Hausdorff topological space, and denote by E the space of real (or complex) valued continuous functions on S, supplied with the compact-open topology. Let E^* denote the space of continuous linear functionals on E, supplied with the strong topology. Our result is that E^* is a complete bornologique space when S is paracompact. Previous results along these lines required that S be sigma-compact. (Received October 13, 1953.)

88. D. O. Ellis: *On the topolattice and permutation group of an infinite set.* II.

Let S be an infinite set and Λ be the lattice of all T_1 topologies on S. Λ is called the *topolattice* (Robert Bagley and David Ellis, *On the topolattice and permutation group of an infinite set*, Bull. Amer. Math. Soc. Abstract 60-1-89) of S. One shows: *The group of permutations of S is isomorphic to the group of automorphisms of Λ. Any automorphism of Λ is completely determined by its behavior on the hyperplanes of Λ.* The methods used are lattice-theoretic. (Received September 3, 1953.)

89. D. O. Ellis and Robert Bagley (p): *On the topolattice and permutation group of an infinite set.*

Let S be an infinite set and let Λ denote the set of all T_1 topologies on S. We consider a permutation f of S and define $f(\alpha, \beta) = 0$ for $\alpha, \beta \in \Lambda$ when f is continuous with α as domain topology and β as range topology in S, and $f(\alpha, \beta) = 1$ otherwise. We then find a necessary and sufficient set of algebraic conditions on a general mapping of $\Lambda \otimes \tilde{\Lambda}$ onto the two-element chain $(0, 1)$ in order that the mapping be induced in the above way by a permutation of S. The methods are essentially lattice-theoretic. (Received June 16, 1953.)

90. D. O. Etter and J. S. Griffin, Jr. (p): *On the metrizability of the bundle space.*

It has been shown by Yu. M. Smirnov (see American Mathematical Society Translation No. 91, *On the metrization of topological spaces*) that if X is a topological space, then X is metrizable if and only if X is paracompact and has an open cover each of whose members is metrizable. This leads immediately to the proposition: if $\{X, B, \pi, Y, \mathcal{U}, \phi, G\}$ is a fibre bundle whose base space B and fibre Y are metrizable, then the bundle space X is also metrizable: since if $U \in \mathcal{U}$ then $\pi^{-1}(U)$ is metrizable, it suffices to show X paracompact. But if $\mathcal{C}$ is any open cover for X, let (1) $\mathcal{V}$ be a locally finite refinement of $\mathcal{U}$, (2) $\mathcal{W}$ be a closure-refinement of $\mathcal{V}$ which is also locally finite, (3) c be a function on $\mathcal{W}$ such that if $W \in \mathcal{W}$ then $c(W) \in \mathcal{V}$ and, $W \subset c(W)$, (4) $\mathcal{D}_V$ be the family of intersections of members of $\mathcal{C}$ with $\pi^{-1}(V)$, (5) $\mathcal{D}_V^*$ be a locally finite refinement of $\mathcal{D}_V$ where $V \in \mathcal{V}$, (6) $\mathcal{E}_W$ be the family of intersections of members of $\mathcal{D}^*_{c(W)}$ with $\pi^{-1}(W)$ where $W \in \mathcal{W}$. Then $\bigcup_{W \in \mathcal{W}} \mathcal{E}_W$ is a locally finite open cover refining $\mathcal{C}$. (Received October 15, 1953.)

91. W. M. Faucett: *A result on mobs irreducibly connected between two points.* Preliminary report.

Let S be a compact connected Hausdorff space together with a continuous, associative multiplication. Further, let S have a unit and be irreducibly connected between a and b, where a^2 and b^2 are idempotents. Then S is commutative if and only if S contains an element O such that $Ox=O=xO$ for all $x\in S$. S can be linearly ordered and in the course of the proof it is shown that the multiplication in S is continuous with respect to the order topology. Also if a and b are the unit and the zero of S respectively, and S has no other idempotents, then S contains a countable dense set. (Received October 13, 1953.)

92. M. K. Fort, Jr.: *A theorem about topological n-cells.*

The main result obtained in this paper is: If G is a nonempty, open, connected subset of Euclidean n-space, then there exists a sequence $E_1\ E_2\ E_3 \cdots$ of closed topological n-cells such that $_{1}E_{i}G$. As a corollary to this result we obtain: If G is a nonempty, open, connected subset of Euclidean n-space, then there exists a subset H of G such that: dimension of H is less than n, $G-H$ is open and connected, all of the homotopy groups of $G-H$ vanish. (Received August 10, 1953.)

93. H. C. Griffith: *The enclosing of cells in 3-space by simple closed surfaces.*

Harrold's property P for arcs [Bull. Amer. Math. Soc. Abstract 59-4-434] is extended to apply to closed k-cells in 3-space for $k=1$, 2, or 3. If E^1 is a segment and $E^k=E^1\times E^{k-1}$, then a k-cell C is the image of E^k under a homeomorphism h and $\mathfrak{T}_h$ denotes the collection of sets $h(x\times E^{k-1})$. C has the uniform disk property relative to $\mathfrak{T}_h$ provided there is a positive δ such that for each positive ϵ and $T\in\mathfrak{T}_h$ there is a 2-cell D such that (1) $d(C$, Boundary of $D)>\delta$, (2) $D\cap C$ is a $(k-1)$-cell, (3) D is locally polyhedral except on C, (4) $\rho(T, D\cap C)<\epsilon$ (where ρ is an appropriate metric on $\mathfrak{T}_h$), and (5) D meets C essentially. Theorem. A cell C with property P is enclosed by a polyhedral 2-sphere $K_\epsilon\subset S(C, \epsilon)$ for every positive ϵ if and only if it has the uniform disk property relative to $\mathfrak{T}_h$. (Received October 14, 1953).

94. R. J. Koch: *Remarks on primitive idempotents in compact mobs with zero.*

Recall that a mob is a Hausdorff space together with a continuous associative multiplication. Denote by X a compact mob with zero, let N be the subset of elements whose powers converge to 0, and denote by E_p the set of primitive idempotents of X. It is shown that if e is a nonzero primitive idempotent of X then XeX is a minimal non-nil ideal. This is the converse of a result of Numakura (Bulletin of the Yamagata University, no. 4, 1951), and is proved by showing these are equivalent: (1) e is a nonzero primitive idempotent. (2) Xe is a minimal non-nil left ideal. (3) $(eXe)\backslash N$ is a (compact) group. (4) eXe is a minimal non-nil bi-ideal (Bull. Amer. Math. Soc. Abstract 58-6-575). From Numakura's results it then follows that $(XE_pX)\backslash N=(XE_p)\backslash N$ is the disjoint union of (compact) groups. Finally, it is shown that if X is connected, then $(XE_p)\cap N$ is dense in XE_p. (Received October 13, 1953.)

95*t*. R. J. Koch: *Theorems on mobs.*

Recall that a mob is a Hausdorff space together with a continuous associative multiplication. If e is an idempotent of the semigroup X then Xe is a minimal left ideal iff eXe is a group. If X is a semigroup containing minimal left ideals and minimal right ideals, then the Suschkewitsch kernel K is a group iff K meets a normal set.

The center of K is the intersection of K with the center of X. A natural partition of X is exhibited which separates the maximal subgroups of K, and if X is a compact mob this decomposition is upper semicontinuous. It is shown that if a left translate of some closed subset of the compact mob X exhausts X, then X has a left unit. A mob is termed monothetic if it has a dense cyclic submob. Among results established for compact monothetic mobs is the fact that a compact monothetic mob with two distinct generators is a group. A locally compact monothetic mob with kernel is compact. A locally compact monothetic zero-dimensional mob with unit is a compact group. Two generalizations of monothetic mobs are considered. Finally, a quotient structure is discussed for compact mobs which contain a closed normal set. (Received October 13, (1953.)

96. L. F. McAuley: *Generalized upper semi-continuous collections as applied to aposyndetic decompositions of continua.*

Suppose that G is a collection of point sets. If $\{x_i\}$ is a sequence of points converging to a point x, $x \in g \in G$, $x_i \in g_i \in G$ for each i, $g_i \neq g_j$ for $i \neq j$, and if $\{y_i\}$ is a sequence of points, $y_i \in g_i$ for each i; then every convergent subsequence of $\{y_i\}$ converges to a point of g. G is said to be an upper semi-continuous collection. Certain subcollections of G are defined to be "regions." Suppose that M is a continuum. Let H denote the collection of all point sets $M(p)$ for $p \in M$ (see Bull. Amer. Math. Soc. Abstract 59-1-77). $S(p, x)$ denotes the set of all points y of $M(p)$ such that there does not exist a subcollection N of H such that N^* is the sum of a finite number of continua and N^* separates x from y in M. Suppose that M is also a completely separable semi-metric topological space and that K is the collection of all point sets $S(p, x)$ for each point p of M and each point x of $M(p)$; then K is an upper semi-continuous collection of mutually exclusive closed point sets filling up M; and furthermore, with respect to its elements as points, K is an aposyndetic, connected, and completely separable Hausdorff topological space. (Received October 14, 1953.)

97. Frances A. Norton: *Mayer-Vietoris sequence for homotopy groups.*

Let X be a fibre space over B with fibre F. Suppose that B is a topological product $B_1 \times B_2$. For $j=1, 2$, the inclusion map $i_j: B_j \rightarrow B$ gives an induced fibre space X_j over B_j with the same fibre F. Assuming X, B, F to be arc-wise connected, it is shown that there exists an exact sequence of the form $\cdots \rightarrow \pi_n(F) \xrightarrow{I} \pi_n(X_1) + \pi_n(X_2) \xrightarrow{J} \pi_n(X) \xrightarrow{\Delta} \pi_{n-1}(F) \rightarrow \cdots$ where I, J, Δ are defined, in a natural manner, by means of some injection homomorphisms and boundary homomorphisms. This sequence of homotopy groups corresponds to the well known Mayer-Vietoris sequence in homology theory. (Received October 19, 1953.)

98. R. A. Roberts: *Duality theorems for generalized manifolds.*

Two topologies are defined for the Čech homology groups using field coefficients, one for locally compact spaces and the other for compact spaces. These topologies enable the author to prove Alexander and Poincaré type duality theorems for generalized manifolds, without restrictions such as perfect normality, and to obtain some preliminary results concerning carriers of Čech cycles. For example, it is shown that if M is a compact space and F is an algebraic field, then there is a topology for $H^s(M, F)$ and a collection of pairs of points of M such that the o-homology classes of the 0-Čech cycles carried by each pair of points have $H^0(M, F)$ as their linear closure in the topology. (Received October 13, 1953.)

99. W. L. Strother: *Fixed points in the space of subsets.*

It is shown that if X is a compact Hausdorff space then $S(X)$ has the fixed point property for functions which are embedded extensions of functions on X to $S(X)$. The proof is based on continuity for certain associated multivalued functions. (Received October 13, 1953.)

100. P. M. Swingle: *Locally n-divisible and related connexes.*

New types of connexes related to those in *Generalization of biconnected sets* (Amer. J. Math. vol. 53 (1931) pp. 385–400) are given. For example a connexe M is locally n-divisible at a point x if and only if, for each region H, $x \in H$, (1) there exists a subregion R of H, $x \in R$, and n subconnexes M_i of M, mutually exclusive and each dense in R, and $M \cdot R = (M_1 + \cdots + M_n) \cdot R$; and (2) if the n subconnexes M_i' of M are such that the $M_i' \cdot R$ are mutually exclusive and each dense in $M \cdot R$, then there does not exist a subconnexe M_{n+1}' of M, mutually exclusive with the M_i''s in $M \cdot R$, such that $R \cdot M_{n+1}' \neq 0$ and $M \cdot R = (M_1' + \cdots + M_{n+1}') \cdot R$. Various existence theorems for these connexes are proven including: There exists in any region T of the plane a widely connected subset M, which, for n an integer >0, is locally n-divisible and M and T have the same closure; For $n > 1$ there exists in the plane an n-biconnexe confined connexe, which contains a subconnexe M, which does not contain a biconnected subset. (Received October 13, 1953.)

101. A. D. Wallace: *Peripheral subgroups of clans.*

Recall that a clan is a compact connected Hausdorff space together with a continuous associative multiplication with unit. If u is the unit of the clan S and if $H(u)$ is the set of elements of S with inverses, then $H(u)$ is a compact (topological) subgroup of S. It is known that, if S is contained in R^{n+1}, $n \geq 1$, then $H(u) \subset F(S)$, the set theoretic boundary of S. Simple examples show that $H(u)$ need not coincide with $F(S)$. In this paper this theorem is proved: Let S be a clan contained in R^{n+1}, $n \geq 1$, and let S not cut R^{n+1}. If $H(u)$ cuts R^{n+1}, then $H(u) = F(S)$ and $H(u)$ irreducibly cuts R^{n+1} into the two components R^{n+1}/S and $S/H(u)$. Moreover $H(u)$ is a compact connected Lie group. The condition "$H(u)$ cuts R^{n+1}" can be replaced by "dim $H(u) = n$." (Received September 1, 1953.)

W. M. WHYBURN,
Associate Secretary

THE NOVEMBER MEETING IN EVANSTON

The four hundred ninety-seventh meeting of the American Mathematical Society included a Symposium on Special Topics in Applied Mathematics and was held on Friday and Saturday, November 27–28, 1953. The Symposium was sponsored jointly by the Society and the National Research Council, under a contract with the National Science Foundation. There was a total registration of 256, including the following 206 members of the Society:

E. S. Allen, W. R. Allen, J. M. Anderson, D. L. Arenson, E. L. Arnoff, Nachman Aronszajn, Robert Baer, R. H. Bardell, W. E. Barnes, R. B. Barrar, A. F. Bausch, G. E. Baxter, J. W. Beach, A. A. Blank, H. D. Block, G. M. Bloom, L. M. Blumenthal, R. P. Boas, W. M. Boothby, G. U. Brauer, Leonard Bristow, F. E. Browder, R. H. Bruck, H. D. Brunk, R. C. Buck, E. L. Buell, R. S. Burington, James William Butler, S. S. Cairns, K. H. Carlson, J. W. Carr, W. B. Caton, Lamberto Cesari, Subrahmanyan Chandrasekhar, Y. W. Chen, S. S. Chern, Gustave Choquet, H. M. Clark, M. D. Clement, L. W. Cohen, Harvey Cohn, S. H. Cohn, Richard Courant, V. F. Cowling, C. W. Curtis, H. J. Curtis, M. L. Curtis, W. F. Darsow, R. B. Deal, Jr., John DeCicco, J. C. E. Dekker, R. J. DeVogelaere, J. A. Dieudonné, N. J. Divinsky, C. L. Dolph, M. D. Donsker, R. J. Driscoll, W. L. Duren, Jr., Meyer Dwass, W. F. Eberlein, J. D. Elder, Robert Ellis, Paul Erdös, H. P. Evans, R. L. Evans, Trevor Evans, G. M. Ewing, Chester Feldman, D. A. Flanders, H. H. Fox, W. C. Fox, J. S. Frame, Evelyn Frank, H. D. Friedman, C. G. Fry, R. E. Fullerton, R. A. Gambill, H. L. Garabedian, P. R. Garabedian, J. J. Gergen, J. H. Giese, David Gilbarg, Leonard Gillman, Casper Goffman, H. E. Goheen, Bernice Goldberg, Malcolm Goldman, Michael Golomb, A. W. Goodman, L. M. Graves, R. L. Graves, L. W. Green, Harold Greenspan, L. W. Griffiths, P. E. Guenther, M. M. Gutterman, T. E. Hagensee, J. K. Hale, P. R. Halmos, P. C. Hammer, L. J. Heider, Melvin Henriksen, E. H. C. Hildebrandt, D. L. Holl, T. C. Holyoke, Eberhard Hopf, B. E. Howard, W. A. Howard, K. C. Hsu, H. K. Hughes, Ralph Hull, A. W. Jacobson, Arno Jaeger, T. J. Jaramillo, W. E. Jenner, E. R. Johnston, Mark Kac, Samuel Kaplan, Leo Katz, N. D. Kazarinoff, D. E. Kearney, R. B. Kellogg, L. M. Kelly, D. E. Kibbey, S. C. Kleene, Jacob Korevaar, P. H. Kratz, A. H. Kruse, M. Z. Krzywoblocki, C. D. LaBudde, H. G. Landau, R. E. Lane, Serge Lang, R. E. Langer, E. H. Lee, G. R. Lehner, Joseph Lehner, Jean Leray, Frank Levin, J. J. Levin, G. G. Lorentz, Saunders MacLane, H. M. MacNeille, Morris Marden, E. P. Merkes, B. E. Meserve, A. N. Milgram, M. D. Morley, W. L. Murdock, Edward Nelson, R. J. Nunke, R. H. Oehmke, H. A. Osborn, L. G. Peck, Pasquale Porcelli, G. B. Price, A. L. Putnam, W. T. Reid, Haim Reingold, R. B. Reisel, P. C. Rosenbloom, M. A. Rosenlicht, Arthur Rosenthal, E. H. Rothe, L. A. Rubel, Hans Samelson, O. F. G. Schilling, Morris Schreiber, W. T. Scott, D. H. Shaftman, M. E. Shanks, J. M. Shapiro, Edward Silverman, R. J. Silverman, H. T. Slaby, F. B. Sloss, J. M. Slye, D. M. Smiley, M. F. Smiley, Jerome Spanier, C. E. Springer, George Springer, W. F. Stinespring, A. H. Taub, J. S. Thale, E. A. Trabant, J. W. Tukey, H. L. Turrittin, C. J. Vanderlin, Jr., Bernard Vinograde, R. D. Wagner, G. L. Walker, Daniel Waterman, L. M. Weiner, C. P. Wells, F. J. Weyl, George Whaples, L. R. Wilcox, G. N. Wollan, F. B. Wright, Jr., F. M. Wright, A. W. Wymore, L. C. Young, J. W. T. Youngs, Daniel Zelinsky, J. L. Zemmer, Jr.

The Committee to Select Hour Speakers for Western Sectional Meetings had invited Professor P. C. Rosenbloom of the University of Minnesota to address the Society. Professor Rosenbloom spoke on *The heat equation* on Saturday, November 28, at 10:30 A.M. The presiding officer was Professor R. P. Boas.

The Symposium was divided into three sessions which met respectively at 10:15 A.M. and 1:30 P.M. on Friday and at 1:30 P.M. on Saturday. The first one was presided over by Professor Richard Courant of New York University, and papers by Professor Jean Leray of the Institute for Advanced Study and the College de France on *The physical fact and the differential equations*, by Dr. L. G. Peck of New York University on *One dimensional shock-wave flows* and by Professor P. R. Garabedian of Stanford University on *Recent advances in the application of conformal mapping to hydro-dynamics* were read.

The second session with Professor J. W. Tukey of Princeton University in the chair introduced Professor D. E. Muller of the University of Illinois, Professor Mark Kac of Cornell University, and Professor S. S. Cairns of the University of Illinois, who spoke respectively on *Boolean algebras in electric circuit design*, *Signal and noise problems* and *Discrete structures and large-scale computers.*

The topics of the final session were *Problems of astro-physical fluid dynamics*, *The theory of lattice vibrations*, and *The geometric structure of shock-wave configurations*, presented respectively by Professor Subrahmanyan Chandrasekhar of the University of Chicago, Professor E. W. Montroll of the University of Maryland and the Office of Naval Research, and Professor A. H. Taub of the University of Illinois, Dr. F. J. Weyl, Office of Naval Research and the National Research Council, presiding.

Sessions for the presentation of contributed papers were held on both Friday and Saturday at 9:00 A.M. and 3:30 P.M. Presiding officers at these sessions were Dr. W. E. Jenner and Professors G. M. Ewing, L. M. Blumenthal and W. T. Reid, respectively.

There was a dinner for members of the Society and their guests in Sargent Hall on Friday evening, at which Professor H. T. Davis acted as toastmaster.

Abstracts of papers presented at the meeting are recorded below. Abstracts whose numbers are followed by "*t*" were read by title. In the case of joint papers presented in person, (p) after a name indicates the author who read the paper. Mr. Gordon was introduced by Professor Antoni Zygmund, Professor Halperin by Professor Norman Miller, Dr. Junkat and Professor Knopp by Professor C. N. Moore.

ALGEBRA AND THEORY OF NUMBERS

102. R. H. Bruck and H. T. Slaby (p): *Central nilpotency of commutative Moufang loops.*

Let G be a commutative Moufang loop. The associator (x, y, z) of elements $x, y, z \in G$ is defined by $xy \cdot z = (x \cdot yz)(x, y, z)$. If H, K, N are normal subloops of G, let (H, K, N) denote the (normal) subloop of G generated by the set of associators (h, k, n), $h \in H$, $k \in K$, $n \in N$. The lower central series of G is the sequence $\{G_i; i = 0, 1, 2, \cdots\}$ defined by $G_0 = G$, $G_{i+1} = (G_i, G, G)$. For all non-negative integers i, j, k, $(G_i, G_j, G_k) \subseteq G_{i+j+k+1}$. If $G_k = 1$ and k is the least integer for which this is true, then G is said to be centrally nilpotent of class k. For $n = 3, 4, 5$, if G is generated by n elements, then G is centrally nilpotent of class at most $n-1$. (Received October 13, 1953.)

103*t*. K. T. Chen: *On a Cartan's theorem and its converse.*

First a local Lie group is constructed from a given Lie algebra by using Hausdorff's formula $(\exp x)\ (\exp y) = \exp z$, where x and y are noncommutative indeterminates and z is an infinite sum of bracket products of x and y. This leads immediately to the Cartan's theorem that, in a canonical coordinate system of the first kind in a nilpotent local Lie group G, the multiplication is defined by polynomials, i.e., if $(\alpha) = (\alpha^1, \cdots, \alpha^n)$ and $(\beta) = (\beta^1, \cdots, \beta^n)$ are two points of G, then each coordinate of their product $(\alpha) \cdot (\beta)$ is a polynomial in the α's and β's. The converse of this theorem is proved by using a theorem due to Engel and the formula $\log \{(\exp -x)(\exp y)(\exp x)\} = y + [y, x] + [[y, x], x]/2! + [[[y, x], x], x]/3! + \cdots$ (Received September 23, 1953.)

104*t*. Harvey Cohn: *Stability properties of cyclotomic units.*

Completing results indicated in an earlier abstract (Bull. Amer. Math. Soc. Abstract 58-6-571) the author considers fields for which infinitesimal variations of the basis preserving the discriminant (determinant) will decrease the norms of the units. These so-called *stable* fields include all nontrivial $R(\cos 2\pi/N)$ (except $N=12$), as well as just those $R(\exp 2\pi i/N)$ for which N is square-free. The first result depends essentially on the fact that the smaller half of the *absolute* residues (mod N) are not closed under multiplication. The second result depends on the existence of a normal basis for the field. (Research sponsored by the U. S. Army Office of Ordnance Research under contract DA-20-018-ORD-12332.) (Received October 9, 1953.)

105. C. W. Curtis: *A note on the representations of nilpotent Lie algebras.*

The following theorems are proved. (1) Let $x \to U_x$ be an indecomposable representation of a nilpotent Lie algebra $\mathfrak{L}$ over a field K. Then for all $x \in \mathfrak{L}$, the minimum polynomial of U_x is a prime power. Moreover, any two irreducible constituents of U are equivalent. (2) Let $x \to U_x$ be an irreducible representation of a Lie algebra $\mathfrak{L}$ over a field K of characteristic $p > 0$. If $\mathfrak{D}$ is a Lie algebra of derivations acting in $\mathfrak{L}$, then there exists an irreducible representation $x \to T_x$ of the semi-direct sum $\mathfrak{L} + \mathfrak{D}$, and a 1-1 linear transformation S of the representation space of U into the representation space of T such that $ST_x = U_x S$ for all $x \in \mathfrak{L}$. (3) If $\mathfrak{L}$ is a nilpotent Lie algebra over a field K of characteristic $p>0$, and if $(x_1, \cdots, x_n)$ is a basis of $\mathfrak{L}$ such that $[x_ix_j] \in (x_{i+1}, \cdots, x_n)$, then for each set $(f_1, \cdots, f_n)$ of irreducible polynomials in $K[X]$, there exists one and only one irreducible representation U of $\mathfrak{L}$ such that the minimum polynomial of U_{x_i} is a power of f_i, $i=1, \cdots, n$. Theorem (3) is an extension to arbitrary fields of prime characteristic of a theorem of Zassenhaus (Abh. Math. Sem. Hamburgischen Univ. vol. 13 (1939)). The ideal-theoretic properties of the universal associative algebra are exploited in the proofs of (2) and (3). (Received October 13, 1953.)

106. N. J. Divinsky: *Pseudo-regularity.*

An element x of a ring A is called right-pseudo-regular (r.p.r.) of degree n if there exists an element y such that $x+x^ny+x^{n+1}y=0$. A study is made of the case $n=1$, and the existence of a maximal r.p.r. ideal P and the existence of a maximal left p.r. ideal P_L are established. It is shown that in general they are different from each other and smaller than the Perlis-Jacobson radical R. Several radical-like properties are established and in the commutative case with descending chain conditions a definite relationship is obtained between P, P_L, and R. The techniques of Brown and McCoy are used to discover the structure of commutative rings modulo their sub-radicals (P). Finally the cases $n>1$ are studied. Use is made of an extension of McCoy's result on commutative subdirectly irreducible rings, namely, that every commutative subdirectly irreducible ring, all of whose elements are divisors of zero, is bound to its maximal nil ideal as well as to its Perlis-Jacobson radical (in the sense of Hall). (Received September 24, 1953.)

107t. S. I. Goldberg: *Lie algebra extensions and enlargements of modules.* II.

Let (E, K, ϕ, π) be a K-extension of the Lie algebra G by the G-module M with factor set g and (ϵ, K, ψ, μ) a K-enlargement of the G-module P by M. Then $\gamma\cdot\mu_m = \mu_{\gamma\cdot m}+\beta_\gamma(m)$, $\gamma\in G$, $m\in M$, $u_m\in\epsilon$, $\beta\in C^1(G, K, \mathrm{Hom}\,(M, P))$, $\delta\beta=0$ where the elements μ_m are the representatives corresponding to the isomorphism $\epsilon/P\approx M$. It is well known that the sequence $\cdots\to H^{r-1}(G, M)\xrightarrow{\Delta} H^r(G, P)\to H^r(G, \epsilon)\to H^r(G, M)\to\cdots$ is exact. The invariant coboundary Δ is described as follows: Let $\psi\colon \epsilon\to M$ be the given homomorphism of ϵ upon its quotient ϵ/P, and g be any cocycle in $C^{r-1}(G, M)$. Select representatives $\bar{g}(\gamma_1, \cdots, \gamma_{r-1})$ at random so that $\bar{g}$ is multilinear and $\psi\bar{g}(\gamma_1, \cdots, \gamma_{r-1})=g(\gamma_1, \cdots, \gamma_{r-1})$. Then $f=\delta\bar{g}$ has values in P and the mapping Δ is the one obtained by sending the cohomology class of g in $H^{r-1}(G, M)$ into that of f in $H^r(G, P)$. It is shown that a necessary and sufficient condition for the existence of a K-extension ϵ^* of G by ϵ with $\epsilon^*/P\approx E$ is $\Delta(g)=\delta\bar{g}$, $\bar{g}\in C^2(G, K, \epsilon)$, that is, $\Delta(g)$ is a relative coboundary modulo K. (Received October 13, 1953.)

108. Arno Jaeger: *Riccati's differential equation in fields of characteristic* 2.

While Riccati's differential equation in a field F of characteristic $p>2$ can always be solved in a quadratic extension of F (cf. F. Kasch, Arch. Math. vol. 4 (1953) pp. 17–22) the case $p=2$ presents a different situation: If D is a regular differentiation in the sense of A. Jaeger (Monatshefte für Mathematik vol. 56 (1952) p. 189) the differential equation (1) $R(y)=Dy+ay^2+by+c=0$ $(a, b, c\in F;\ a\neq 0)$ implies (2) $D(R(y))=bDy+(Da)y^2+(Db)y+Dc=0$. By elimination of Dy resp. y^2 from (1) and (2) one obtains an at most quadratic equation in y resp. a linear differential equation in y, and it has to be checked how far their common solutions solve (1), too. With the abbreviation $P(y)=by+Dy$ one obtains: (a) If $P(b)\neq 0$ or if $P(b)=P(a)=P(c)=0$ there exist two distinct solutions in a separable extension of F or in F. (b) If $P(b)=0$ and $P(a)\neq 0$ there is at most one solution, possibly in an inseparable, but D-admissible, extension of F, and the vanishing of a certain differential polynomial of order 3 in a, b, c is necessary and sufficient for solvability. (c) If $P(b)=P(a)=0$, but $P(c)\neq 0$, there is no solution. (Received November 28, 1953.)

109*t*. A. R. Schweitzer: *On the transition from Grassmann's extensive algebra to modern number theory.*

In his article, *Applications of Grassmann's extensive algebra* (Amer. J. Math. vol. 1, pp. 350–358) W. K. Clifford has developed a system of complex units. In his treatise, *Untersuchungen ueber die Summen von Quadraten* (Bonn, 1886) R. Lipschitz asserts (p. 2) that a system of units due to him and reported in C. R. Acad. Sci. Paris (October, 1880) agrees with Clifford's system. In this paper the author interprets Lipschitz's article in the Comptes Rendus as an introduction to the treatise mentioned above. In the latter treatise Lipschitz successively aims to determine all linear transformations with rational coefficients which leave invariant (1) the sum of two squares, (2) the sum of three squares, (3) the sum of n squares ($n>3$); see pp. 7, 54, 79. To attain his object Lipschitz uses respectively as a tool (1) Gauss's theory of integral complex numbers, (2) a theory of integral quaternions, (3) a theory of integral complex expressions of the nth order ($n>3$). In his treatise Lipschitz discusses (p. 26) a genesis of quaternions to which the author makes transition from his determination of the quaternion in Grassmann's extensive algebra (Math. Ann. vol. 69 (1910)). An interesting commentary on Lipschitz's theory of integral quaternions is due to L. E. Dickson, *Algebras and their arithmetics* (Chicago, 1923). (Received October 14, 1953.)

110. L. M. Weiner (p) and Pasquale Porcelli: *A derivation of Cauchy's inequality for polynomials.*

The following lemmas and theorem are established: Lemma 1: If $m-1$ is a positive integer, $r_1=1$, $r_2, \cdots, r_m$ the m mth roots of unity, p a positive integer less than $m+1$, s a positive integer less than m such that T_s denotes the sum of the products of the r_i, excluding r_p, taken s at a time, then T_s has absolute value 1. Lemma 2: Under the hypotheses of Lemma 1, $\prod_{i=1, i\neq p}^{m} |r_p-r_i| = m$. Theorem: If n is a positive integer, $\{a_p\}_{p=0}^{n}$ a sequence of numbers, and each of M and R a positive number such that $\left|\sum_{p=0}^{n} a_p z^p\right| < M$ for $z=Re^{2\pi i t/n+1}$ and $t=1, 2, \cdots, n+1$, then $|a_p| < M/R^p$ for $p=0, 1, \cdots, n$. (Received October 9, 1953.)

ANALYSIS

111. Nachman Aronszajn: *Linear functionals on proper functional spaces and reproducing kernels. Applications to Green's functions.*

Linear functionals $L(f)$ on a proper functional space $\mathcal{F}$ on a basic set $\mathcal{E}$ are always limits of finite linear combinations $L_S(f)=\sum \alpha_k f(x_k)$ for systems $S=\{x_k; \alpha_k\}$, $x_k\in\mathcal{E}$, α_k real or complex. If $\mathcal{F}$ is a Hilbert space, $K(x, y)$ its reproducing kernel, and L a linear functional given as limit of L_S, then a necessary and sufficient condition in order that $L(f)$ exists and be continuous on $\mathcal{F}$ is that $L_S^{(x)}L_S^{(y)}K(x, y)$ converge to a finite limit $L^{(x)}L^{(y)}K(x, y)$. If we have in $\mathcal{F}$ two equivalent quadratic norms, $\|f\|$ and $\|f\|_1$, the above property for the corresponding kernels K and K_1 will be simultaneously true or false. This allows us to prove regularity properties of Green's functions and their derivatives for a large class of differential problems. (Received October 15, 1953.)

112. F. E. Browder: *The eigenfunction expansion theorem for the general singular self-adjoint elliptic system of differential operators.*

Let D be a domain in Euclidean n-space (or in a sufficiently differentiable manifold of dimension n). Suppose that K is a formally self-adjoint system of r differential operators of order t acting on r-vectors and with coefficients defined on D. Suppose that on every compact subdomain of D, K is elliptic and suitably differentiable. Let K_0 be the symmetric operator in the r-vector $L^2(D)$ obtained by restricting K to the vector functions whose components are infinitely differentiable and vanish outside compact subsets of D. Suppose that K_1 is a self-adjoint extension of K_0. Then, it is shown that K_1 has an eigenfunction expansion in the sense defined by F. Mautner (Proc. Nat. Acad. Sci. U.S.A. (1953) pp. 49–53) and the properties of this expansion are discussed. The basic part of the proof rests in showing for Im $(c)\neq 0$, $k>n/2t$, that $(K_1-cI)^{-k}$ is a bounded system of integral operators of Carleman type. The theorem contains as a special case the results of Weyl, Kodaira, and others on eigenfunction expansions for systems of ordinary differential equations. (Received October 9, 1953.)

113. Gustave Choquet: *On a class of set-functions.*

The author studies the properties of positive and subadditive capacities defined on an additive and hereditary set of compact subsets of a space E. (1) Several conditions equivalent to the fact that the intersection of two capacitable sets is always capacitable are given: strict increase of f is, roughly speaking, the condition in order that this be verified. (2) Comparison of several classes of subsets of E, the approximable, perfectly capacitable and accessible subsets whose interest is their invariance under most of the ordinary operations: difference, union, and intersection (finite or denumerable). (3) Detailed study of the *simple capacities* f defined as follows: If ϕ is a continuous application of E into F measured by $\mu\geqq 0$, for every compact $K\subset E$, let $f(K)=\mu(\phi(K))$. When E and F are compact and metric, and f is of finite total variation, f can be considered as a superposition of a sequence of Radon measures, and there is identity between accessible, approximable, and perfectly capacitable subsets. (Received October 15, 1953.)

114. R. A. Gambill: *Criteria of stability and instability for linear systems of differential equations with periodic coefficients.*

Consider the differential system (A) $y_i''+\sigma_i^2y_i+\lambda\sum_{k=1}^{n}\psi_{ik}(x)y_k=0$, $i=1,\cdots,n$, where λ is a small real parameter, $\sigma_1,\cdots,\sigma_n$ are distinct positive numbers, and (1) $\psi_{ik}(x)$ are real periodic functions of period $T=2\pi/\omega$, of mean value zero; (2) $\psi_{ik}(x)$ have absolutely convergent Fourier series; (3) $m\omega\neq\sigma_i\pm\sigma_j$, $i, j=1,\cdots,n$; $m=0,1,\cdots$. The solution (B) $[y_i=0, i=1,\cdots,n]$ of (A) is said to be parametrically stable (Liapounoff) if, given $\epsilon>0$, there exists a $\delta>0$ such that $|\lambda|<\delta$, $|y_i(0,\lambda)|<\delta$ implies $|y_i(t,\lambda)|<\epsilon$ for all $t\geqq 0$ [or $t\leqq 0$]. If (4) $\|\psi_{ik}\|=\|a_{ik}\|+\|b_{ik}\|$, where $\|a_{ik}\|=\mathrm{diag}\,(A_1, A_2,\cdots,A_h)$, b_{ik} may be $\neq 0$ only if they are all below [or above] the matrices A_l; (5) for each $A_l=\|\psi_{pq}\|$, either A_l is symmetric or the functions ψ_{pq} are even, then (B) is parametrically stable (in both directions). In particular, conditions (4), (5) are satisfied if $\psi_{ik}=0$ for all $i<k$ [$i>k$]. This theorem is proved by means of a variant, studied by L. Cesari [Mem. Accad. Italia (4) vol. 11, pp. 634–692], of Poincaré's method of casting out the secular terms in the solution of (A).

For a given system (A), it may happen that the solution (B) is nonparametrically stable for every ω. Criteria are given for this behavior. These criteria state that if any of certain expressions $P, Q, \cdots$, depending on ψ_u and the numbers σ_i, are $\neq 0$, then necessarily this is the expected behavior. Thus these criteria show that this behavior is rather the rule. Condition (2) can be replaced by L-integrability in $(0, T)$ (J. K. Hale). (Received October 13, 1953.)

115*t*. P. R. Garabedian and Max Shiffman: *On the solution of partial differential equations by the Hahn-Banach theorem.*

The following construction of the Green's function and the Neumann's function for the elliptic partial differential equation $\Delta u - pu = 0$ in a domain D of n-dimensional space is developed, where p is a positive, suitably regular function. Consider the class of functions v in D which vanish on the boundary of D and have continuous second derivatives in D. For z in D, let L denote the linear functional defined by the formula $v(z) = L[\Delta v - pv]$ over the space of functions of the form $\Delta v - pv$. By the maximum principle, the functional L is bounded, and it can therefore be extended to the whole space of continuous functions in D. Let $S(z, w)$ be a parametrix for $\Delta u - pu = 0$ which vanishes on the boundary of D. It is shown that $G = S - L[\Delta S - pS]$ is the desired Green's function. This existence proof has broader applications than the earlier one due to Lax (Proc. Amer. Math. Soc. vol. 3 (1952) pp. 526–531) because only a parametrix, and not a fundamental solution, is required in the construction. (Received October 5, 1953.)

116. A. W. Goodman: *On regions omitted by univalent functions.* II.

Let S denote the family of functions $f(z) = z + a_2z^2 + \cdots$ regular and univalent in $|z| < 1$, let A_f denote the area of the intersection of the circle $|w| < 1$, and the image of $|z| < 1$ under $f(z)$, and let $A = \inf_{f \in S} \{A_f\}$. Jenkins (Amer. J. Math. vol. 75 (1953) pp. 406–408) has proved that $A > .5387\pi$. With a slight refinement of Jenkins' methods, it is possible to prove that $A > .6023\pi$. It is known that $A < .7728\pi$. (Received August 3, 1953.)

117*t*. L. I. Gordon: *Fractional anti Laplacians.* Preliminary report.

Consider functions $f(p) = f(x, y)$ of two real variables, periodic (say, of period 1) in x and y. By L^r, denote the class of functions f such that $|f|^r$ is integrable over the unit square. Then for $\beta > 0$, $\sum'_{m,n} (m^2+n^2)^{-\beta/2}(2\pi i)^{-\beta}e^{2\pi i(mx+ny)}$ is the Fourier series of a function $\psi_\beta(x, y) \in L(= L^1)$. Let $f \in L$, $f \sim \sum_{m,n} c_{mn}e^{2\pi i(mx+ny)}$, and let $I_\beta f$ denote the convolution of f with ψ_β. Then $I_\beta f \sim \sum_{m,n} (m^2+n^2)^{-\beta/2}(2\pi i)^{-\beta}c_{mn}e^{2\pi i(mx+ny)}$ and one has $I_\alpha I_\beta f = I_{\alpha+\beta} f$. For sufficiently smooth f, $\sum_{m,n} (m^2+n^2)^{\beta/2}(2\pi i)^{\beta}c_{mn}e^{2\pi i(mx+ny)}$ is the Fourier series of a function in L which we denote by $I^\beta f$ and, automatically, $I_\beta I^\beta f = I^\beta I_\beta f = f$. Thus if $\Delta(I_{2-\beta}f)$ is continuous, where $0 \leqq \beta \leqq 2$ and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, then $f^\beta = \Delta(I_{2-\beta}f)$. As is the case with fractional integration of functions of one variable, there are relations between the function classes to which f, $I_\beta f$, and $I^\beta f$ belong. For example: (i) If $f = f(x, y) \in L^p$, $p > 1$, $\beta < 2/p$, then $I_\beta f \in L^q$ where $1/p - 1/q = \beta/2$. If $p \geqq 1$ and $2/p < \beta < 2/p + 1$, then $I_\beta f \in \text{Lip}\ (\beta - 2/p)$. (ii) If $f \in \text{Lip}\ \alpha$, $0 < \alpha < 1$, then $I_\beta f \in \text{Lip}\ (\alpha + \beta)$ for $\alpha + \beta < 1$ and $I^\beta f \in \text{Lip}\ (\alpha - \beta)$ for $0 < \beta < \alpha$. (Received September 18,1953.)

118. J. K. Hale: *Periodic solutions of systems of nonlinear differential equations.*

Consider systems of the form (1) $\ddot{x}_j + a_j\dot{x}_j + \sigma_j^2 x_j = \epsilon q_j(x, \dot{x}; \epsilon)$, $j = 1, 2, \cdots, n$, where $\sigma_j>0$, $a_j\geqq 0$, ϵ is a small real parameter, $x=(x_1, \cdots, x_n)$, $\dot{x}=(\dot{x}_1, \cdots, \dot{x}_n)$, and q_j are analytic functions for $|\epsilon|<\epsilon_0$, $|x_j|<A$, $|\dot{x}_j|<A$. A variant of Poincaré's method of casting out the secular terms is studied and the convergence is proven. If $a_k=0$ for some k, and the equation (2) $\int_0^{2\pi} q_k(B\sigma_k^{-1}\sin t, B\cos t, 0, \cdots, 0; 0)\cos t\,dt=0$ has a simple root for some $B=b$, then there is a periodic solution (limit cycle) to (1) of the form C_k: $x_k=B_0\sigma_k^{-1}\cos(\tau t+\phi)+\epsilon W_k(\tau t+\phi;\epsilon)$, $x_j=\epsilon W_j(\tau t+\phi;\epsilon)$, $j\neq k$, $j=1, 2, \cdots, n$, where φ is an arbitrary constant and all W_j are analytic functions of ϵ with coefficients periodic in t, and $\tau=\sigma_k+O(\epsilon)$, $B_0=b+O(\epsilon)$ are constants. Thus, the limit cycle C_k is characterized by a frequency τ close to σ_k, and by the fact that x_k may be large and $x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n$ are small. In particular, for systems (3) $\ddot{x}_j+\sigma_j^2x_j=\epsilon q_j(x, \dot{x};\epsilon)$, we have n analogous limit cycles C_j. If we suppose in (1) that $a_k=0$ and $q_k(x, \dot{x};\epsilon)=f(x_k)\dot{x}_k+g(x_k, \dot{x}_k)+h(x, \dot{x})+\epsilon q^*(x, \dot{x};\epsilon)$, where $f(x_k)=a_0+a_1x_k^2+\cdots+a_mx_k^{2m}$, $a_0a_m<0$, $g(-x_k, \dot{x}_k)=-g(x_k, \dot{x}_k)$, $h(x_k, \dot{x}_k, 0, \cdots, 0)=0$, then (2) certainly has a root $B=b$. Furthermore, if one considers (3) with $q_j(x, -\dot{x};\epsilon)=q_j(x, \dot{x};\epsilon)$, then there is a periodic solution C_k to (3) for every B, $|B|<A$ and for $k=1, 2, \cdots, n$ (n two-dimensional manifolds of cycles). (Received October 13, 1953.)

119*t*. J. K. Hale: *Systems of linear differential equations with L-integrable periodic coefficients.*

Consider the system (1) $d^2y_i/dx^2+\sigma_i^2y_i+\lambda\sum_{k=1}^{n}\phi_{ik}(x)y_k=0$, $i=1, 2, \cdots, n$, where $\sigma_1, \sigma_2, \cdots, \sigma_n$ are distinct real positive numbers, λ is a small real parameter, the $\phi_{ik}(x)$ are real functions, periodic of period $T=2\pi/\omega$, $\int_0^T\phi_{ik}(x)dx=0$, and possessing absolutely convergent Fourier series. By using a variant of Poincaré's method of casting out the secular terms in the solutions of the above system, L. Cesari (Mem. Accad. Italia (6) vol. 6, pp. 634–692) proved the following result: If $m\omega\neq\sigma_i\pm\sigma_k$, $i, k=1, 2, \cdots, n$, $m=0, \pm 1, \pm 2, \cdots$, and if either $\phi_{ik}(x)=\phi_{ik}(-x)$ or $\phi_{ik}(x)=\phi_{ki}(x)$, $i, k=1, 2, \cdots, n$, then for λ sufficiently small, the solutions of (1) are bounded. In the present study, the same result is obtained by assuming only that the $\phi_{ik}(x)$ are periodic, L-integrable, and $\int_0^T\phi_{ik}(x)dx=0$. A concept of mean value $m[f]$ is discussed for functions of the class C_ω of functions $f=\sum_{i=1}^{h}e^{\alpha_i x}\phi_i(x)$, with the α_i complex constants and the $\phi_i(x)$ of period T, L-integrable in $[0, T]$. Various theorems are proven concerning $m[f]$ and the primitives $F(x)$ of the functions $f\in C_\omega$. In particular, a function $f\in C_\omega$ has a primitive $F\in C_\omega$ if and only if $m[f]=0$. These theorems are applied in the discussion of the problem above and particularly in the proof of the convergence of the mentioned method of successive approximations under the new conditions. (Received October 13, 1953.)

120*t*. Israel Halperin: *Reflexitivity in function spaces.*

That $L^p(B)$ is reflexive if $1<p<\infty$ and B is reflexive has been shown by R. S. Phillips, *On weakly compact subsets of a Banach space*, Amer. J. Math. vol. 65 (1943) pp. 108–136. The $L^p(B)$ are a special, nontypical case of the $L^\lambda(B)$ spaces discussed in a paper by H. W. Ellis and Israel Halperin, *Function spaces determined by a levelling length function*, to appear in the Canadian Journal of Mathematics vol. 5 (1953). It is now shown that $L^\lambda(B)$ is reflexive if and only if both L^λ and B are reflexive. (Received September 21, 1953.)

121. W. B. Jurkat: *A function theoretic inclusion theorem for Nörlund means.*

Let $N(p_n)$ and $N(q_n)$ be two Nörlund methods with the corresponding power series $p(x)=\sum p_n x^n$, $q(x)=\sum q_n x^n$. For the inclusion $N(p_n)\subseteq N(q_n)$ exact conditions are stated depending only on the singularities of the function $q(x)/p(x)=k(x)=\sum k_n x^n$: Suppose that $k(x)$ is regular and $\neq 0$ for $x=0$ and that to each point $\xi\neq 0$ of $|x|\leqq 1$ there exists a real number α such that $k(x)(1-x/\xi)^\alpha$ is regular and $\neq 0$ at the point $x=\xi$. Then the following conditions are necessary and sufficient for the inclusion $N(p_n)\subseteq N(q_n)$ with regular $N(p_n)$: (i) $k(x)$ must be regular for $|x|<1$; (ii) the exponent α_0 corresponding to $\xi_0=1$ must be not negative; (iii) if there are points $\xi_\mu\neq 1$, $|\xi_\mu|=1$, for which the corresponding α_μ are positive, then $\alpha_0\geqq\alpha_\mu$ must hold for $\mu=1,\cdots,m$. (Points with $\alpha\leqq 0$ are not essential.) By considering the case $p(x)=1$ one sees that (i), (ii), (iii) are at the same time the exact conditions for regularity of $N(k_n)$. (Received October 16, 1953.)

122t. W. B. Jurkat: *Questions of signs in power series.*

In extending the well known theorem of Kaluza and Szegö concerning the signs in a reciprocal power series there are obtained some results on the distribution of signs in the quotient $k(x)=\sum k_n x^n$ of the two power series $q(x)=\sum q_n x^n$ and $p(x)=\sum p_n x^n$. Let us set formally $k(x)=q(x)/p(x)$ and $q_0=p_0=1$. Then it is sufficient for $k_n\geqq 0$ resp. $k_n\leqq 0$ for all $n\geqq 1$, that the two following conditions are satisfied: $0<p_{n+1}/p_n\nearrow$, $q_n/p_n\nearrow$ resp. $\searrow$. Another but simpler pair of sufficient conditions is $p_n\searrow$, $q_n-p_n\nearrow$ resp. $\searrow$. There are some more general results using the differences of higher order of p_n and q_n. (Received November 2, 1953.)

123t. Konrad Knopp: *On the proof of the main Tauberian theorem for the C_k- and H_k-methods.*

For the now classical theorem "*If a_ν are real, if $\sum a_\nu$ is summable C_k or H_k to s* $(k>0)$, *and if $\nu a_\nu\leqq M$, then $\sum a_\nu=s$*" an inductive proof is given which is, for general k, of simpler structure than the previous proofs. The main difference from these lies in the fact that the series-to-series transform is used instead of the sequence-to-sequence transform. The proof is especially simple for the H_k-methods. A remark of Dr. Jurkat is added with respect to a corresponding theorem for general iterated methods. (Received November 3, 1953.)

124t. Jacob Korevaar: *A very general form of Littlewood's theorem.*

Let $f(u)$ denote the sum of a power series $\sum a_n e^{-nu}$ which is convergent for $u>0$. Littlewood and Hardy have shown that the conditions $a_n>-K/n$ $(n=1, 2, \cdots)$ and $f(u)\to s$ $(u\downarrow 0)$ together imply that $s_n=\sum_{k\leq n} a_k\to s$ $(n\to\infty)$. Now let $L(t)$ denote a slowly oscillating function, that is, a positive continuous function defined for $t>0$ for which $L(ct)/L(t)\to 1$ as $t\to\infty$ for every $c>0$. Let α be real, and denote $t^\alpha L(t)$ by $\phi(t)$. Let $\omega(u)$ be defined, positive and bounded for $u>0$, while $\omega(u)\downarrow 0$ as $u\downarrow 0$. The following general Tauberian theorem is proved. The conditions $a_n>-\phi(n)$ $(n=1, 2, \cdots)$ and $|f(u)-s|<\omega(u)$ $(u>0)$ together imply that $|s_n-s|\leqq\rho(n)$ $(n=0, 1, \cdots)$, where $\rho(0)=K_1$, $\rho(n)=\min_{p=1,2,\dots}\{K_2 n\phi(n)/p+K_3^p\omega(p/n)\}$ $(n\geqq 1)$ if $\liminf_{u\downarrow 0} u\log\omega(u)>-\infty$, and $\rho(n)=0$ $(n=0, 1, \cdots)$ if $\liminf_{u\downarrow 0} u\log\omega(u)=-\infty$. Here the numbers K_i depend only on the functions ϕ and ω $(K_3>1)$. It is shown by various examples that the above estimates are essentially best possible. For example, if $\omega(u)=\exp(-1/u)$, then $a_n>-o(n^{-1/2})$ implies convergence of $\sum a_n$. However, the function $f(\log 1/x)=\int_0^x(1-x)^{-2}\{1-2x+2x^4-2x^9+\cdots\}dx$ is $O\{\exp(-1/(1-x))\}$ as $x\uparrow 1$ while the corresponding series $\sum a_n$ for which a_n

$=O(n^{-1/2})$ is easily seen to diverge. The above results improve and complete earlier results of the author [Duke Math. J. vol. 18 (1951) pp. 723–734; Proc. Nederl. Akad. Wetensch. Ser. A vol. 56 (1953) pp. 281–293]. (Received September 28, 1953.)

125. Jacob Korevaar: *Another numerical Tauberian theorem for power series.*

The power series associated with the series $\sum a_n$: $(0)+1-1/2+1/3-1/4+\cdots$ is $f(u)=\sum(-1)^{n-1}n^{-1}e^{-nu}=\log(1+e^{-u})$. Clearly $|f(u)-\log 2|<u$ $(u>0)$, and $a_n\geqq -1/n$. Thus the Tauberian theorem of the preceding paper gives the estimate $|s_n-\log 2|\leqq C/\log n$ $(n\geqq 2)$. A much stronger result, however, is known in this case; one actually has $|s_n-\log 2|\leqq C/n$ $(n\geqq 1)$. A similar stronger result holds whenever $f(u)=\sum a_n e^{-nu}$ (where the series is assumed to converge for $u>0$) is analytic at $u=0$. This hypothesis together with the condition $a_n\geqq-\phi(n)=-n^{\alpha}L(n)$, $L(n)$ a slowly oscillating function, implies that $|s_n-f(0)|\leqq C\phi(n)$ $(n\geqq 1)$. A more general result is the following. Let the series $f(u)=\sum a_n e^{-nu}$ converge for $u>0$, and let there be a function $g(u)$ analytic at $u=0$ such that $|f(u)-g(u)|\leqq\omega(u)$ on some interval $0<u<\delta$, where $\omega(u)\downarrow 0$ as $u\downarrow 0$. Let $a_n\geqq-\phi(n)$. Then $|s_n-g(0)|\leqq\rho(n)=\min_p\{C_1n\phi(n)/p+C_2^p\omega(p/n)\}$ $(n\geqq n_0)$, where the minimum is taken over all p satisfying $C_3\leqq p\leqq C_4n$. In case $\liminf_{u\downarrow 0} u\log\omega(u)=-\infty$ one has the stronger conclusion $f(u)\equiv g(u)$, hence $|s_n-g(0)|\leqq C_5\phi(n)$. The proofs of these results depend upon a further refinement of the theory of best L_1 approximation from one side by polynomials of given degree as developed by Geza Freud and by the author. (Received November 13, 1953.)

126. M. Z. Krzywoblocki: *On some peculiarities of partial differential systems in two independent variables.*

By means of a non-one-to-one transformation a system of partial differential equations in two independent variables is transformed onto an ordinary differential system with a parameter. The following lemma was proved: Each of all the curves lying in the intersections of each of all the integral surfaces of a system of nonlinear partial differential equations of the kth order in two independent variables with the planes perpendicular to x- (or y-) axis represents a singular invariant solution of the corresponding differential form of this system, which solution is invariant under the k-enlargement of some one-parameter continuous group of transformations with one pair of independent variables being variables and the other being parameters. These curves represent the peculiar solutions of the system in question and they are some extremities on the integral surfaces. (Received October 9, 1953.)

127. R. E. Lane: *The integral of a function with respect to a function.* II.

Infinite subdivisions (and refinements of infinite subdivisions) are used with the trapezoidal formula to generalize the author's previous definition of $\int_a^b u(x)\,dv(x)$, still retaining integration by parts and other properties of the Stieltjes integral. An example is given in which each of u and v is totally discontinuous. If u is bounded and Lebesgue integrable and $v(x)=x$ in $[a,b]$, then $\int_a^b u(x)dv(x)$ is equal to the Lebesgue integral. (Received October 13, 1953.)

128. Norman Levinson and J. J. Levin (p): *Singular perturbations of nonlinear systems of differential equations and an associated boundary layer equation.*

The relationship of the solutions of a certain "full" system of differential equations (1) as $\epsilon \to 0+$, to a particular solution of a related "degenerate" system (2) is investigated. The systems are: (1) $dx/dt = f(x, y, t, \epsilon)$, $\epsilon^r dy/dt = g(x, y, t, \epsilon)$; (2) $dx/dt = f(x, y, t, 0)$, $0 = g(x, y, t, 0)$; where x and f are real vectors of dimension $n_0 \geqq 1$, y and g are real vectors of dimension $n \geqq 1$, $\epsilon > 0$, and $r > 0$ and fixed. Systems containing higher powers of ϵ and more dependent variables are also investigated. A "boundary layer equation" related to (1) is discussed. The results generalize those of Tihonov (Mat. Sbornik N.S. vol. 27 (1950) pp. 147–156). (Received October 13, 1953.)

129. G. G. Lorentz: *The Borel property of summability methods.*

For a function $\Omega(n)$ tending to $+\infty$ with n, and a regular summability method A, the following statements are equivalent: (1) Each sequence s_n with $s_1 + \cdots + s_n = o(\Omega(n))$ is A-summable; (2) for each convergent series $\sum c_n$, A-lim $c_n\Omega(n) = 0$; (3) $R(\lambda_n, 1) \subset A$, where $\lambda_n = \exp(\Omega(1)^{-1} + \cdots + \Omega(n)^{-1})$. For this, $\sum_\nu \Omega(\nu)\left|\Delta_\nu a_{n\nu}\right| = O(1)$ is necessary and sufficient. It is written then $\Omega(n) \in S_1(A)$. Such Ω were discussed by the author [Canadian Journal of Mathematics vol. 1 (1949) pp. 305–319; vol. 3 (1951) pp. 236–256]. They are applied here in the following way: A has the Borel property if there is an $\Omega \in S_1$ with $\sum \exp(-\delta\Omega(n)) < +\infty$ for each $\delta > 0$. For some special methods, necessary and sufficient conditions are given. Also, A has the "Banach property" if A-lim $\Phi_n(x) = 0$ a.e. for each orthonormal system $\Phi_n(x)$. Sufficient for this is that there exists an $\Omega \in S_1(A)$ with $\sum \Omega(n)^{-1} \log^2 n < +\infty$. (Received October 14, 1953.)

130*t*. K. S. Miller and L. A. Zadeh: *On an integral equation occurring in the theory of prediction.*

In the theory of prediction one frequently encounters the integral equation $f(t) = \int_0^T R(t, \zeta)g(\zeta)d\zeta$ where t ranges in the interval $[0, T]$ and the kernel R is of the form $R(t, x) = \int_{-\infty}^{\infty} M_\zeta \overline{H}(t, \zeta) M_\zeta \overline{H}(x, \zeta)d\zeta$. Here M_ζ is a linear differential operator operating on the variable ζ and $\overline{H}$ is essentially the one-sided Green's function of some other linear differential operator. A solution of the integral equation is obtained and the results discussed and interpreted in terms of prediction theory. (Received September 29, 1953.)

131*t*. V. L. Shapiro: *Cantor-type uniqueness of multiple trigonometric integrals.*

Let $c(u)$ be a complex-valued function integrable on every bounded domain in n-dimensional euclidean space E_n, and for which $c(u)|u|^{-2}$ is in L_1 on E_n. Suppose that the integral $\int_{E_n} e^{ix \cdot u}c(u)du$ is spherically summable $(C, 1)$ to zero almost everywhere and that the $(C, 1)$ spherical mean of this integral of rank R, $\sigma_R^{(1)}(x)$, is such that $\limsup_{R\to\infty} \left|\sigma_R^{(1)}(x)\right| < \infty$ in $E_n - Z$ where Z is a closed set of vanishing capacity. It is then shown in this paper that $c(u)$ is equal to zero almost everywhere. The proof is brought about by considering generalized Laplacians of the first kind in connection with Riemann summability of trigonometric integrals. Further results of the above kind are obtained when $c(u)|u|^{-2}$ is in L_2 on E_n and almost everywhere $(C, 1)$ summability is replaced by summability in $E_n - Z$. In particular for the plane, it is shown that if $c(u) = O(u^{1-\epsilon})$, $\epsilon > 0$, is integrable on every bounded domain, if $\int_{E_2} e^{ix\cdot u}c(u)du$ is spherically summable $(C, 1)$ to zero in $E_2 - Z$, and if $\int_{E_2 - D_2(0,1)} e^{ix\cdot u}c(u)u^{-2}du$ is spherically convergent to a continuous function in $E_2 - W$ where W is a closed denumerable set such that $ZW=0$, then $c(u)$ is equal to zero almost everywhere. (Received October 13, 1953.)

132*t*. H. S. Wall: *Concerning harmonic matrices.*

The statement that the $n\times n$ matrix M is a *harmonic* matrix means that M_{ij} is a complex-valued function from the sensed pairs (x, y) of real numbers which, for each number y (x), is continuous and of bounded variation in x (y) on every interval, and, for each x, y, z, $M(x, y)M(y, z)=M(x, z)$ and $M(x, x)=I$. Theorem A. The $n\times n$ matrix M is harmonic if and only if there exists an $n\times n$ matrix F of complex-valued functions from the real numbers, continuous and of bounded variation in every interval, such that $F(0)=0$ and $M(x, y)=I+\int_y^x dF(s)\cdot M(s, y)$. This correspondence $M\leftrightarrow F$ is reversible, $M_{ij}(x, y)$ is continuous in (x, y) and $M(x, y)=I+\int_y^x M(x, s)\cdot dF(s)$. Theorem B. If $n=2$, det $M(x, y)=1$ if and only if $F_{11}=-F_{22}$. Theorem C. Suppose f is an n-rowed column vector of functions and z an n-rowed column vector of numbers. The equation $f(x)=z+\int_y^x dF(s)\cdot f(s)$ has the unique solution $f(x)=M(x, y)\cdot z$. If $n=2$, the quotient f_1/f_2 is the unique solution w of the Stieltjes integral equation $w(x)-w(y)=\int_y^x w^2 dF_{21}+\int_y^x w\,d(F_{22}-F_{11})-\int_y^x dF_{12}$. (Received August 28, 1953.)

133. Daniel Waterman: *On an integral of Littlewood and Paley.*

Littlewood and Paley defined $g(\theta)=\{\int_0^1(1-\rho)\left|\phi'(\rho e^{i\theta})\right|^2 d\rho\}^{1/2}$ for $\phi(z)$ of class H_p, $p\geqq 0$. They showed that $\|g(\theta)\|_p\leqq A_p\|\phi\|_p$. For $p\geqq 1$ they showed the converse result, $\|\phi\|_p\leqq A_p\|g\|_p$. Here both results are shown to hold for the Hille-Tamarkin class $\mathfrak{H}_p$ in the half-plane, with the definition $g(\tau)=\{\int_0^\infty\sigma\left|\phi'(\sigma+i\tau)\right| d\sigma\}^{1/2}$, superseding the result $\|g(\tau)\|_p\leqq A_p\|\phi\|_p$ previously given by this author for the definition $g(\tau)=\{\int_0^\infty(1-e^{-\sigma})\left|\phi'(\sigma+i\tau)\right|^2 d\sigma\}^{1/2}$. (Received October 13, 1953.)

134. F. M. Wright: *A transformation for S-fractions.*

It is proved that, if the formal power series $Q(w)=\sum_{n=0}^\infty\mu_n w^{n+1}$ and $P(w)=\sum_{n=0}^\infty\lambda_n w^{n+1}$ are both the corresponding power series of *S*-fractions, then $P(w)=\lambda_0 w+wQ(w)$ if and only if there is a set of parameters in terms of which the *S*-fraction expansions of $Q(w)$ and $P(w)$ have certain prescribed forms. This result is then used to obtain results relative to the backward extension of Stieltjes and Hausdorff moment sequences. Many of the results developed relative to the backward extension of a given Stieltjes moment sequence $\{\mu_n\}$ such that the *S*-fraction expansion of the formal power series $\sum_{n=0}^\infty\mu_n w^{n+1}$ is nonterminating have been proved previously by H. S, Wall [Trans. Amer. Math. Soc. vol. 31 (1929) pp. 91–116], but it is believed that the present development, which avoids the use of determinants, is simpler and more direct. (Received October 1, 1953.)

135*t*. R. E. Zink: *Direct unions of measure spaces.*

Let $\{(Y_x, \Sigma_x, \nu_x)\}$ be a family of measure spaces such that the Y_x are pairwise disjoint point sets and such that the parameter x ranges over the set X of the measure space (X, Σ, μ). Then if (X, Σ, μ) is a σ-finite measure space, and if the function $\nu_x(Y_x)$ is defined for all x, finite valued, and measurable (Σ), it is possible to construct a σ-ring of subsets of the space $S=\bigcup_{x\in X}Y_x$ and a measure, τ, on this σ-ring. Furthermore the resulting measure space, which is called the direct union of (Y_x, Σ_x, ν_x) with respect to (X, Σ, μ), has the property that if E is an arbitrary measurable set, E_x belongs to Σ_x for almost all x, the function $\nu_x(E\cap Y_x)$ is, when arbitrarily defined on the remaining null set, measurable (Σ) and $\tau(E)=\int\nu_x(E\cap Y_x)d\mu(x)$. The method of construction is also applicable to the cartesian product of two measure spaces and it is possible by this technique to establish an extension of the Lebesgue planar measure. In connection with the above, a theorem on integration in spaces which are related to direct unions of measure spaces can be established. (Received October 13, 1953.)

APPLIED MATHEMATICS

136. R. L. Evans: *REAC solution in the large of a linear differential equation.*

Under Office of Ordnance Research Contract DA-11-022-ORD-489 the differential equation (1) $0=y^{(2)}-y(3/8\alpha^3+1/\alpha^2r^{11}-11/8\alpha^2r^8-(2\lambda+1)/2\alpha+k(k+1)/r^2)$ was studied. Eigenvalues were sought which would lead to solutions of (1) $\ni y(0)=y(+\infty)=0$. Since 0 and ∞ are both irregular singular points of (1) this involves solution in the large—the interrelation of solutions about different singular points. The case in which $k=1$ and $\alpha=.01$ was treated on the Reeves Electronic Analog Computer (REAC) as follows. Let $y_1(1)=1$, $y_1'(1)=0$, $y_2(1)=0$, $y_2'(1)=1$, $\lim_{r\to+\infty}(y_3\ \&\ y_3')=0$, and $\lim_{r\to+\infty}(y_4\ \&\ y_4')\neq 0$; and let (2) $R_1(\lambda)\cdot y_1+y_2=(\text{multiple of } y_3)$ define $R_1(\lambda)$. Let $r=1/x$, $y(r)=Y(x)$, $\lim_{x\to+\infty}(Y_1\ \&\ Y_1')=0$, $\lim_{x\to+\infty}(Y_2\ \&\ Y_2')\neq 0$, and let (3) $R_2(\lambda)\cdot y_1+y_2=(\text{multiple of } Y_1)$ define $R_2(\lambda)$. Desired λ's are $\ni R_1(\lambda)=R_2(\lambda)$. In (2) (or (3)) a term in y_4 (or Y_2) is also present except when R_1 (or R_2) is $\ni$ the extra term changes sign. These values are determined by trial and error on the REAC but are continuous functions of λ. Values for this case are: $(3/8-(2\lambda+1)/200)$: (.2, .05, .01); R_1: (−2.26, −4.58, −11.36); and R_2: (−.077, −.92, −2.15) in corresponding sequence. The REAC method is the best exploratory one but may need supplementing at near-critical eigenvalues (small values in the first sequence above). Also, for the case treated here no satisfactory eigenvalue was found. The method can be generalized. (Received October 15, 1953.)

137*t*. H. E. Salzer: *Osculatory quadrature formulas.*

By integrating $f(x)$ in Hermite's n-point osculatory interpolation formula (1) $f(x_0+ph)=\sum_{i=-[(n-1)/2]}^{[n/2]}\{L_i^{(n)}(p)\}^2\{1-2L_i^{(n)\prime}(i)(p-i)\}f_i+\sum_{i=-[(n-1)/2]}^{[n/2]}\{L_i^{(n)}(p)\}^2\cdot(p-i)hf_i'+R_{2n}(p)$, where $x=x_0+ph$, $x_i=x_0+ih$, $f(x_i)=f_i$, $f'(x_i)=f_i'$ and where (2) $L_i^{(n)}(p)=\prod'^{[n/2]}_{j=-[(n-1)/2]}\{(p-j)/(i-j)\}$, $j=i$ absent from $\prod'$, and (3) $R_{2n}(p)=f^{(2n)}(\xi)h^{2n}\{\prod_{j=-[(n-1)/2]}^{[n/2]}(p-j)\}^2/(2n)!$, $x_{-[(n-1)/2]}\leqq\xi\leqq x_{[n/2]}$, one obtains an extremely accurate series of integration formulas which utilize both the integrand and its first derivative at n fixed equally spaced points. They are of the form (4) $\int_{x_0+rh}^{x_0+sh}f(x)dx=\sum_{i=-[(n-1)/2]}^{[n/2]}\{hA_if_i+h^2B_if_i'\}+R$. These n-point "osculatory quadrature formulas" are not only exact when $f(x)$ is a polynomial of degree $\leqq 2n-1$, but can also be shown to be much more accurate than a $2n$-point formula which is obtained by integrating Lagrange's formula and which employs the integrand alone at equally spaced arguments at intervals of h. The exact fractional values of A_i and B_i are tabulated for $n=2(1)7$ and for every combination of (r, s) within the range $(-[(n-1)/2], [n/2])$. A supplementary schedule tabulates the exact values of the coefficients $C(n, r, s)$ in the remainder $R=C(n, r, s)h^{2n+1}f^{(2n)}(\xi_1)$, $x_{-[(n-1)/2]}\leqq\xi_1\leqq x_{[n/2]}$, which holds whenever $f^{(2n)}(\xi)$ in (3) is a continuous function of p. The coefficients A_i and B_i are shown to satisfy $A_{-[(n-1)/2]+j}$, $B_{-[(n-1)/2]+j}$ for interval $(-[(n-1)/2]+k, -[(n-1)/2]+k+m)=A_{[n/2]-j}$, $-B_{[n/2]-j}$, resp. for interval $([n/2]-k-m, [n/2]-k)$ for every n, $1\leqq m\leqq n-1$, $j=0, 1, \cdots, n-1$. (Received October 2, 1953.)

GEOMETRY

138. R. B. Deal: *Union differentiation and union correspondence.*

Recently, T. K. Pan has stated in an abstract certain results for what he terms *relative geodesics*. It is shown in the present work that his relative geodesics are union curves and that what he refers to as *relative first curvature* is what Springer and other authors have called union curvature. Some additional generalizations of well known results for geodesics are obtained for union curves. For instance, the limit of the quotient of the angle between the union curves tangent to a given curve at two points and the arc length between these points is shown to be the union curvature of the curve. Generalized intrinsic derivatives are defined in two different manners. Properties of each are studied. A generalization of the theorem stating that the divergence of the unit normal to a curve in a surface is the negative of the geodesic curvature is obtained for union curvature. Two theorems are obtained for union correspondence. The first gives a set of necessary and sufficient conditions for two surfaces to be in union correspondence, whereas the second theorem extends the analysis of the first in order to find conditions on the congruence which allow the existence of a second surface whose geodesics are in union correspondence with union curves on the first surface. (Received October 14, 1953.)

139. P. C. Hammer: *Maximal convex sets.*

Let L be an arbitrary linear vector space over the real or complex numbers. A set S is a *semispace* at the origin θ of L if and only if (a) S does not contain θ, (b) S is convex, (c) S contains one of the open half-lines determined by θ on every line through θ. All semispaces are obtained by translations. The semispaces with L form the minimal intersection basis for all convex sets in L. If C_0 and C_1 are nonvacuous disjoint convex sets in L, then there exists a dichotomy of L into convex sets H_0, H_1, such that $H_0 \supset C_0$ and $H_1 \supset C_1$. Every such H_0 is an intersection set of a class of translates of a semispace. In degenerate cases the boundary of a semispace is a hyperplane. (Received September 24, 1953.)

LOGIC AND FOUNDATIONS

140. J. C. E. Dekker: *A theorem on hypersimple sets.*

This paper gives an affirmative answer to the following question raised by Post: "Can a creative set be Turing reducible to a hypersimple set?" Let E be the class of all recursive sets and F the class of all recursively enumerable sets. "α t-red β" denotes: α is reducible to β by truth tables; "α red β" denotes: α is Turing reducible to β. Two sets are called *Turing equivalent* if they are Turing reducible to each other. *Theorem.* For every set $\alpha \in F-E$ one can effectively find a set β such that: (a) β is hypersimple, (b) α red β and β red α. *Corollary* 1. There exist sets σ and τ such that σ red τ is true, but σ t-red τ is false. *Corollary* 2. There exist two sets in $F-E$ which are not Turing equivalent if and only if there exist two hypersimple sets which are not Turing equivalent. The proof of part (a) does not use the fact that hypersimple sets exist. As a side result a new proof for the existence of a hypersimple set is therefore obtained. (Received October 13, 1953.)

141*t*. David Nelson: *Contraposition with strong negation.* Preliminary report.

A predicate calculus and arithmetic are presented which make a further separation of constructible concepts beyond that of P-realizable systems (J. Symbolic Logic vol. 14 (1949) pp. 16–26). A truth definition is given. A formula $A\supset B$ is $\mathfrak{P}$-realizable just in case there exist a pair of partial recursive functions ϕ and ψ such that, if a $\mathfrak{P}$-realizes A, then $\phi(a)$ $\mathfrak{P}$-realizes B, and if b $\mathfrak{P}$-realizes B, then $\psi(b)$ $\mathfrak{P}$-realizes A. Other clauses are as for P-realizability. A predicate calculus shown to satisfy the definition is obtained from Schütte's K_3 (Math. Ann. vol. 122 (1950) pp. 47–65) by replacing "Zusammenziehungen" by the weak rule "$A\supset(A\supset B)$ is deducible from $A\supset(A\supset(A\supset B))$," adding an inference rule and axioms (including contraposition) for strong negation, and a modification of the right hand V and $\exists$ introduction rules. The arithmetic system replaces the induction schema by a rule of inference. The nonprovability of certain formulas is established. The new systems enjoy symmetries not found in the previously discussed constructive ones. A further paper in preparation establishes a kind of extensionality and discusses deductive extent of the systems. (Received October 13, 1953.)

Statistics and Probability

142. H. D. Brunk: *The strong law of large numbers for asymptotically almost periodic stochastic processes.* Preliminary report.

The concept of a stochastic process whose random variables have an asymptotically almost periodic joint distribution is introduced, generalizing the concept of a stationary stochastic process. Doob (Duke Math. J. vol. 6 (1940) pp. 290–307) adapted Birkhoff's individual ergodic theorem to obtain a strong law of large numbers for stationary processes. A strong law of large numbers is obtained for asymptotically almost periodic stochastic processes using the individual ergodic theorem as generalized by Riesz (Comment. Math. Helv. vol. 17 (1944–45) pp. 221–239) and an invariant measure (Dowker, Duke Math. J. vol. 14 (1947) pp. 1051–1061) based on averages (Dowker, loc. cit., Dunford and Miller, Trans. Amer. Math. Soc. vol. 60 (1946) pp. 538–549). Examples are given of asymptotically almost periodic processes. (Received October 9, 1953.)

Topology

143t. L. E. Pursell: *On multiplicative semigroups of non-negative semicontinuous functions.*

The set of all non-negative upper semicontinuous functions on a topological space X with the multiplication operation forms a semigroup $U(X)$. The idempotents of $U(X)$ are those functions on X which vanish on an open set and are one on its complement. The nonzero idempotents of $U(X)$ are ordered as follows: $f>g$ if $f\cdot g=g$. $M(U, X)$ consists of all minimal nonzero idempotents with the topology: $A\subset M(U, X)$ is closed if A consists of all g in $M(U, X)$ which precede some idempotent f. If X is a T_1 topological space, then X is homeomorphic to $M(U, X)$. If X and Y are T_1 spaces and $U(X)$ and $U(Y)$ are isomorphic, then X and Y are homeomorphic. One can obtain similar results for the multiplicative semigroups of all non-negative lower semicontinuous functions, all bounded non-negative lower semicontinuous functions, and all bounded non-negative upper semicontinuous functions. (Received October 16, 1953.)

144t. C. T. Yang: *Topological generalizations of theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson.*

Let X be a compact Hausdorff space and let T be an involution on X without fixed point such that there is some element of the nth special homology group of $(X; T)$ in the sense of Smith (Lefschetz, *Algebraic topology*, Appendix B), which is not annihilated by any homomorphism induced by a map of $(X; T)$ into an arbitrary T-space $(Y; T)$. Let E, F be closed subsets of X^2 such that (i) $E \cup F = X^2$, (ii) $(x, y) \in E \leftrightarrow (y, x) \in E \leftrightarrow (T(x), y) \in F \leftrightarrow (x, T(y)) \in F$, (iii) $\{(x, x) : x \in X\} \subset E - F$. Two points x, y of X are called orthogonal if $(x, y) \in E \cap F$. It is proved that a continuous real-valued function on X maps some n mutually orthogonal points of X together with their T-images into a single value. This result is topological and generalizes a previous generalized Dyson's theorem of the author (Bull. Amer. Math. Soc. Abstract 59-6-720). The proof is roughly as follows: The nerves of certain finite open coverings of X are imbedded into euclidean spaces in such a way that the previous generalized Dyson's theorem can apply. Then a limit procedure yields the desired assertation. Topological generalizations of this kind of Borsuk-Ulam's and Kakutani-Yamabe Yujobô's theorems are also established. (Received October 7, 1953.)

F. J. Weyl,
National Research Council
J. W. T. Youngs,
Associate Secretary

THE NOVEMBER MEETING IN PASADENA

The four hundred ninety-eighth meeting of the American Mathematical Society was held at the California Institute of Technology, Pasadena, California on Saturday, November 28. Attendance was approximately 90, including the following 70 members of the Society:

T. M. Apostol, L. A. Aroian, M. S. Barnes, R. A. Beaumont, M. M. Beenken, Clifford Bell, L. D. Berkovitz, W. W. Bledsoe, R. R. Christensen, Randolph Church, L. M. Coffin, P. H. Daus, C. R. DePrima, R. J. Dickson, Jr., S. P. Diliberto, R. P. Dilworth, Roy Dubisch, O. J. Dunn, D. E. Edmondson, M. P. Epstein, Arthur Erdélyi, W. J. Firey, Harley Flanders, G. E. Forsythe, A. L. Foster, R. E. Graves, J. W. Green, C. J. A. Halberg, Jr., H. J. Hamilton, A. R. Harvey, M. R. Hestenes, D. H. Hyers, C. G. Jaeger, M. L. Juncosa, J. L. Kelley, G. J. Kleinhesselink, P. A. Lagerstrom, L. C. Lay, J. E. LeBel, M. A. Lee, Solomon Lefschetz, T. C. Littlejohn, J. L. McGregor, J. C. Miller, T. S. Motzkin, A. B. Neale, E. D. Nering, T. E Oberbeck, James Pacheres, D. H. Potts, W. T. Puckett, Jr., R. M. Redheffer, Edgar Reich, E. B. Roessler, M. M. Schiffer, I. M. Singer, I. S. Sokolnikoff, F. L. Spitzer, M. L. Stein, Robert Steinberg, E. G. Straus, A. C. Sugar, Irving Sussman, J. D. Swift, A. E. Taylor, Elmer Tolsted, C. B. Tompkins, F. A. Valentine, L. E. Ward, L. E. Ward, Jr., W. R. Wasow, J. V. Whittaker.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor M. M. Schiffer of Stanford University delivered an hour address entitled *Variational methods for domain functionals*. Professor Schiffer was introduced by Professor Arthur Erdélyi, who also presided at the morning session for contributed papers. The afternoon sessions were presided over by Professor R. P. Dilworth and Dr. G. E. Forsythe.

Those attending the meetings were entertained afterwards at tea at the Athanaeum by the Mathematics Department of the California Institute of Technology.

Abstracts of the papers presented are listed below. In the case of joint authorship of a paper, the name of that author who presented it is followed by "(p)." Papers presented by title have their numbers followed by "*t*." Dr. Stein was introduced by Professor J. W. Green, Dr. Mackie by Professor Arthur Erdélyi, Mr. Marcus by Professor S. P. Diliberto, and Mr. Scott by Professor Alfred Tarski.

ALGEBRA AND THEORY OF NUMBERS

145. R. A. Beaumont (p) and R. P. Peterson, Jr.: *Set-transitive permutation groups.*

A permutation group G on n symbols is s set-transitive if for every pair of s element subsets S and T of the symbols, there is a permutation in G which carries S into T. The group G is set-transitive if G is s set-transitive for all s. If s is greater than

1, an s set-transitive group is primitive, and using classical results on primitive groups and a theorem on the distribution of primes, the authors show that $[n/2]$ set-transitive groups, not containing the alternating group, are possible only when $n=5$, 6, and 9. All $[n/2]$ set-transitive groups of degree 5, 6, and 9 are determined, and these groups are seen to be set-transitive. (Received October 13, 1953.)

146*t*. Chen-chung Chang: *Two general theorems on direct products of algebras.*

Consider *algebras* $\mathfrak{A}=\langle A, O_0, \cdots, O_\xi, \cdots\rangle$, $\xi<\alpha\neq 0$, formed by a set A and operations O_ξ under which A is closed. The *order* of $\mathfrak{A}$ is the power $\mathfrak{a}$ of the ordinal α, $\mathfrak{a}=\mathfrak{p}(\alpha)$. Each O_ξ is an operation on sequences of a definite type $\beta_\xi>0$; the *rank* of O_ξ is $\mathfrak{b}_\xi=\mathfrak{p}(\beta_\xi)$. A set $S\subseteq A$ is *isomorphically embeddable* in another algebra $\mathfrak{A}'=\langle A', O_0', \cdots, O_\xi', \cdots\rangle$ if there is a 1-1 mapping f of S into A' such that $a=O_\xi(b_0, \cdots, b_\zeta, \cdots)$ iff $f(a)=O_\xi'(f(b_0), \cdots, f(b_\zeta), \cdots)$ for all $a, b_0, \cdots, b_\zeta, \cdots \in S$ and $\xi<\alpha$. Theorem I. Assumptions: $\mathfrak{A}_i$, with $i\in I$, are algebras of finite order with finitary operations; C is a set of elements of the direct product $\mathfrak{P}_{i\in I}\mathfrak{A}_i$ with $\mathfrak{p}(C)<\mathfrak{c}$, $\mathfrak{c}\geq\aleph_0$. Conclusion: C is isomorphically embeddable in $\mathfrak{P}_{i\in J}\mathfrak{A}_i$ for some $J\subseteq I$, $\mathfrak{p}(J)<\mathfrak{c}$. Theorem II. Assumptions: $\mathfrak{A}_i$, with $i\in I$, are algebras of order $\mathfrak{a}$; $\mathfrak{b}=$l.u.b. of ranks of operations in $\mathfrak{A}_i$; C is a set of elements of $\mathfrak{P}_{i\in I}\mathfrak{A}_i$ with $\mathfrak{p}(C)\leq\mathfrak{c}$, $\mathfrak{c}\geq\aleph_0$. Conclusion: C is isomorphically embeddable in $\mathfrak{P}_{i\in J}\mathfrak{A}_i$ for some $J\subseteq I$, $\mathfrak{p}(J)\leq\max(\mathfrak{a}, \mathfrak{c}^{\mathfrak{b}})$ (and $\mathfrak{p}(J)\leq\max(\mathfrak{a}, \mathfrak{c})$ if all operations of $\mathfrak{A}_i$ are finitary). Both theorems apply to systems with relations instead of operations. (Received October 14, 1953.)

147*t*. Chen-chung Chang: *A characterisation of equational classes of algebras with arbitrary operations.*

The following theorem is analogous to those stated in abstract 148 but concerns algebras with operations of arbitrary rank (finite or infinite): Let K be a class of (similar) algebras of order $\mathfrak{a}$; let $\mathfrak{b}=$l.u.b. of ranks of operations in each algebra of K; and let $\mathfrak{c}\geq\max(\mathfrak{b}, \aleph_0)$. Then K is equational iff it satisfies the following conditions: (i) if $\mathfrak{A}_i\in K$ for every $i\in I$, where $\mathfrak{p}(I)\leq\max(\mathfrak{a}, \mathfrak{c}^{\mathfrak{b}})$, then $\mathfrak{P}_{i\in I}\mathfrak{A}_i\in K$; (ii) if $\mathfrak{A}\in K$ and $\mathfrak{A}$ is homomorphic to $\mathfrak{B}$, then $\mathfrak{B}\in K$; (iii) if every set S of elements of an algebra $\mathfrak{A}$ with $\mathfrak{p}(S)\leq\mathfrak{c}$ is isomorphically embeddable in an algebra of K, then $\mathfrak{A}\in K$. (Received October 14, 1953.)

148*t*. Chen-chung Chang, Herman Rubin, and Alfred Tarski: *A characterization of equational classes of algebras with finitary operations.*

For notations see abstracts 146 and 152. The following two theorems generalize the result stated in abstract 152. Theorem I: Let K be a class of algebras of finite order with finitary operations; let $\mathfrak{c}\geq\aleph_0$. Then K is equational iff it satisfies the following conditions: (i) if $\mathfrak{A}_i\in K$ for every $i\in I$, where $\mathfrak{p}(I)<\mathfrak{c}$, then $\mathfrak{P}_{i\in I}\mathfrak{A}_i\in K$; (ii) if $\mathfrak{A}\in K$ and $\mathfrak{A}$ homomorphic to $\mathfrak{B}$, then $\mathfrak{B}\in K$; (iii) if every set S of elements of an algebra $\mathfrak{A}$ with $\mathfrak{p}(S)<\mathfrak{c}$ is isomorphically embeddable in an algebra of K, then $\mathfrak{A}\in K$. Theorem II: Let K be a class of algebras of arbitrary order $\mathfrak{a}$ with finitary operations; let $\mathfrak{c}\geq\aleph_0$. Then K is equational iff it satisfies condition I(i) with "$<\mathfrak{c}$" replaced by "$\leq\max(\mathfrak{a}, \mathfrak{c})$," condition I(ii), and condition I(iii) with "$<$" replaced by "$\leq$". In case $\mathfrak{c}>\aleph_0$ in Theorem I or $\mathfrak{c}\geq\mathfrak{a}$ in Theorem II, the phrase "set S of elements" in (iii) can be replaced by "subalgebra S." The original proof of the theorems was based upon the result in abstract 152 and upon properties of free algebras; some ideas from

Birkhoff's paper mentioned in abstract 152 were used. The proofs were later simplified with the help of the results in abstract 146. (Received October 14, 1953.)

149. M. P. Epstein: *On the theory of Picard-Vessiot extensions.*

A Galois theory of homogeneous linear ordinary differential equations is developed, extending results of E. R. Kolchin. Let F be an ordinary differential field of characteristic 0 with field of constants C. A differential field G (with field of constants D) is a Picard-Vessiot extension of F if (1) there exists $L(y)=y^{(n)}+p_1y^{(n-1)}+\cdots+p_ny$ ($n\geq 1$, each $p_i\in F$) with a fundamental system of zeros ($\eta_1, \cdots, \eta_n$) such that $G=F\langle\eta_1, \cdots, \eta_n\rangle$; (2) D is a normal algebraic extension of C. For given F and $L(y)$ the existence of a Picard-Vessiot extension G is demonstrated. An isomorphism is established between the group $\mathfrak{G}$ of all automorphisms of G over F and an algebraic matrix group $c(\mathfrak{G})$ over C, and the dimension of $c(\mathfrak{G})$ is computed. The Galois subgroups of $\mathfrak{G}$ are characterized using algebraic properties of $c(\mathfrak{G})$ and a topology is introduced in G in which the set of fixed fields becomes the set of closed differential fields. A one-one correspondence between these sets of groups and differential fields is demonstrated and the relations between the algebraic and topological structures of G are studied. (Received October 13, 1953.)

150. T. E. Oberbeck: *Results on pseudo-periodic functions of any number of variables.*

Trigonometric series for $\theta_\alpha(x_1+\cdots+x_n)[\prod_{i=1}^{n}\theta_1(x_i)]^{-1}$, $n\geq 3$, $\alpha=1, 2, 3, 4$ are obtained in two ways, leading to paraphrasable identities [E. T. Bell, Trans. Amer. Math. Soc. vol. 22 (1921) pp. 1–30 and pp. 198–219]. In contrast with the method of Basoco [Amer. J. Math. vol. 54 (1932) pp. 242–252] which treats $\theta_\alpha(x_1+x_2)[\theta_1(x_1)\cdot\theta_1(x_2)]^{-1}$ directly as functions having *poles* at $x_i=0$, $i=1, 2$, the method of this paper is to introduce functions F_n^α which are *regular* in the region (R) defined by $|\mathrm{Im}\, x_i|<\mathrm{Im}\, \pi\tau$, $i=1, \cdots, n$, and which involve $\Phi_n^\alpha=\theta_\alpha(x_1+\cdots+x_n)[\prod_{i=1}^{n}\theta_1(x_i)]^{-1}$. Then for $n\geq 3$, F_n^α is expressed in terms of F_j^α, $j=1, \cdots, n-1$, and an associated function $\overline{F}_{n-1}^\alpha$. The function $\overline{F}_n^\alpha$ is also regular in (R) and it involves $\Phi_n^1C_\alpha(x_1+\cdots+x_n)$ where $c_\alpha(u)$ is cot u for $\alpha=1, 2$ and csc u for $\alpha=3, 4$. $\overline{F}_n^\alpha$ is expressible in terms of $\overline{F}_2^\alpha$ and F_j^1, $j=1, \cdots, n-1$. The method is essentially that of the author's doctoral thesis [*Trigonometric expansions of doubly-periodic functions of the second kind in any number of variables*, California Institute of Technology, 1948] which considers only F_n' and $\overline{F}_n^1$, a somewhat special case inasmuch as $F_1^1=1$. Besides extending the results of the thesis, its basic techniques are presented in a much improved notation. (Received October 14, 1953.)

151. Irving Sussman: *On the local semigroup decomposition of associate rings.*

The class of associate rings as an ultimate generalization of the Boolean ring concept, which was introduced by the author with A. L. Foster at the 1953 Summer Meeting of the Society, is further analyzed via its decomposition into disjoint multiplicative domains. Using certain of these sets generated by idempotents together with the nonzero divisors of the ring, an ideal theoretic structure is organized and related to the subdirect structure of the associate ring. A number of interesting inner relationships within the rings are established; for example, the equivalence of the following two conditions: each maximal compatible set considered as a Boolean ring is complete and atomistic, and each idemideal contains a subideal whose only proper idempotent element is its generator. (Received October 13, 1953.)

152*t*. Alfred Tarski: *On equational classes of algebras with finitary operations.*

For notations see Abstract 146. A class K of algebras is called *equational* if there is a set E of algebraic equations (involving only elements and fundamental operations of an algebra) such that K consists of all those algebras in which every equation of E is identically satisfied. Birkhoff in Proc. Cambridge Philos. Soc. vol. 31, pp. 433–454 gives a purely mathematical criterion for a class K to be equational. For algebras of finite order and with finitary operations another criterion can be given which, as opposed to Birkhoff's criterion, does not involve direct products of arbitrary infinite systems of algebras. In fact, a class K of such algebras is equational iff it satisfies the following conditions: (i) if $\mathfrak{A}\in K$ and $\mathfrak{B}\in K$, then $\mathfrak{A}\times\mathfrak{B}\in K$; (ii) if $\mathfrak{A}\in K$ and $\mathfrak{A}$ is homomorphic to $\mathfrak{B}$, then $\mathfrak{B}\in K$; (iii) if every finite set of elements of an algebra $\mathfrak{A}$ is isomorphically embeddable in an algebra of K, then $\mathfrak{A}\in K$. The original proof of this theorem was based upon results in Abstract 59-4-500 and McKinsey, Journal of Symbolic Logic vol. 8, p. 66. C. C. Chang pointed out that the proof can be simplified by applying Theorem I of Abstract 60-1-146. (Received October 14, 1953.)

ANALYSIS

153. S. P. Diliberto: *Bounds for periods of periodic solutions.*

Let $\dot{x}=f(x)$, x a two vector, $f\in C^1$ in the plane. Let S be the set of singular points of f plus the point at infinity. Let B be a region bounded away from S. Let $U(t)$ be a periodic solution of the equation *lying in* B. Let $m=\min \|f(x)\|$ for $x\in B$ ($m>0$), let $K=\max\ (|\partial f_i/\partial x_j|,\ \|f\|)$ for $x\in B$. Then the period of $U(t)$ is less than a constant depending only on B, m, K. This is used as a "uniform convergence theorem" for periodic solutions of $\dot{x}=f(x,\lambda)$. (Received October 19, 1953.)

154. W. J. Firey: *Ballistically closed regions.*

Definition: a closed set K, having an interior point, is ballistically closed with respect to a system of second order, ordinary differential equations if: (a) For any two points P and P' of K, and every interval of the independent variable, there is a unique solution to the system having P and P' as boundary values; (b) The path-segment of the solution joining P to P' lies wholly in K. Let the system be (E) $d^2y/dt^2=Ay$, y being a vector in a real Hilbert space and A a bounded, linear operator, independent of t. All limits are in the sense of the strong topology. An operator B is said to expand K if: (a) B has a bounded inverse; (b) The image of K under B includes K. Theorem: For K to be ballistically closed with respect to (E), it is necessary and sufficient that: (i) K is convex; (ii) There is a solution to $Ax=0$ in K (after a translation which does not affect the problem, this may be taken to be the origin); (iii) The transformation $I+s^2A$ expands K for all real s (assuming the translation of (ii) to have been performed). For certain choices of K, conditions on A, free of the parameter s, are found. (Received October 13, 1953.)

155*t*. H. J. Fletcher and C. J. Thorne: *Sine and cosine transforms.*

A list of inverse sine and cosine transforms is presented, together with various methods used in obtaining the table. These methods are sufficiently general to extend the table to all rational functions of the transformed variable that have an inverse transform, and many nonrational functions. Sponsored by the Office of Ordnance Research, U. S. Army. (Received October 15 1953.)

156. M. D. Marcus: *An invariant surface theorem for a nondegenerate system.*

An invariant surface Γ of dimension r for (1) $\dot{x}=f(x, t, \lambda)$, x, f n-vectors, λ a real parameter, $f\in C^r$, is a C^r manifold in E^n such that if $u(t)$ is a solution of (1) and $u(t_0)\in\Gamma$, then $u(t)\in\Gamma$ for all $t\geqq t_0$. Assume Γ_0 is an invariant surface of (1) for $\lambda=0$. Sufficient conditions are obtained for the existence of a family of such surfaces, Γ_λ, for λ small such that $\lim_{\lambda\to 0}$ dist $(\Gamma_\lambda, \Gamma_0)=0$, by reduction of (1) to normal coordinates and use of the Schauder fixed point theorem. Reduction of certain perturbation problems for second order equations are indicated. (Received October 19, 1953.)

157. R. M. Redheffer: *Approximation by enumerable sets.*

Let $\{r_n\}$ be an enumerable set of real numbers and let $\{d_n\}$ be a sequence of positive real numbers. We say that a set E of real numbers is approximated by $\{r_n\}$ within $\{d_n\}$ if, whenever x is in E, the inequality $|x-r_n|<d_n$ holds for infinitely many n. A similar notion of mean approximation is defined, and several results are established. For example if $\sum d_n=\infty$ the approximation is possible for every E, but if $\sum d_n<\infty$ it is necessary that $m(E)=0$. (Received October 15, 1953.)

158t. R. M. Redheffer: *The dependence of reflection on angle.*

Consider a stratified dielectric medium, that is, one for which $\epsilon/\epsilon_0=e(x)$, $\mu/\mu_0=m(x)$, complex-valued functions of the distance x to one interface. A *thin, solid absorber* is a thin medium of this type which is terminated by a conducting plate and satisfies $|me|\gg 1/2$. The fundamental theorem is the following: For a given polarization, let a thin, solid absorber be such that the amplitude reflection, $|R|$, as a function of incidence angle θ, assumes a minimum of value R_0 at the angle $\theta=\theta_0\neq 0$. Then the complex reflection R at every incidence-angle θ and at both polarizations is completely determined by θ_0 and R_0. The explict relations are derived, together with detailed criteria for optimum design. The results follow readily from the behavior of certain Riccati equations containing an arbitrary function and a parameter. (Received October 15, 1953.)

159t. Edgar Reich: *On a conformal mapping constant defined by Study.*

Let S_c be the class of functions $f(z)=z+a_2z^2+\cdots$ regular and schlicht for $|z|<1$, each mapping $\{|z|<1\}$ onto a convex domain $D(f)$. Study (*Konforme Abbildung einfach-zusammenhängender Bereiche*, Leipzig, Teubner, 1913) proved the existence of a universal constant μ, $1/2<\mu<1$, such that $f\in S_c$, $\sup_{|z|<1}|f(z)|=\infty$ imply $\{|z|\leqq\mu\}\not\subset D(f)$. It is shown that $0.87<\mu<0.92$ for the lowest such constant. (Received September 23, 1953.)

160. Edgar Reich: *On values omitted by schlicht analytic functions.*

Let $w=f(z)=z+a_2z^2+\cdots$ be regular and schlicht for $|z|<1$, and denote the image of $\{|z|<1\}$ under $f(z)$ by $D(f)$. Let $A(f)=$area $D(f)\cap\{|w|<1\}$. Goodman

[Bull. Amer. Math. Soc. vol. 55 (1949) pp. 363–369] showed that $A(f) > .50\pi$ for any admissible f. Jenkins [Amer. J. Math. vol. 75 (1953) pp. 406–408] improved this bound to $A(f) > .53\pi$. By a refinement of Jenkins' technique the author shows that $A(f) > .62\pi$. (Received September 17, 1953.)

Applied Mathematics

161. H. F. Bohnenblust and G. E. Latta (p): *A note on the Wiener-Hopf integral equation.*

Wiener and Hopf solved the integral equation $\int_0^\infty k(x-y)f(y)dy = f(x)$ by utilizing the regularity properties of the Laplace transforms of the functions k and f. The assumptions on k and f were such that a definite overlap of the regularity regions was obtained. This method can also be used sometimes in the case where this overlap reduces to the boundary of the regularity regions, as well as for equations of the type $g(x) = \int_0^\infty k(x-y)f(y)dy$ for given g, k for $x > 0$. Three examples arising from mixed boundary value problems are discussed, one of them being the Oseen equations for flow past a semi-infinite plate. (Received November 28, 1953.)

162. G. E. Hudson and D. H. Potts (p): *On a class of solutions of Maxwell's equations.*

Solutions of Maxwell's equations in which the electric and magnetic vectors are taken to be real vectors multiplied by complex-valued functions are considered. This form of solution is a generalization of the usual plane wave solution. Maxwell's equations then furnish the necessary condition that the phases of the two vectors are not only functionally related but their phase difference satisfies a certain nonlinear ordinary differential equation of second order. The amplitude ratio is also seen to depend on this phase difference indicating that the concept of a characteristic impedance of the medium would need to be generalized for such waves. The spatial part of the phase function of the waves satisfies an eikonal equation and the electric and magnetic vectors are orthogonal to each other and to the direction of propagation of the wave. Furthermore, the vector amplitudes of these vectors are proportional to gradients of two functions which themselves are solutions of eikonal equations and are incidentally also potential functions. These results can be summarized in the statement that the problem of finding this type of solution of Maxwell's equations is identical with the problem of finding a class of axially similar isometric orthogonal curvilinear coordinate systems. (Received October 13, 1953.)

163. A. G. Mackie: *The one-dimensional unsteady motion of a gas initially at rest and the analogous problem of the breaking of a dam.*

The object of this paper is to discuss the one-dimensional unsteady adiabatic motion of a gas which is initially at rest with a prescribed density distribution such that the specific entropy is uniform. The contour integral methods which Copson developed recently for even analytic functions are extended to apply to general analytic initial conditions. The solution is valid in the range $1 < \gamma < 3$ where γ is the adiabatic index of the gas. Of particular interest, in view of the hydraulic analogy, is the case $\gamma = 2$ for which real variable methods cannot readily be adapted. The motion of the front of a water column flowing into a dry, horizontal stream bed is discussed. A curious type of solution, corresponding to a particular choice of initial distribution, which was established by Pack for a countable sequence of values of γ, is verified to hold over the whole range and is interpreted in terms of the dam-break problem. The methods are also found to be applicable to other types of boundary value problems. (Received October 13, 1953.)

164t. F. E. Maud and C. J. Thorne: *Thin plates under combined loads.* I.

General expressions for the deflection of thin rectangular plates with constant edge thrusts are obtained for cases in which two opposite edges have arbitrary but given deflections and moments. Six important cases of boundary conditions on the remaining two edges are treated. The sine transform is used to obtain the solution as a single trigonometric series. Numerical solutions are obtained in special cases. Some of the questions proposed by Conway (Journal of Applied Mechanics vol. 71 (1949) pp. 301–309) are answered. Sponsored by the Office of Ordnance Research, U. S. Army. (Received October 15, 1953.)

165. E. D. Nering: *Symmetric solutions for symmetric 4-person games.*

The author provides a family of symmetric solutions for all general-sum symmetric 4-person games. For each game one or more families of solutions are given, each of which depends on one parameter which usually may take on an interval of values. The families are given in different ways for various ranges of the game defining parameters, but seven basic types of configurations are used as the building blocks for them all. The solutions specialize into the known symmetric solutions in the zero-sum case. (Received October 15, 1953.)

166t. L. E. Payne: *Axially symmetric crack and punch problems in a medium with transverse isotropy.*

The author has shown (J. Soc. Ind. Appl. Math. (1953)) that the complete solution to the distribution of stress in an isotropic medium for an axially symmetric crack or punch problem may be expressed in terms of a potential and its stream function. In this paper the author shows that the complete solution to the same problems in the case of transverse isotropy can be derived from two modified potentials with their corresponding modified stream functions. By the appropriate substitution these latter problems can be reduced to the analogous isotropic problems. Hence the solution to the crack or punch problem for a medium with transverse isotropy may be obtained directly from the solution to the corresponding isotropic problem. A number of new problems are considered. (Received September 25, 1953.)

167t. P. C. Suppes and Muriel Winet: *An axiomatization of utility based on the notion of utility differences.*

This paper is concerned with an axiomatization of utility based on the following three primitive notions. K is a set, interpreted as a set of alternatives. Q is a binary relation whose field is K, with the intended interpretation that xQy if and only if x is not preferred to y. R is a quaternary relation whose field is K, with the intended interpretation that x, yRz, w if and only if the difference in preference between x and y is not greater than that between z and w. The following defined notions are also used: $xIy =_{df} xQy$ and yQx; $xPy =_{df}$ not yQx; $B(y, x, z) =_{df} xPy$ and yPz, or zPy and yPx; $x, yMu, v =_{df} yIu$, $B(y, x, v)$, x, yRy, v and y, uRx, y. An ordered triple $\langle K, Q, R\rangle$

is said to be a *utility difference structure* if the following axioms are satisfied: A1. Q is transitive and connected in K; A2. R is transitive and connected in $K \times K$; A3. x, yRy, x; A4. There is a z such that x, zRz, y and z, yRx, z; A5. If xIy and x, zRu, v, then y, zRu, v; A6. If $B(y, x, z)$, then not x, zRx, y; A7. If $B(y, x, z)$, $B(w, u, v)$, x, yRu, w and y, zRw, v, then x, zRu, v; A8. If not u, vRx, y, then there is a z such that $B(z, u, v)$ and x, yRu, z; A9. If x, yRu, v and not xIy, then there are elements s and t and a positive integer k such that $u, sM^k t, v$ and u, sRx, y, where M^k is the kth power of the relation M. The expected adequacy theorem is established, that is: (A) there exists a real-valued function f defined on K such that (i) xQy if and only if $f(x) \leq f(y)$, and (ii) x, yRu, v if and only if $|f(x) - f(y)| \leq |f(u) - f(v)|$; (B) the function f of (A) is unique up to a linear transformation. (Received October 15, 1953.)

GEOMETRY

168*t*. Rafael Artzy: *4-webs and Möbius' net.*

In a plane 4-web consisting of 3 families of parallel lines and one line-pencil a minimum net is constructed. Its points are in one-one correspondence to all pairs of rational numbers, i.e. to the points of Möbius' net, if closing of the following two configurations is postulated: Brianchon's hexagon consisting of lines from the 3 parallel families, and a specialization of Reidemeister's theorem Σ.3 [cf. *Grundlagen der Geometrie*, 1930, p. 93]. These postulates are shown to be also necessary. By introducing translations of pencil-lines the whole of Möbius' net (including its lines) can be obtained from the minimum net in the 4-web. The above mentioned postulates are shown to follow in Möbius' net from Moufang's D_9 and thus another proof of Moufang's statement [Math. Ann. vol. 105 (1931)] concerning the characterization of Möbius' net is established. This paper will be published in Riveon Lematematika vol. 7. (Received October 9, 1953.)

169*t*. Rafael Artzy: *Minimum net in* 4 *pencils of straight lines*. Preliminary report.

The paper deals with a minimum net in a web which consists of 4 pencils of straight lines in the projective plane. No 3 vertices of these pencils are to be collinear. The closing of two configurations is shown to be sufficient and necessary for building upon this minimum net, establishing in it a coordinate system, and showing that all its points have rational coordinates. These configurations are rational specializations of configurations which the author has shown (in a paper which will appear shortly in Math. Ann.) to characterize 4-webs in general, and which in turn are generalizations of well known configurations in webs with 3 or 4 collinear pencil vertices, as Reidemeister's and Thomsen's configurations [cf. Blaschke-Bol, *Geometrie der Gewebe*, 1938] and Reidemeister's Σ.3 [Reidemeister, *Grundlagen der Geometrie*, 1930]. (Received October 9, 1953.)

170. Harley Flanders: *An extension theorem for solutions of* $d\omega = \Omega$.

Let Ω be a closed differential form on an open set U in euclidean space. Let α be a form on an open subset V such that $d\alpha = \Omega$. It is shown, under appropriate topological conditions, that if the set V is cut down to a slightly smaller set W, then there is a form ω on U such that $d\omega = \Omega$ on U and ω coincides with α on W. An analogous result is given for the case of isolated singularities. (Received October 13, 1953.)

LOGIC AND FOUNDATIONS

171*t*. Jan Kalicki: *A problem concerning recursive truth-tables.*

It has been proved that there exists a method to find in a finite number of steps whether or not two truth-tables with finitely many elements have the same set of tautologies (cf. Kalicki, Journal of Symbolic Logic vol. 17, pp. 161–163). It is proved now that no such method exists in case of truth-tables which are not necessarily finite. This is true even when the problem is restricted to truth-tables which are recursive, i.e., to the truth-tables for which there is a decision method to test in a finite number of steps whether or not an arbitrary wff is a tautology of any of them. The method used consists of a reduction of the problem to the question of testing whether or not any given recursive set of positive integers is empty, and proving that this last problem is undecidable. (Received October 13, 1953.)

172*t*. Dana Scott: *The theorem on maximal ideals in lattices and the axiom of choice.*

For terminology see Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. 25, rev. ed. In a lattice with a unit element and with at least two elements the existence of a maximal ideal can be proved using the axiom of choice (in the form of Zorn's Lemma, for example). Conversely the existence of maximal ideals in lattices implies an equivalent form of Zermelo's Axiom. For let X be a nonempty set, $\leqq$ be a partial ordering of X, and C be the class of all subsets of X that are chains under the relation $\leqq$. It is to be shown that C has a maximal element. Assume $X \notin C$. The set $C \cup \{X\}$ forms a lattice $\mathfrak{L}$ with a unit and at least two elements under the relation of set-theoretical inclusion. The union of every proper ideal of $\mathfrak{L}$ is an element of C, whence there is a correspondence between maximal ideals of $\mathfrak{L}$ and maximal elements in C. Thus the existence of maximal ideals in certain lattices implies the existence of maximal chains included in partial orderings, which in turn implies the axiom of choice. (Received October 15, 1953.)

173*t*. Alfred Tarski: *The axiom of choice and the existence of a successor for every cardinal.*

The aim of the paper is to exhibit two closely related statements from the arithmetic of cardinals, one of which can be proved without the help of the axiom of choice while the other is equivalent to this axiom. (For another example of two statements with these properties see Lindenbaum-Tarski, Comptes Rendus Soc. Sci. Vars. vol. 29 (1926) Class III, p. 312.) The well known theorem to the effect that every cardinal has a successor can be formulated in several different ways. Consider, e.g., the following three formulations. S_1: For every cardinal α there is a cardinal β such that (i) $\alpha<\beta$ and (ii) the formula $\alpha<\gamma<\beta$ holds for no cardinal γ. S_2: For every cardinal α there is a cardinal β such that (i) $\alpha<\beta$ and (ii) the formula $\alpha<\gamma$ implies $\beta \leqq \gamma$ for every cardinal γ. S_3: For every cardinal α there is a cardinal β such that (i) $\alpha<\beta$ and (ii) the formula $\gamma<\beta$ implies $\gamma \leqq \alpha$ for every cardinal γ. It turns out that *the proof of* S_1 *does not require the axiom of choice* and that S_3 *is equivalent to this axiom*. The relation between S_2 and the axiom of choice has not yet been cleared up. (Received October 14, 1953.)

TOPOLOGY

174*t*. V. L. Klee and W. R. Utz: *Some remarks on continuous transformations.*

Suppose f is a transformation of the metric space M onto the metric space fM. It is well known that if f is continuous, it must have certain other properties also. This paper considers the extent to which some of these properties imply continuity. Under consideration are (1) fX is compact for every compact $X \subset M$; (2) fY is connected for every connected $Y \subset M$; (3) $f^{-1}p$ is closed for each $p \in fM$. Among the results established are: (A) Every 1, 3-map is continuous. (B) M is locally connected at the point $q \in M$ if and only if every 1, 2-map on M is continuous at q. The paper will appear in the Proceedings of this Society. (Received October 5, 1953.)

175. Sherman Stein: *Families of curves.*

A. Forrester proved in Proc. Amer. Math. Soc. (1952) pp. 333–334 that if ϕ is an involution without fixed points of the Euclidean sphere then the chords joining P to $\phi(P)$ for all points P of the sphere fill the interior of the sphere. This theorem can be generalized to the following elementary and purely topological theorem: If ϕ is an involution without fixed points of the topological sphere S^n bounding the topological cell E^{n+1} and I the unit interval and F: $S^n \times I \rightarrow E^{n+1}$ satisfies (a) $F(P, 0) = P$ and (b) $F(P, 1-t) = F(\phi(P), t)$, then F is onto E^{n+1}. The proof consists of assuming the contrary and thereby producing a homotopy between two unhomotopic maps of the sphere. (Received October 13, 1953.)

176. L. E. Ward, Jr.: *On invariant sets.*

Let X be a continuum ($=$compact connected Hausdorff space) and $f(X) \subset X$ continuous. It is shown that (1) *there exists a minimal non-null invariant subcontinuum, K, without cut points*, and (2) *if X contains an end point, e, f is monotone onto, and $f(e) = e$, then K may be found such that* $K \subset X - e$. (1) generalizes a result due to J. L. Kelley; (2) is an extension of a theorem of Schweigert. (Received August 21, 1953.)

J. W. GREEN,
Associate Secretary

BOOK REVIEWS

Typical means. By K. Chandrasekharan and S. Minakshisundaram. Oxford University Press, 1952. 10+140 pp. $6.50.

This book, which is the first of a new series of monographs to be published under the auspices of the Tata Institute of Fundamental Research, Bombay, is concerned entirely with the theory and applications of Riesz summability. In view of the fact that Hardy's *Divergent series* devoted only a little space to Riesz summability, there was room for another book dealing more fully with this particular method. While reasonably complete, the book is not exhaustive; indeed, an account of everything that is known on the subject would be impossible in a book of this size. However, a useful series of notes at the end of each chapter contains numerous references, and also the statement of certain theorems for which room was not found in the main text. The book collects in a convenient form much material which has hitherto been available only in the original papers, and it should prove of great use to anyone who wants to work in this particular field. It is a pity that its usefulness should be diminished by the errors which occur in it.

In the remaining remarks, we use the following notation (which is in line with that used in the book). If $\{\lambda_n\}$ is any sequence of positive numbers increasing to infinity, and if $\sum a_n$ is any given series, we write

$$(1)\quad A(t) = A_\lambda(t) = \sum_{\lambda_\nu \leq t} a_\nu;$$

$$(2)\quad A^k(t) = A_\lambda^k(t) = \sum_{\lambda_\nu \leq t} (t-\lambda_\nu)^k a_\nu = k\int_0^t (t-\tau)^{k-1} A(\tau)\,d\tau \quad (k > 0);$$

$A_\lambda^k(t)$ is the Riesz sum, and $t^{-k}A_\lambda^k(t)$ the Riesz mean, of order k and type λ associated with the series $\sum a_n$. If $t^{-k}A_\lambda^k(t) \to s$ as $t \to \infty$, we say that the series is summable (λ, k) to s. [This is more usually termed summability $(R;\ \lambda,\ k)$, but as we are concerned only with Riesz summability there is no harm in omitting the "R."]

Apart from some introductory material, Chapter I deals, broadly speaking, with relations between Riesz means of the same type but different orders. Such topics as limitation theorems and M. Riesz's convexity theorem are dealt with adequately. There follows a section (§1.8) on Tauberian theorems. It may be remarked that this section deals only with those Tauberian theorems in which the hypotheses involve only Riesz sums. Those (closely related) Tauberian theorems in which we deduce the convergence or summability (λ, k) of the series $\sum a_n$ from hypotheses about the behaviour of

$$\sum_{n=0}^{\infty} a_n e^{-\lambda_n \sigma} \qquad (\sigma > 0),$$

together with some Tauberian condition are dealt with separately in Chapter III. The treatment of §1.8 has certain shortcomings. Thus in one theorem (Theorem 1.81) one of the hypotheses is that $A(x)$ satisfies a Tauberian condition of the "slowly oscillating" type:

$$A(x) - A(x - t) = O(t^{\gamma} V(x)) \tag{3}$$

for all x and $0<t<F(x)$, where $\gamma>0$, and where $V(x)$, $F(x)$ are certain positive functions of x. While the proof is logically valid, the truth of (3) for all x and all sufficiently small t implies that $A(t)$ is continuous, so that if (as appears to be taken for granted throughout the book) $A(t)$ is taken to be of the form (1), the hypotheses can be satisfied only in the trivial case in which $a_n = 0$ for all n. If we take $A(t)$ to be, say, any function which is Lebesgue integrable in any finite interval ($A^k(t)$ then being defined by the integral in (2)) the proof remains valid, and the theorem becomes significant. Perhaps we are intended to take the theorem in this more general sense, but there is no indication of this in the enunciation. Somewhat similar remarks apply to certain other theorems.

Chapter II deals with the "second theorem of consistency," and related matters. Without going into details, the two main theorems in this field can be roughly stated as follows:

(A). Let $\phi(x)$ be a positive function increasing to infinity as $x \to \infty$, which is such that, for some Δ, $\phi(x) = O(x^{\Delta})$, and which satisfies certain other conditions which ensure that, roughly speaking, it should behave in a "reasonably regular" way. Then any series summable (λ, k) to s is also summable $(\phi(\lambda), k)$ to s. This theorem, due in its original form to Hardy, was later proved under less restrictive conditions on $\phi(x)$ by K. A. Hirst.

(B). Let $\phi(x)$ be a positive function increasing to infinity as $x \to \infty$, which is such that, for any δ (however small), $\phi(x) = O(x^{\delta})$, and which satisfies similar "regularity conditions." Suppose that a given series, while not necessarily summable (λ, k), is summable (λ, k') to s for some k', and that its (λ, k) mean is $o(F(x))$, where $F(x)$ is a certain function depending on $\phi(x)$. Then the series is summable $(\phi(\lambda), k)$ to s. This theorem is due to Zygmund.

These two theorems have usually been regarded as quite separate theorems, albeit belonging to the same field of ideas and proved by similar methods. An interesting feature of the treatment in this book is that the authors have unified the subject by giving a general theorem which includes both (A) and (B) as special cases. This is done at the cost of imposing conditions on $\phi(x)$ which are slightly more restrictive than those imposed by Hirst, but this difference is not of great importance. Unfortunately, there is an error in the argument which renders the proof invalid as it stands. (The line immediately above equation (2.59) does not follow from the previous line if $p<0$.) It would, however, be possible to construct a correct proof without any very extensive alterations to the text.

One other remark may be made. The authors assert that, under the conditions of (B), the method $(\phi(\lambda), k)$ is more powerful than (λ, k). While this is no doubt the case for any "reasonable" sequence $\{\lambda_n\}$, it is surely not *always* true. For if λ_n increases so rapidly that the conditions of the "high indices theorem" are satisfied not only by $\{\lambda_n\}$, but also by $\{\phi(\lambda_n)\}$, then (λ, k) and $(\phi(\lambda), k)$ are both equivalent to convergence.

Chapter III is concerned with the summability (λ, k) and (e^{λ}, k) of the Dirichlet series

$$\sum_{n=0}^{\infty} a_n e^{-\lambda_n s}.$$

It was, of course, inevitable that this chapter should contain a certain amount of material which is also to be found in Hardy and M. Riesz's Cambridge Tract on Dirichlet series. But much other material is also included. For example, a fair amount of space is devoted to results concerning absolute Riesz summability, the theory of which has been developed only since the publication of the Tract. Again, the problem of obtaining sufficient conditions for the summability of the series at a point lying actually on the abscissa of summability, which is only briefly mentioned in the Tract, is dealt with at some length.

The title of Chapter IV (*Applications to Fourier series*) is perhaps slightly misleading; one might expect it to refer to single Fourier series, whereas the chapter is throughout concerned with the "spherical" summability of multiple Fourier series. But, of course, any theorem on k-fold Fourier series which is valid for all k will yield results on single Fourier series on taking $k=1$, and a number of important results on single Fourier series are, in fact, included as special cases of the theorems of this chapter. The greater part of this chapter is concerned with the results of quite recent research, and I found it of much interest. Unfortunately, it is marred by an error which renders the proof of one of the theorems invalid as it stands. (Equation (4.54) does not follow from the line above, and is, in any case, clearly false, for its truth for arbitrarily small t would imply that $S(x)$ was continuous.)

Apart from the errors which have been mentioned, the book contains a number of slips of minor importance.

B. KUTTNER

Volume and integral. By W. W. Rogosinski. New York, Interscience, 1952. 10+160 pp. $1.75.

This little book is an exposition of the classical Lebesgue theory of Euclidean n-space, following the outer-inner measure approach to measure theory, and the ordinate set approach to integration theory. The book is intended as an introduction to modern abstract integration theory. More specifically, the author has in mind the incorporation of Lebesgue theory into the undergraduate honors programs of English universities. He has therefore chosen to present a detailed exposition of a restricted portion of the subject.

The first chapter takes up the necessary amount of Boolean algebra (with countable operations) and point set topology in Euclidean n-space. The third chapter develops measure theory for n-space by first defining measure for countable unions of "intervals" and then using these sets to define outer and inner measure for general sets. The fifth chapter presents the ordinate set approach to integration, wherein the integral of a non-negative function is defined as the "volume" under its graph in $(n+1)$-space. Included are the basic convergence theorems and the Fubini theorem, but not L^p theory. The sixth chapter takes up the theory of differentiation in one dimension, based on the Vitali covering theorem.

The most novel feature of the book is the inclusion, in chapters two and four, of an independent parallel development of the theory of content and the theory of Riemann integration based on the theory of content. The student is thus led through the historical evolution of the subject, and can better appreciate the tremendous stride taken by Lebesgue.

Proofs of theorems are given in detail. Scattered throughout the book is a collection of 35 exercises, indications of their proofs being collected at the ends of chapters. The book is carefully written, and should admirably serve its purpose as an undergraduate text.

L. H. LOOMIS

Introduction to measure and integration. By M. E. Munroe. Cambridge, Addison-Wesley, 1953. 10+310 pp. $7.50.

In the last years, in particular since Halmos' book on measure theory, modern theory of integration has become a regular part of the mathematics curriculum. Munroe's book is one of the first designed specifically for students. It begins with the basic concepts of real function theory and covers the by now classical part of measure theory. The organization is logical in the sense that measure, measurability, integration, and differentiation are studied in that order. The exercises are selected so as to make the text better understood as well as to suggest further study. The presentation is clear and concise.

Chapter I introduces the concepts in use throughout the book: algebra of sets, additive classes, metric spaces, limits and continuity. Chapters II and III are devoted to the investigation of measures: construction and particular types. Beginning with properties of additive set functions and their Jordan decomposition, the author procedes to the construction of measures by means of outer, then metric outer, measures. Lebesgue-Stieltjes, Hausdorff, and Haar measures are considered in some detail. Measurable functions are introduced in Chapter IV. Operations on and approximations of these functions are carefully presented. Chapter V is devoted to integration. The definition of integrals begins with that of integrals of simple functions, then of non-negative measurable functions. The study of indefinite integrals leads to the Radon-Nikodým theorem. The Fubini theorem closes the chapter. Chapter VI, entitled convergence theorems, starts with the study of various types of convergence of measurable functions, and continues with that of L_p spaces, linear functionals on Banach spaces, orthogonal expansions in Hilbert spaces, and the mean ergodic theorem. The final Chapter VII on differentiation covers differentiation of additive set functions in euclidean spaces, metric density, and differentiation with respect to nets. The book contains also a few general concepts of probability theory.

There is no doubt that the author did a remarkable job in about three hundred pages. Yet, it seems to the reviewer that for students who are introduced for the first time to measure theory, the presentation is too abstract, too fast, and too soon. The reviewer's own teaching experience and bias would lead him to a somewhat different order of presentation. He would start with the algebra of sets and Borel sets and proceed at once to measurable functions, integration, and convergence theorems. Construction of measures and metric spaces would follow. The reviewer feels that the basic property—to be emphasized—of the family of measurable functions is its closure under the "usual" operations of analysis. The student is familiar with continuous functions and with the difficulties due to the fact that limits of convergent sequences of such functions are not necessarily continuous. It is thus easy to make him understand the importance and value of the closure property. Furthermore, it seems useful to avoid the possibility of the student's acquiring the misleading idea that the measurability concept is based on the measure concept, and also to point out that these two concepts are related only through the background of completely additive classes. However, the main reason for the suggested approach is didactic. This approach requires a few concepts only, hence it is easier to digest for the student who is not overburdened with new concepts. Also, since convergence theorems are basic in the modern theory of integration and in its use, it seems preferable to get to them as fast as possible and not to have to cover the first 220 pages. Construction of measures and study of metric spaces, which are far more abstract and involved, would come later.

Stripped of the concepts and details unnecessary to the approach outlined above, the part of the book relative to measurability, integration, and convergence theorems would make an excellent text for a one-semester course, possibly an undergraduate one. The foregoing remarks reflect mainly the fact that this well-written and well-rounded book may be used successfully in various ways.

M. Loève

Contributions to the theory of games. Vol. 2. Ed. by H. W. Kuhn and A. W. Tucker. (Annals of Mathematics Studies, no. 28.) Princeton University Press, 1953. 396 pp. $4.00.

This collection of various papers on the theory of games is the second such volume edited by Professors Kuhn and Tucker to form an Annals of Mathematics Study. The merit of this form of publication seems to the reviewer to be considerable, because the great extent and development of mathematics make a volume dedicated to a specialized topic an efficient way to reach specialists. A good many of the twenty-one papers in the volume are good and indicate that the subject has not yet lost its vitality or momentum.

The first section of the volume contains five papers on finite zero-sum two-person (z-s t-p) games. Paper no. 1, by v. Neumann, discusses the optimal assignment problem, i.e., the assignment of n persons to n jobs so as to maximize the value of the assignment. He shows that the problem is equivalent to solving a z-s t-p game which is simpler computationally than the original assignment problem. Paper no. 2 by Gillies, Mayberry, and v. Neumann, discusses two variants of poker. In paper no. 3 Motzkin, Raiffa, Thompson, and Thrall give a method for computing optimal solutions. In paper no. 4 Dresher and Karlin construct an algorithm for the solutions of games where the mixed strategies of the players constitute compact convex sets (in Euclidean space); the solutions are fixed points of appropriate mappings. In paper no. 5 Arrow, Barankin, and Blackwell discuss "admissible" points of a convex set.

The second section of the book is devoted to z-s t-p games where the players dispose of infinitely many pure strategies, e.g., where the pure strategies of a player are points in an interval. Shiffman in paper no. 6 and Karlin in paper no. 7 study games of timing, i.e., games on the unit square, say, where the payoff function is increasing in one variable and decreasing in the other. The name comes from the fact that the pure strategies may be interpreted as times at which the players act, and it is profitable for each player to delay his own action as long as possible, provided he acts in advance of his opponent. In paper no. 8 Karlin studies a class of games where the payoff function has a positive nth partial derivative with respect to one of the variables; this is a natural extension of the notion of a convex game. Glicksberg and Gross in paper no. 9 exhibit some games with rational payoff functions whose pathologic behavior disappoints the hope that certain properties of finite games could be extended to games with rational payoff functions. In the problem of statistical decision as formulated by Wald (and in certain other z-s t-p games), several writers have shown under various hypotheses the equivalence of mixing strategies before and after the chance variable (say) is observed. In paper no. 10 Blackwell studies a special case where any randomized strategy is equivalent to a mixture in fixed proportions of countably many pure strategies.

The third section of the volume is devoted to games in extensive form. In paper no. 11 Kuhn discusses a new formulation of the extensive form of a general n-person game which covers a larger class of games than the formulation of v. Neumann. He uses a geometric model, the successive presentation of alternatives in the game appears as the branching of a topological tree, and the patterns of information appear as partitions of the vertices of the tree. Several general results are proved for certain games in this form. (In his excellent book *Introduction to the theory of games*, McKinsey made use of Kuhn's formulation.) Dalkey (paper no. 12) adopts Kuhn's formulation and studies "variations in the structure of a game in extensive form which leave the major strategic properties of the game invariant irrespective of the payoff function." His paper extends to general games results of Krentel, McKinsey, and Quine (Duke Math. J. vol. 18 (1951) pp. 885–900). In paper no. 13 Gale and Stewart study z-s t-p games with perfect information (chess is such a game). It was proved by v. Neumann (e.g., v. Neumann and Morgenstern, *Theory of games and economic behavior*, 2d. ed., Princeton University Press, 1947, p. 112; see also E. Zermelo, *Ueber eine Anwendung der Mengenlehre auf die Theorie des Schachspiels*, Proceedings of the Fifth International Congress of Mathematicians, Cambridge, 1912, vol. II, p. 501) that every such game whose number of moves is finite has a solution in pure strategies. Gale and Stewart consider games where the number of moves is infinite, and among their results is the construction of a game with perfect information which has no solution in pure strategies. In paper no. 14, G. L. Thompson studies signaling strategies and applies his results in paper no. 15 to a model of the game of bridge. A signaling strategy for a player is "a pure strategy for that player restricted to that subset of his information sets which prevent him from having perfect recall." In paper no. 16, J. W. Milnor analyzes a situation which occurs in certain games, where "one can measure the 'incentive' to move at any particular configuration by imagining the possibility of passing instead."

The fourth and final section is devoted to *n*-person games. In paper no. 17 L. S. Shapley proposes to evaluate the equities of the players of an arbitrary *n*-person game. Whether such a game has a solution in the sense of v. Neumann and Morgenstern is an unsolved problem. In papers no. 18, no. 19, and no. 20, R. Bott, D. B. Gillies, and L. S. Shapley, respectively, introduce interesting classes of games for which they obtain solutions. In paper no. 21 H. Raiffa proposes "arbitration conventions" for choosing an imputation from the solution set of v. Neumann and Morgenstern.

Each section of the volume is preceded by an excellent editorial introduction which summarizes the various papers and indicates lines of further research. The volume itself is indispensable for students of the subject.

J. WOLFOWITZ

NEW JOURNALS

Mathematica Scandinavica. Vol. 1, no. 1. Copenhagen, 1953. 192+12 pp. 40 Danish crowns per volume of two numbers; 20 crowns to members of the sponsoring societies and members of societies (including the American Mathematical Society) having reciprocity agreements with them.

Nordisk Matematisk Tidskrift. Vol. 1, no. 1. Oslo, 1953. 64 pp. Foreign subscriptions, 22 Norwegian crowns per volume of four numbers.

These two journals replace Matematisk Tidsskrift A and B and Norsk Matematisk Tidsskrift. They are published by the five Scandinavian mathematical societies, the second in conjunction with the Scandinavian associations of teachers of mathematics. They are intended to take, more or less, the places in Scandinavia which are occupied in the U.S.A. by this Bulletin together with Proceedings of the American Mathematical Society and by the American Mathematical Monthly, respectively. Thus Mathematica Scandinavica publishes short research and expository papers (in English, French or German); as far as this portion goes, it differs from Proceedings of the American Mathematical Society chiefly in that the papers are supposed to be mostly by Scandinavian authors (all but one in this issue are); they are fewer in number (17 in this issue), more restricted in subject matter (reflecting the special interests of Scandinavian mathematicians, they are mostly in analysis), and more interesting (possibly reflecting the lesser pressure for publication which distinguishes European from American mathematical life). Mathematica Scandinavica also contains problems, book reviews, and notes on mathematics in Scandinavia; the latter include, in the first issue, a useful directory of mathematicians at Scandinavian institutions of higher education and research. The supplementary pages at the end contain, on one side, abstracts of the papers in this issue (suitable for cutting out and mounting on cards); on the other side, announcements of new publications.

Nordisk Matematisk Tidskrift corresponds more closely to its British counterpart, the Mathematical Gazette, than to the American Mathematical Monthly; it covers secondary-school as well as college mathematics. The contents range from short research papers on subjects of general interest to discussions on teaching problems. The languages of this journal are Danish, Norwegian and Swedish, although English, French and German are also admitted, and articles in the Scandinavian languages are provided with English summaries.

R. P. BOAS, JR.

BRIEF MENTION

Les systèmes axiomatiques de la théorie des ensembles. By H. Wang and R. McNaughton. (Collection de Logique Mathématique, Série A.) Paris, Gauthier-Villars, 1953. 56 pp. 750 fr.

The purpose of this booklet is to facilitate the comparison of various popular approaches to axiomatic set theory. In 27 pages the authors describe Cantor's "naive" set theory, the theory of types, Zermelo's axiom system and Fraenkel's modification of it, the von Neumann-Bernays axiom system, and the two systems (*New foundations and mathematical logic*) presented by Quine, and give a brief discussion of some of the pertinent problems of relative consistency. It is not surprising that the style of presentation is somewhat breathless. Its very conciseness, however, makes the book useful; students of the subject will appreciate the bird's-eye-view that the authors provide. The book concludes with a bibliography and several useful indexes. The bibliography is long enough to contain most of the helpful references but not so long as to be discouraging. (The order of the items is slightly bewildering; it appears to be roughly chronological.) The authors, and Mme. Destouches-Fevrier, the editor of the series, are to be congratulated on having produced a valuable addition to the library of every student (and teacher) of axiomatic set theory.

PAUL R. HALMOS

Topologie. Vol. 2. By C. Kuratowski. (Monografie Matematyczne, vol. 21.) 2d ed. Warsaw, Polish Mathematical Society, 1952. 8+444 pp. $6.00.

The first edition (1950) was reviewed in this Bulletin, vol. 58, p. 265; the second edition apparently differs only by the correction of some errors.

Recherches arithmétiques. By Ch.-Fr. Gauss. Trans. by A.-C.-M. Poullet-Delisle. Paris, Blanchard, 1953. 22+502 pp.

This is a facsimile of the 1807 translation of *Disquisitiones Arithmeticae*.

Table of arctan x. (National Bureau of Standards Applied Mathematics Series, vol. 26.) Washington, Government Printing Office, 1953. 14+170 pp. $1.75.

This is a reissue of Table 16 of the New York Mathematical Tables Project (1942), reviewed in this Bulletin vol. 49, p. 531.

A survey of modern algebra. By G. Birkhoff and S. MacLane. Rev. ed. New York, Macmillan, 1953. 12+472 pp. $6.50.

The first edition (1941) was reviewed in this Bulletin vol. 48,

p. 342. "We have added several important topics (equations of stable type, dual spaces, the projective group, the Jordan and rational canonical forms for matrices, etc.)."

Symposium sobre algunos problemas matemáticas que se están estudiando en Latino América. Punta del Este, 19–21 *Diciembre* 1951. Montevideo, Centro de Cooperacion Científica para América Latina, n.d. 184 pp.

This volume contains 14 lectures on topics in functional analysis, integral geometry, calculus of variations, group theory, ergodic theory, differential equations, electromagnetic theory, functions of several complex variables, relativity, and Laplace transforms.

RESEARCH PROBLEMS

The department of RESEARCH PROBLEMS will publish the statements of problems whose solution would make a significant contribution to mathematical research. Problems which are suitable for publication in the problem department of the American Mathematical Monthly will not be accepted for publication in RESEARCH PROBLEMS. Only problems whose solutions are unknown to the author should be submitted. Furthermore, the problems desired are those for which the solution will take the form of a research paper to be accepted on its merits and published in a research journal; since the BULLETIN does not accept contributed papers, it will not publish the solutions of its research problems. An attempt will be made, however, to publish references to papers which contain solutions.

The readers of the BULLETIN are invited to contribute problems to the department of RESEARCH PROBLEMS. Each problem should carry the name of the author and a brief title and should be written in a single paragraph in a form similar to an abstract, and in nontechnical language if possible. Relevant references should be included. All problems intended for publication should be sent to G. B. Price.

1. Einar Hille: *On the zeros of a certain class of Fourier transforms.*

In the theory of analytic continuation in the meromorphic star developed by H. von Koch in Arkiv för Matematik, Astronomi och Fysik vol. 12, no. 23 (1917) he encountered the integrals

$$f_n(x) = \int_0^\infty \frac{t^n}{\Gamma(t)} e^{itx}dt.$$

For his theory it was essential to show that $f_n(x) \neq 0$ for $n = 0, 1, 2, \cdots$, but this he was unable to do in spite of several efforts in which his pupils also shared. Prove or disprove this conjecture as well as the more general one that $f_n(x) \neq 0$ for real values of x. (Received October 8, 1953.)

2. R. E. Johnson: *Quotient rings.*

If R is a subring of the ring S having the property that $aR \cap R \neq 0$ for each nonzero $a \in S$, then S is called a (right) ***quotient ring*** of R. The set $Q(R)$ of all quotient rings of R has maximal elements by Zorn's lemma. *Question* 1. If $S \in Q(R)$, is $Q(S) \subseteq Q(R)$? *Question* 2. Are the maximal elements of $Q(R)$ isomorphic to each other? If the answer in either case is no, then it would be interesting to know for what types of rings the answer is yes. A general theory of quotient rings would make a significant contribution to the structure theory of rings. A few results on quotient rings are to be found in Proc. Amer. Math. Soc. vol. 2 (1951) p. 895 and in Duke Math. J. vol. 18 (1951) p. 808. (Received October 12, 1953.)

3. R. P. Boas, Jr.: *The absolute value of an absolutely convergent Fourier series.*

If $f(x)$ has an absolutely convergent Fourier series, does $|f(x)|$ have an absolutely convergent Fourier series? A negative answer would supply a simple example of a phenomenon discovered by Marcinkiewicz [Mathematica, Cluj vol. 16 (1940) pp. 66–73]: that there exist functions f and g with absolutely convergent Fourier series such that $g(f)$ does not have an absolutely convergent Fourier series. (Received October 16, 1953.)

4. R. P. Boas, Jr.: *The limit of the absolute value of an entire function.*

Let $f(z)$ be an entire function of exponential type. It is well known, and easily proved by Phragmén-Lindelöf arguments, that if $f(x)$ is bounded or approaches a limit as $x \to \infty$, then $f(x+iy)$ is bounded or approaches a limit, for each y, as $x \to \infty$. If $|f(x)|$ approaches a nonzero limit, does $|f(x+iy)|$ necessarily approach a limit? (Received October 16, 1953.)

5. R. P. Boas, Jr.: *Power series with positive coefficients.*

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has $|z| = 1$ as its circle of convergence, and $a_n \geq 0$, then in a general way the behavior of $f(z)$ on an arc of the circle near $z = 1$ governs the behavior of the function on the whole circle. If $f(z)$ has boundary values $f(e^{i\theta})$ belonging to L^2 on a small arc $|\theta| < \epsilon$, N. Wiener has shown (unpublished) that boundary values exist for all θ and $f(z)$ belongs to L^2 on the whole circle. Does this theorem extend to L^p for other values of p, and in particular for $p = 1$? (Received October 16, 1953.)

6. R. M. Thrall: *Simultaneous congruence of skew symmetric matrices over a finite field.*

Let F be a finite field, and let A, B, C, D be skew symmetric F-matrices of degree n. Find necessary and sufficient conditions for the existence of a matrix P such that $PAP^{tr} = C$, $PBP^{tr} = D$. This problem arises in the study of metabelian groups. (Received October 19, 1953.)

7. R. M. Thrall: *Multiplication of Schur functions.*

In the theory of group characters, certain symmetric functions called *Schur functions* are important [Cf. D. E. Littlewood, *Theory of group characters*, Oxford, 1940, p. 82 ff.]. Each partition $\lambda: \lambda_1 + \cdots + \lambda_n = n$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, of n determines a homogeneous symmetric function $\{\lambda\}$ of degree n. If μ is a partition of m, then

$$\{\lambda\}\{\mu\} = \sum k(\lambda, \mu, \nu)\{\nu\}$$

where the sum is over all partitions ν of $m+n$. Find an analytic formula for the coefficients $k(\lambda, \mu, \nu)$. Combinatorial rules are known for the determination of these coefficients. (Received October 19, 1953.)

NOTES

The editors of the Transactions wish to acknowledge the services of the following persons, not members of the Editorial Committee, who have been consulted regarding papers submitted for publication: R. P. Agnew, Frederick Bagemihl, P. T. Bateman, R. E. Bellman, R. P. Boas, Paul Civin, J. A. Dieudonné, Arthur Erdélyi, N. J. Fine, P. R. Garabedian, Hilda Geiringer, Abe Gelbart, G. A. Hedlund, M. H. Heins, Edwin Hewitt, I. I. Hirschman, J. A. Jenkins, Meyer Jerison, R. E. Johnson, A. W. Jones, Bjarni Jónsson, Mark Kac, Shizuo Kakutani, Samuel Karlin, Kunihiko Kodaira, J. L. Koszul, Joseph Lehner, G. G. Lorentz, L. H. Loomis, E. J. McShane, Morris Marden, Wilhelm Magnus, W. T. Martin, A. P. Morse, M. E. Munroe, Zeev Nehari, David Nelson, Ivan Niven, R. S. Phillips, G. B. Price, Hans Rademacher, C. E. Rickart, M. S. Robertson, Alex Rosenberg, H. L. Royden, Arthur Sard, Leo Sario, M. M. Schiffer, I. E. Segal, Atle Selberg, Max Shiffman, N. E. Steenrod, Walter Strodt, R. J. Walker, A. L. Whiteman.

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Applications for the predoctoral fellowship of Sigma Delta Epsilon, Graduate Women's Scientific Fraternity, for the year 1954–1955 should be submitted before February 1, 1954. Applicants must have the equivalent of a Master's degree. Application blanks may be secured from Dr. Esther S. Anderson, Geography Department, University of Nebraska, Lincoln 8, Nebraska.

The Research Corporation awarded a grant to the Canadian Mathematical Congress to enable groups of younger mathematicians from smaller Canadian colleges to further their independent studies at the Summer Research Institute which was held at Queen's University.

The National Science Foundation has announced grants in mathematics to the following institutions, to support studies by the professors indicated: Haverford College, R. C. James; Illinois Institute of Technology, L. R. Wilcox; Northwestern University, Alex Rosenberg and Daniel Zelinsky; Purdue Research Foundation, C. R. Putnam; University of Virginia, E. J. McShane; Wayne University, Casper Goffman; University of Wisconsin, R. H. Bing.

Dr. W. J. Coles of Duke University has been awarded a Fulbright Scholarship for 1953–54 and is at Cambridge University.

Professor E. J. Gumbel of Columbia University has been elected a member of the International Statistical Institute at The Hague.

Professor A. E. Heins of Carnegie Institute of Technology has been awarded a Guggenheim Fellowship and is studying in Denmark.

Professor Dora McFarland of the University of Oklahoma has received a $500 University of Oklahoma Foundation Teaching Award for extraordinary excellence in student counciling and teaching of freshmen and sophomores.

Dr. D. B. Ray of Cornell University has been awarded a Jewett Postdoctoral Fellowship and is at Princeton University.

Professor J. B. Rosser of Cornell University has been awarded a Fulbright Grant and is studying at the Sorbonne.

Professor Ernst Snapper of the University of Southern California has been awarded a National Science Foundation Postdoctoral Fellowship and is at Harvard University.

Professor J. G. van der Corput of the University of Amsterdam is on leave of absence and has been appointed to a visiting professorship at the University of California, Berkeley.

Dr. S. H. Khamis of the Statistical Office, United Nations, has been appointed to an associate professorship at the American University of Beirut.

Associate Professor Elisha Netanyahu of the Hebrew Technical College is on leave of absence and has been appointed to a visiting professorship at Stanford University.

Associate Professor Richard Arens of the University of California, Los Angeles, is on leave of absence at the Institute for Advanced Study.

Mr. H. W. Baeumler of the University of Buffalo has been appointed to an assistant professorship at Marshall College.

Dr. V. N. Behrns of the University of Buffalo has accepted a position as senior operations analyst with the Consolidated Vultee Aircraft Corporation, Fort Worth, Texas.

Dr. S. E. Benesch of the University of Illinois has accepted a position as research engineer with the Jet Propulsion Laboratory, California Institute of Technology. (This name was incorrectly given as S. E. Banach in the November, 1953 issue of the Bulletin.)

Dr. Joseph Bram of the University of Chicago has accepted a position as analyst with the Engineering Research Associates, Arlington, Virginia.

Dr. C. C. Buck of the University of Michigan has been appointed to an assistant professorship at the University of Alabama.

Associate Professor A. P. Calderón of the Ohio State University is on leave of absence at the Institute for Advanced Study.

Mr. K. V. Casey has accepted a position as programming engineer with the General Electric Company, Cincinnati, Ohio.

Associate Professor C. E. Clark of Emory University has accepted a position as operations analyst with the Operations Research Office, Johns Hopkins University, Chevy Chase, Maryland.

Dr. J. H. Curtiss of the National Bureau of Standards has been appointed to an adjunct professorship at New York University.

Dr. Elizabeth Cuthill of the University of Maryland has accepted a position as mathematician with the David Taylor Model Basin, Carderock, Maryland.

Dr. Leila A. Dragonette of the University of Chicago has accepted a position as mathematician with the Office of Naval Research, Washington, D. C.

Dr. M. S. Edwards of the Ohio State University has accepted a position as mathematician at the Wright-Patterson Air Force Base, Dayton, Ohio.

Assistant Professor Bernard Epstein of the University of Pennsylvania is on leave of absence and has been appointed a research associate at the Institute for Mathematics and Mechanics, New York University.

Dr. Paul Erdős of the National Bureau of Standards, Los Angeles, has been appointed to a visiting professorship at the University of Notre Dame.

Associate Professor A. C. Eringen of the Illinois Institute of Technology has been appointed to an associate professorship at Purdue University.

Assistant Professor Trevor Evans of Emory University is on leave of absence and has been appointed a research associate at the University of Chicago.

Mr. A. V. Fend of the University of Illinois has been appointed to an assistant professorship at Florida State University.

Miss Bernice Goldberg of the Air Force Cambridge Research Center has accepted a position as technical engineer with the General Electric Company, Evandale, Ohio.

Dr. Arthur Grad of the Office of Naval Research is on leave of absence and has been appointed a mathematician at the Institute of Mathematical Sciences, New York University.

Mr. R. R. Hare, Jr. of the Air Force Missile Test Center, Patrick Air Force Base, Cocoa, Florida, has accepted a position as operations analyst with the Operations Research Office, Johns Hopkins University, Chevy Chase, Maryland.

Dr. Joseph Hashisaki of the University of Illinois has been appointed to an assistant professorship at the Montana State University.

Mr. P. S. Herwitz of the University of North Carolina has accepted a position as research associate with the Institute for Cooperative Research, Baltimore, Maryland.

Dr. T. J. Jaramillo of the Illinois Institute of Technology has accepted a position as senior research engineer with the Chicago Midway Laboratories, University of Chicago.

Associate Professor T. L. Jordan, Jr. of Wofford College has accepted a position as staff member with the Los Alamos Scientific Laboratory.

Assistant Professor Dora E. Kearney of Westminster College has been appointed to an associate professorship at Upper Iowa University.

Dr. Erwin Kleinfeld of the University of Chicago has been appointed to an assistant professorship at the Ohio State University.

Dr. R. J. Lambert of the Defense Department has been appointed to an assistant professorship at the Iowa State College of Agriculture and Mechanic Arts.

Associate Professor Mary Ann Lee of Sweet Briar College is on leave of absence and has accepted a position as assistant mathematician at the Rand Corporation.

Professor Walter Leighton of Washington University has been appointed to a professorship at Carnegie Institute of Technology.

Dr. Benjamin Lepson of the Office of Naval Research has been appointed to an assistant professorship at the Catholic University of America.

Professor H. B. Leonard of the University of Arizona has retired with the title professor emeritus.

Professor Eugene Lukacs of Our Lady of Cincinnati College has accepted a position as head of the Statistical Branch, Office of Naval Research, and has been appointed a lecturer at American University.

Dr. John McCarthy of Princeton University has been appointed to an acting assistant professorship at Stanford University.

Mr. J. E. McGaughy of Columbia University has accepted a position as mathematician with the Vitro Corporation.

Dr. J. D. McKnight of Purdue University has accepted a position as senior aerophysics engineer with the Consolidated Vultee Aircraft Corporation, Fort Worth, Texas.

Dr. Murray Mannos of the National Bureau of Standards has accepted a position as mathematician with the Raytheon Manufacturing Company, Waltham, Massachusetts.

Mr. J. M. Marr of the University of Tennessee has been appointed to an assistant professorship at the Kansas State College of Agriculture and Applied Science.

Dr. N. M. Martin of the University of California, Los Angeles, has accepted a position as research associate with the Willow Run Research Center, University of Michigan.

Dr. O. B. Moan of the International Business Machines Corporation has accepted a position as staff quality control engineer with the Hughes Aircraft Company, Culver City, California.

Professor C. T. Molloy of the Polytechnic Institute of Brooklyn has accepted a position as head of physics research, Vitro Corporation of America.

Professor G. D. Mostow of Johns Hopkins University is on leave of absence and has been appointed to a visiting professorship at the Instituto de Matemática Pura e Aplicada, Rio de Janeiro.

Dr. D. M. Nead of Indiana University has accepted a position as scientist with the Rohm and Haas Company, Redstone Arsenal, Huntsville, Alabama.

Mr. A. B. Neale of the Marquardt Aircraft Company has accepted a position as mechanical engineer with Hydro-Aire, Incorporated, Burbank, California.

Dr. E. S. Northam of Michigan State College has accepted a position as mathematician with the Bendix Aviation Corporation, Detroit, Michigan.

Dr. R. S. Novosad of the Tulane University of Louisiana has been appointed to an assistant professorship at the Pennsylvania State University.

Mr. E. M. Olson of Columbia University has accepted a position a mathematician with the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

Dr. R. H. Owens of Brown University has accepted a position as physical sciences coordinator with the Office of Naval Research, Pasadena, California.

Assistant Professor R. P. Peterson of the University of Washington has been appointed to an assistant professorship at the University of California, Riverside.

Mr. S. L. Pollak of the National Bureau of Standards has accepted a position as mathematician with the Raytheon Manufacturing Company, Waltham, Massachusetts.

Professor Emeritus George Pólya of Stanford University has been appointed to a visiting professorship at the Swiss Federal Institute of Technology.

Mr. M. B. Ritterman of Long Island University has accepted a position as senior engineer with Sylvania Electric Products, Incorporated, Whitestone, New York.

Mr. Calvin A. Rogers of the University of Washington has been appointed to an assistant professorship at the Colorado Agricultural and Mechanical College.

Dr. Mary Estill Rudin of Duke University has been appointed to an assistant professorship at the University of Rochester.

Associate Professor M. A. Sadowsky of the Illinois Institute of Technology has been appointed to a professorship at the Rensselaer Polytechnic Institute.

Miss Jean E. Sammet of the Metropolitan Life Insurance Company has accepted a position as assistant project engineer with the Sperry Gyroscope Company, Great Neck, New York.

Assistant Professor J. W. Sawyer of the University of Georgia has been appointed to an associate professorship at the University of Richmond.

Associate Professor I. E. Segal of the University of Chicago is on leave of absence and has been appointed to a visiting associate professorship at Columbia University.

Dr. V. L. Shapiro of Rutgers University is on leave of absence at the Institute for Advanced Study.

Dr. K. S. Shih of the University of Illinois has been appointed to a visiting assistant professorship at Washington University.

Professor L. L. Smail of Lehigh University has retired with the title professor emeritus.

Mr. G. T. Thompson of the Oregon State College has accepted a position as mathematician with the Jet Propulsion Laboratory, California Institute of Technology.

Associate Professor Emeritus J. I. Tracey of Yale University has been appointed to a professorship at Texas Christian University.

Miss Jane A. Uhran of Indiana University has been appointed to an assistant professorship at Butler University.

Dr. D. H. Wagner of the Operations Evaluation Group, Navy Department, is on leave of absence at the Massachusetts Institute of Technology.

Associate Professor Marion Wetzel of Denison University is on leave of absence at Oregon State College.

Mr. E. L. Whitney of Dartmouth College has accepted a position as scientific services officer with the Defense Research Board, Ottawa, Ontario, Canada.

Dr. N. A. Wiegmann of the National Bureau of Standards has been appointed to a visiting professorship at the Catholic University of America.

Professor W. L. G. Williams of McGill University has retired.

Professor Antoni Zygmund of the University of Chicago is on leave of absence in England.

The following promotions have been announced:

F. J. Arena, North Dakota State College, to an assistant professorship.

J. L. Brenner, State College of Washington, to an associate professorship.

Jewell H. Bushey, Hunter College, to a professorship.

P. A. Clement, State College of Washington, to an assistant professorship.

H. K. Crowder, Case Institute of Technology, to an assistant professorship.

Mary H. Cummings, University of Rhode Island, to an associate professorship.

Frederic Cunningham, University of New Hampshire, to an assistant professorship.

M. D. Donsker, University of Minnesota, to an associate professorship.

J. E. Eaton, Queens College, to an associate professorship.

Harley Flanders, University of California, Berkeley, to an assistant professorship.

Bernard Friedman, New York University, to a professorship.

B. R. Gelbaum, University of Minnesota, to an associate professorship.

E. L. Godfrey, Defiance College, to a professorship.

Christopher Gregory, University of Hawaii, to a professorship.

J. G. Hacker, State College of Washington, to director of the J. R. Jewett Observatory.

B. F. Hadnot, Florida State University, to an assistant professorship.

G. A. Hunt, Cornell University, to an associate professorship.

C. B. Lindquist, University of Minnesota, to a professorship.

W. R. Mann, University of North Carolina, to an associate professorship.

G. W. Medlin, Wake Forest College, to an assistant professorship.

J. H. Mulligan, Jr., New York University, to a professorship.

Russell Remage, University of Delaware, to an associate professorship.

H. B. Ribeiro, University of Nebraska, to a professorship.

C. L. Riggs, Texas Technological College, to an associate professorship.

Abraham Seidenberg, University of California, Berkeley, to an associate professorship.

M. E. Shanks, Purdue University, to a professorship.

G. L. Spencer, University of Maryland, to an assistant professorship.

F. J. Weyl, Office of Naval Research, to director of the Mathematical Sciences Division.

P. A. White, University of Southern California, to a professorship.

J. H. Wolfe, University of Utah, to an associate professorship.

The following appointments to instructorships have been announced: Brigham Young University: Mr. Harvey Fletcher; University of California, Berkeley: Dr. Anne C. Davis, Dr. M. P. Epstein, Dr. J. P. Roth; University of California, Davis; Dr. R. L. Blair; Carnegie Institute of Technology: Dr. D. H. Shaffer; Case Institute of Technology: Dr. E. B. Leach; University of Chicago: Dr. Maurice Auslander, Dr. Serge Lang; University of Colorado: Mr. L. F. Boron, Dr. Aboulghassem Zirakzadeh; University of Connecticut: Mr. A. E. Nussbaum, Dr. E. E. Osborne; Cornell University: Dr. Stevan Orey; Dartmouth College: Mr. Karel DeLeeuw; University of Delaware: Dr. E. J. Pellicciaro; Duke University: Dr. L. R. Ford, Jr.; Indiana University: Dr. J. R. Blum; Iowa State College of Agriculture and Mechanic Arts: Dr. W. D. Lindstrom, Dr. D. E. Sanderson, Dr. R. D. Stalley, Dr. F. M. Wright; Lehigh University: Dr. R. C. Carson; University of Minnesota: Mr. G. U. Brauer, Dr. L. W. Green, Dr. J. M. Slye; Ohio University: Mr. J. G. Elliott; University of Oregon: Dr. R. L. San Soucie; Princeton University: Mr. D. A. Buchsbaum, Mr. R. H. Crowell, Mr. H. A. Forrester, Mr. G. A. Hufford, Mr. R. J. Semple; Purdue University: Dr. Robert Baer, Mr. N. D. Kazarinoff, Dr. J. J. Levin, Dr. R. E. Zink; Reed College: Dr. D. D. Aufenkamp; Washington University: Dr. Anne E. Scheerer; State College of Washington: Dr. W. J. Firey, Dr. W. M. Gilbert; Yale University: Mr. Louis Auslander.

Professor J. J. Corliss of the University of Illinois died on October 29, 1953 at the age of fifty-seven years. He had been a member of the Society for twenty-three years.

Professor Daniel Resch of Northwestern University died in July, 1953 at the age of twenty-nine years.

Mr. T. E. Snook of New York City died on August 13, 1953. He had been a member of the Society for thirty-three years.

NEW PUBLICATIONS

Bellman, R. *An introduction to the theory of dynamic programming*. Santa Monica, Rand Corporation, 1953. 10+99 pp.

Bochner, S. See Yano, K.

Booth, A. D., and Booth, K. H. V. *Automatic digital calculators*. New York, Academic Press, 1953. 231 pp., 1 plate. $6.00.

Booth, K. H. V. See Booth, A. D.

du Boscq de Beaumont, H. *Contribution au dernier théorème de Fermat*. Paris, Editions Industrielles, Techniques et Littéraires, 1952.

Cafiero, F. *Funzioni additive d'insieme ed integrazione negli spazi astratti*. Naples, Liguori, 1953. 178 pp.

Corson, E. M. *Introduction to tensors, spinors, and relativistic wave-equations (relation structure)*. New York, Hafner, 1953. 12+221 pp. $10.00.

Courant, R., and Hilbert, D. *Methods of mathematical physics*. Vol. I. New York, Interscience, 1953. 15+561 pp. $9.50.

Eves, H. *An introduction to the history of mathematics*. New York, Rinehart, 1953. 15+422 pp. $6.00.

Fejes Tóth, L. *Lagerungen in der Ebene, auf der Kugel und im Raum*. Berlin, Springer, 1953. 10+197 pp. 24.00 DM; bound 27.00 DM.

Fluid dynamics. (Proceedings of Symposia in Applied Mathematics, vol. 4.) New York, McGraw-Hill, 1953. 6+186 pp. $7.00.

Gonseth, F. *La géométrie et le problème de l'espace*. V. *Les géométries non euclidiennes*. Neuchâtel, Griffon, 1952. 110 pp.

Goodstein, R. L., and Primrose, E. J. F. *Axiomatic projective geometry*. Leicester, University College, 1953. 11+140 pp. 15s.

Gromeka, I. S. *Sobranie sočinenii*. Moscow, Izdatel'stvo Akad. Nauk SSSR, 1952. 296 pp. 23.40 rubles.

Gross, B. *Mathematical structure of the theories of viscoelasticity*. (Actualités Scientifiques et Industrielles, no. 1190.) Paris, Hermann, 1953. 74 pp. 600 fr.

Hansen, M. H., Hurwitz, W. N., and Madow, W. G. *Sample survey methods and theory*. Vol. I. *Methods and applications*. New York, Wiley, 1953. 22+638 pp. $8.00.

Hansen, M. H., Hurwitz, W. N., and Madow, W. G. *Sample survey methods and theory*. Vol. II. *Theory*. New York, Wiley, 1953. 8+332 pp. $7.00.

Hilbert, D. *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*. New York, Chelsea, 1953. 26+282 pp. $4.50.

Hilbert, D. See Courant, R.

Hoffman, O., and Sachs, G. *Introduction to the theory of plasticity for engineers*. New York, McGraw-Hill, 1953. 14+276 pp. $6.50.

Hurwitz, W. N. See Hansen, M. H.

Kline, M. *Mathematics in western culture*. New York, Oxford University Press, 1953. 12+484 pp., 27 plates. $7.50.

Lietzmann, W. *Riesen und Zwerge im Zahlenreich*. Stuttgart, Teubner, 1953. 60 pp. 2.40 DM.

Lindley, D. V., and Miller, J. C. P. *Cambridge elementary statistical tables*. Cambridge University Press, 1953. 36 pp.

LOOMIS, E. S. *Original investigation or how to attack an exercise in geometry.* Columbus, Ohio, The Bonded Scale and Machine Co., 1952. 12+63 pp. Gratis.

LUDFORD, G. S. S. *Riemann's method of integration: its extensions with an application.* (Institute for Fluid Dynamics and Applied Mathematics, Lecture Series, no. 26.) College Park, University of Maryland, 1952. 6+34 pp. (mimeographed)

LORENTZ, G. G. *Bernstein polynomials.* (Mathematical Expositions, no. 8.) University of Toronto Press, 1953. 10+130 pp. $5.75.

LYAPUNOV, A. A. *R-množestva.* (Trudy Mat. Inst. Steklov., vol. 40.) Moscow, Izdatel'stvo Akad. Nauk SSSR, 1953. 67 pp. 2.80 rubles.

MADOW, W. G. See HANSEN, M. H.

MILLER, J. C. P. See LINDLEY, D. V.

MOSTOWSKI, A., and STARK, M. *Algebra Wyższa.* First part. (Biblioteka Matematyczna, vol. 1.) Warsaw, Polskie Towarzystwo Matematyczne, 1953. 7+308 pp.

NARAYAN, S. *A text book of matrices.* Delhi, Chand, 1953. 7+289 pp.

NASVYTIS, A. *Die Gesetzmässigkeiten kombinatorischer Technik.* Berlin, Springer, 1953. 8+103 pp. 9.00 DM.

NEVANLINNA, R. *Uniformisierung.* Berlin, Springer, 1953. 10+391 pp. 46.50 DM; bound, 49.50 DM.

NORTHCOTT, D. G. *Ideal theory.* (Cambridge Tracts in Mathematics and Mathematical Physics, no. 42.) Cambridge University Press, 1953. 8+111 pp. $2.50.

PARS, L. A. *Introduction to dynamics.* Cambridge University Press, 1953. 22+501 pp. $6.00.

PRIMROSE, E. J. F. See GOODSTEIN, R. L.

Proceedings of the Third Midwestern Conference on Fluid Mechanics, University of Minnesota, March 23, 24, *and* 25, 1953. Minneapolis, University of Minnesota, 1953. 634 pp. $6.00.

RIESZ, F., and SZ.-NAGY, B. *Leçons d'analyse fonctionnelle.* 2d ed. Budapest, Akadémiai Kiadó, 1953. 8+455 pp. 100 ft.

ROHRBERG, A. *Theorie und Praxis des logarithmischen Rechenstabes.* Stuttgart, Teubner, 1953. 64 pp. 2.40 DM.

SACHS, G. See HOFFMAN, O.

SALZER, H. E. *Tables of coefficients for the numerical calculation of Laplace transforms.* (National Bureau of Standards Applied Mathematics Series, no. 30.) Washington, Government Printing Office. 2+36 pp. $0.25.

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1953

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[1] Professor Feller was on leave of absence from February through August 1953. His place on the Mathematical Reviews Editorial Committee was filled by Professor R. P. Boas.

[2] Professor Eilenberg was on leave of absence starting September, 1953. His place on the Board of Editors of the American Journal of Mathematics was filled by Professor E. R Kolchin.

[3] Professor Wilder was elected a member of the Colloquium Editorial Committee for the period 1950–52. He was elected a member of the Executive Committee for the period 1952–53 and thus remained on the Council during 1953.

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SOME NEW ALGEBRAIC METHODS IN TOPOLOGY

W. S. MASSEY

1. **Introduction.** The purpose of this address is solely expository. It is intended to give the mathematician who is not an expert in algebraic topology a picture of some of the newer algebraic techniques and machinery which have recently become common in that subject. The expert is warned that this exposition contains no methods or results which have not already been published.

We shall concentrate attention on the spectral sequence, a topic initiated by Leray. By its use, one can investigate the homology structure of a fibre space in terms of the base space and fibre. This method has been applied with considerable success to the study of the topological structure of Lie groups and homogeneous manifolds. Other important applications have been made in the subject of differential geometry in the large.

2. **Graded groups.** In algebraic topology, one associates with each topological space certain algebraic structures, such as groups, rings, vector spaces, modules, etc. In this address, we shall for the sake of simplicity restrict our attention mainly to certain abelian groups which are associated with topological spaces. Almost everything we shall say could be equally well applied to the case of vector spaces over a given field, or more generally, to modules over a given commutative ring. And with a little additional effort, we could consider the various rings that are associated with a space.

Usually it turns out that one associates with a topological space X not a single abelian group, but a whole sequence of abelian groups. The most important examples are the following:

The n-dimensional homology group of X with coefficients in an arbitrary abelian group G, denoted by $H_n(X, G)$ $(n=0, 1, 2, \cdots)$.

The n-dimensional cohomology group of X with coefficients in an arbitrary abelian group G, denoted by $H^n(X, G)$ $(n=0, 1, 2, \cdots)$.

The n-dimensional homotopy group of X, denoted by $\pi_n(X)$ $(n=1, 2, 3, \cdots)$. These groups were introduced by Hurewicz in 1935. $\pi_1(X)$ is the ordinary fundamental group, which need not be abelian. However, for $n>1$, $\pi_n(X)$ is abelian.

These groups are the very basis of algebraic topology. Many im-

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portant topological properties of a space X are reflected in certain algebraic properties of these groups. Many questions of a purely topological nature about spaces can be reduced to questions of a purely algebraic nature about the homology, cohomology, or homotopy groups of X. For these, and various other reasons, it is important to be able to determine as much as possible about their structure, and to determine any relations that may exist between the groups associated with different spaces under various conditions.

When one has to consider sequences of abelian groups, as in the examples just mentioned, it is often more convenient to collect all the groups together into one bigger group, and then consider that one bigger group instead. One does this by forming the (weak) direct sum of all the groups of the given sequence. The larger group which results from this process is now generally called a *graded group*; it is the direct sum of a certain indexed sequence of subgroups. Thus as examples we have the graded groups

$$H(X, G) = \sum_{n=0}^{\infty} H_n(X, G), \qquad H^*(X, G) = \sum_{n=0}^{\infty} H^n(X, G),$$

associated with any pair consisting of a topological space X and an abelian group G.[1]

3. Groups with a differential operator. One naturally asks at this point how the homology, cohomology, and homotopy groups of a space are defined. To go into this question in detail would take us too much time. However, there is a standard algebraic procedure by which the homology and cohomology groups of a space are derived from what are called a group of chains and a group of cochains respectively. It is the purpose of this section to explain this procedure.

Let A be an abelian group. An endomorphism, $d: A \to A$, of A is called a *differential operator* if $d^2 = 0$, i.e., for any $x \in A$, $d[d(x)] = 0$. A pair (A, d) consisting of an abelian group A and a differential operator d on A is called a *differential group*. Given a differential group (A, d), we shall denote by $\mathcal{Z}(A)$ the kernel of d, and by $\mathcal{B}(A)$ the image,[2] $d(A)$. The condition $d^2 = 0$ implies that $\mathcal{B}(A) \subset \mathcal{Z}(A)$, and hence one may form the factor group $\mathcal{Z}(A)/\mathcal{B}(A)$. This factor group will be denoted by $\mathcal{H}(A)$, and called the *derived group*.

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[1] It is customary to extend the above definitions by defining $H_n(X, G)$ and $H^n(X, G)$ for n negative to be the trivial group consisting of the zero element alone.

[2] In topological applications, $\mathcal{Z}(A)$ is generally called the group of cycles or cocycles. Similarly, $\mathcal{B}(A)$ is called the group of boundaries, or coboundaries, or bounding cycles.

For any topological space X and coefficient group G one may define in various ways two differential groups: the group of chains, $(C(X, G), \partial)$, and the group of cochains, $(C^*(X, G), \delta)$. The differential operators $\partial: C(X, G) \to C(X, G)$ and $\delta: C^*(X, G) \to C^*(X, G)$ are called the boundary and coboundary operators respectively. Although one can define the groups of chains and cochains in many different ways, it is a fundamental theorem (or rather, set of theorems) of algebraic topology that the derived groups are always the same (up to an isomorphism) for "reasonably nice" spaces. These derived groups are by definition the homology and cohomology groups, $H(X, G)$ and $H^*(X, G)$ respectively.

The graded structure on $H(X, G)$ and $H^*(X, G)$ came from the fact that $C(X, G)$ and $C^*(X, G)$ have a naturally defined graded structure:

$$C(X, G) = \sum_n C_n(X, G),$$

$$C^*(X, G) = \sum_n C^n(X, G). \qquad \text{(direct sum)}$$

Here $C_n(X, G)$ is called the group of n-dimensional chains, and $C^n(X, G)$ is called the group of n-dimensional cochains. With respect to these graded structures, the differential operators ∂ and δ always turn out to be homogeneous of degrees -1 and $+1$ respectively, i.e.,

$$\partial[C_p(X)] \subset C_{p-1}(X), \qquad \delta[C^p(X)] \subset C^{p+1}(X).$$

From this it follows immediately that the subgroups of cycles and bounding cycles also split up into direct sums of sequences of subgroups. Therefore the factor group, cycles modulo bounding cycles, also has a graded structure.

This process of forming the derived group, $\mathcal{H}(A)$, from a given differential group, (A, d), is very common in algebraic topology. Moreover, it is also becoming common in some other branches of mathematics.

4. Exact triangles of groups. Another frequently occurring algebraic notion in algebraic topology is that of an "exact triangle" of groups and homomorphisms. Suppose we have three abelian groups, A, B, and C, and homomorphisms $f: A \to B$, $g: B \to C$, $h: C \to A$. These groups and homomorphisms may be conveniently exhibited by a triangular diagram, as follows:

$$\begin{tikzcd}
A \arrow[rr] & & B \arrow[dl] \\
& C \arrow[ul] &
\end{tikzcd}$$

This triangle is said to be *exact* in case the image of each homomorphism is precisely the kernel of the following homomorphism, i.e.,

$$f(A) = g^{-1}(0), \qquad g(B) = h^{-1}(0), \qquad h(C) = f^{-1}(0).$$

The notion of an exact triangle has the advantage that it considerably shortens the statement of many results, making them seem more natural and easier to remember.

A purely algebraic situation of frequent occurrence which gives rise to an exact triangle is the following (we shall need to make use of this example shortly). Let (A, d) be a differential group, and let B be a subgroup of A which is "stable" under d, i.e., $d(B)\subset B$. Then the restriction of d to the subgroup B defines a differential operator on B; also, d maps cosets of A (modulo B) into cosets, and defines an endomorphism of the factor group A/B which is obviously a differential operator also. Hence we can form the derived groups, $\mathcal{H}(A)$, $\mathcal{H}(B)$, and $\mathcal{H}(A/B)$. The inclusion homomorphism of B into A induces a homomorphism i: $\mathcal{H}(B)\rightarrow\mathcal{H}(A)$ of the derived groups in a natural way. Similarly, the natural homomorphism $A\rightarrow A/B$ which assigns to each element $x\in A$ its coset, $x+B$, induces a homomorphism j: $\mathcal{H}(A)\rightarrow\mathcal{H}(A/B)$. Finally, by using the differential operator d, it is possible to define[3] a homomorphism d': $\mathcal{H}(A/B)\rightarrow\mathcal{H}(B)$ in a natural way. These three homomorphisms, i, j, and d', fit together to form a triangle,

$$\begin{tikzcd}
\mathcal{H}(B) \arrow[rr] & & \mathcal{H}(A) \arrow[dl] \\
& \mathcal{H}(A/B) \arrow[ul] &
\end{tikzcd}$$

which may be easily proved to be exact.

5. Spectral sequences of differential groups. The algebraic apparatus we have described so far has been more or less standard in this subject for many years now. We shall now take up an idea of much more recent vintage, which has been of central importance in much of the rapid progress that algebraic topology has made in the last five years.

DEFINITION. An infinite sequence of differential groups, (A_n, d_n), $n=1, 2, 3, \cdots$, is called a *spectral sequence*[4] in case each group in the sequence is the derived group of its predecessor, i.e.,

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[3] If $u\in\mathcal{H}(A/B)$, then $d'(u)$ is defined as follows: Choose a cycle z belonging to the homology class u. Then z is a coset, $z=y+B$, where $y\in A$. The fact that z is a cycle implies that $d(y)\in B$. Then $d'(u)$ is defined to be the homology class of $d(y)$. This definition can be shown to be independent of the choices of z and y.

[4] The notion of a spectral sequence is due mainly to J. Leray. For references to early work on the subject, see [2].

$$A_{n+1} = \mathcal{H}(A_n).$$

Associated with any spectral sequence is a certain *limit group*. Roughly speaking, this limit group is defined as follows. Let $\mathcal{Z}(A_n)$ denote the kernel of d_n, as before, and let

$$\kappa_n : \mathcal{Z}(A_n) \to A_{n+1}$$

be the natural homomorphism which is defined by assigning to each element of the subgroup $\mathcal{Z}(A_n)$ its coset modulo $\mathcal{B}(A_n)$. Then κ_n is a homomorphism of a subgroup of A_n onto A_{n+1}. Consider the sequence of groups A_n and homomorphisms κ_n, $n = 1, 2, \cdots$. This sequence of groups and homomorphisms almost satisfies the standard definition of a "direct sequence" of groups; it fails only because the homomorphisms κ_n are not defined over all of A_n. By a slight modification of the usual definition of the limit group of a direct sequence of groups, one defines the limit group[8] of the sequence $\{A_n, \kappa_n\}$.

6. **An example.** We shall now illustrate by means of an example how some of the main results of algebraic topology can be best stated in terms of spectral sequences.

One of the most important concepts of present day topology is that of a *fibre space*. Not only does this concept occur in many different situations in topology itself, but it also is of central importance in some other branches of mathematics. Fibre spaces arise everywhere in differential geometry in the consideration of tangent spaces and tensor spaces over differentiable manifolds; and it is a natural concept in considering a coset space of a Lie group. The notion of a fibre space is a generalization of the familiar idea of the Cartesian product of two spaces. In fact, Pontrjagin has called them "skew products."

Various different definitions are now current for the term "fibre space." For our purposes it will suffice if we say that a fibre space is locally a product space. To be precise, a fibre space is a quadruple (B, p, X, F), where B, X, and F are topological spaces, called respectively the "total space," the "base space," and the "fibre," and p is a continuous map of B onto X, called the "projection." These four things are required to satisfy the following condition: For any point $x \in X$ there exists a neighborhood V of x and a homeomorphism ϕ of $V \times F$ onto $p^{-1}(V)$ such that

$$p[\phi(v, y)] = v$$

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[8] For the benefit of the interested reader, we give the precise details in Appendix I.

for any points $v \in V$ and $y \in F$.

Given any two topological spaces X and F, one can always construct the following trivial fibre space having X for base space and F for fibre: choose B to be the product $X \times F$, and let p be the projection of $X \times F$ onto X. The main interest attaches to those fibre spaces which are *not* product spaces "in the large." An example of a nontrivial fibre space is obtained by letting B be a Möbius strip, X be the circumference of a circle, and F a segment of the real line. The definition of the projection p: $B \to X$ is left to the reader. Another example is obtained by taking B to be a covering space of X, and p: $B \to X$ the usual projection of the covering space onto the base space. In this case the fibre is a discrete space. In any case, one can usually construct many different (i.e., non-isomorphic) fibre spaces with a given base space and fibre.[6]

It is an important problem for many different reasons to determine what relations exist between the homology (or cohomology) groups of the total space, base space, and fibre in a fibre space. Various partial results along this line were obtained as long ago as the 1930's. However, the first nearly complete answer to this problem was obtained by J. Leray[7] about 1947. His final results for cohomology groups may be stated as follows. Let (B, p, X, F) be a fibre space, and G an abelian group. Then there exists a spectral sequence (E_n, d_n), $n = 2, 3, \cdots$, and a decreasing sequence of subgroups of $H^*(B, G)$,

$$H^*(B, G) = D^0 \supset D^1 \supset D^2 \supset \cdots \supset D^p \supset \cdots$$

such that the following two facts are true:

(a) The first term of the spectral sequence, E_2, is naturally isomorphic[8] to $H^*(X, H^*(F, G))$.

(b) Let E_∞ denote limit group of this spectral sequence. Then E_∞ is naturally isomorphic to the weak direct sum, $\sum_p D^p/D^{p+1}$.

At first sight, this result does not look as if it would be very helpful in deducing precise relations between the cohomology groups of B, X, and F. However, it is much more useful than would at first appear due to the following additional facts:

(c) As mentioned above, cohomology groups of topological spaces are graded groups. Therefore $E_2 = H^*(X, H^*(F, G))$ is "bi-graded," i.e., is the direct sum of the *doubly* indexed family of subgroups, $H^p(X, H^q(F, G))$. We shall denote $H^p(X, H^q(F, G))$ by $E_2^{p,q}$.

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[6] A closely related concept is that of a fibre bundle. A fibre bundle is a fibre space which has certain additional elements of structure attached to it.

[7] This result of Leray's was first announced in various notes in C. R. Acad. Sci. Paris, cf. the references given in [2] and [3].

[8] To be strictly correct, E_2 is isomorphic to the cohomology group of X with the local coefficient system $H^*(F, G)$. However the formulation of the text is correct in most cases in which spectral sequences are actually applied to fibre spaces (e.g., the case in which X is simply connected).

(d) In most cases that occur in practice, $H^*(X, H^*(F, G))$ can be expressed in terms of $H^*(X, G)$ and $H^*(F, G)$ by means of tensor products and other purely algebraic operations, provided G is chosen suitably.

(e) The group E_3 inherits a bi-graded structure from the bi-graded structure on E_2. That is, E_3 is the direct sum of a doubly indexed family of subgroups, $E_3^{p,q}$, where $E_3^{p,q}$ is the image of a subgroup of $E_2^{p,q}$ under the homomorphism κ_2 described in §5 above. One now sees inductively that all the groups E_n are bi-graded, and the homomorphism κ_n is compatible with the bi-graded structures on E_n and E_{n+1} (i.e., κ_n maps $E_n^{p,q}$ into $E_{n+1}^{p,q}$). This implies that the limit group, E_∞, is also bi-graded.

(f) For any pair of integers p and q, there exists an integer N (depending on p and q) such that the groups $E_n^{p,q}$ are isomorphic for all $n > N$. This implies that the component $E_\infty^{p,q}$ of E_∞ is determined in N steps. Naturally, this fact that any given component of E_∞ can be determined by a finite process is of essential importance.

Practically all the known results about the cohomology groups of a fibre space can be derived from this fundamental theorem of Leray. And this is in spite of the fact that the spectral sequence says nothing about the various group extensions involved if one tries to determine $H^*(B, G)$ from E_∞. There is also the further difficulty that, in various applications, the problem of computing the successive E_r's and d_r's is not effectively solved.

An analogous result about the homology groups of a fibre space is due to J. P. Serre [5].

7. **Exact couples.** The question naturally arises, how are spectral sequences of a fibre space defined? Here again it would take us too long to go into details, but there is a certain algebraic mechanism which gives rise to a spectral sequence which can be easily described. This algebraic mechanism seems to be applicable to many problems of topology.

DEFINITION. An *exact couple*[9] is an exact triangle in which two of the groups are the same; to be precise, an exact couple consists of two abelian groups, A and C, and three homomorphisms, $f: A \to A$, $g: A \to C$, and $h: C \to A$. These homomorphisms are required to satisfy the following "exactness" conditions:

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[9] Exact couples were introduced by the author; cf. [4].

$$f(A) = g^{-1}(0),\ g(A) = h^{-1}(0), \text{ and } h(C) = f^{-1}(0).$$

These three conditions can be easily kept in mind if one makes the following triangular diagram,

$$\begin{tikzcd} A \arrow[rr] & & A \arrow[dl] \\ & C \arrow[ul] & \end{tikzcd}$$

and observes that the kernel of each homomorphism is required to be the image of the preceding homomorphism. We shall denote such an exact couple by the notation $\langle A, C; f, g, h\rangle$, or more briefly $\langle A, C\rangle$.

There is an important operation which assigns to an exact couple $\langle A, C; f, g, h\rangle$ another exact couple $\langle A', C'; f', g', h'\rangle$, called the *derived* exact couple. This derived exact couple is defined as follows.

Define an endomorphism d: $C \to C$ by $d = g \circ h$. Then $d^2 = d \circ d = g \circ h \circ g \circ h = 0$, since $h \circ g = 0$ by exactness. Therefore d is a differential operator on C. Let $C' = \mathcal{H}(C)$, the derived group of the differential group (C, d). Let $A' = f(A)$, which is a subgroup of A. Define $f' = f | A'$, the restriction of f to the subgroup A'. The homomorphism h': $C' \to A'$ is induced by h; it is readily verified that $h[\mathcal{Z}(C)] \subset A'$, and $h[\mathcal{B}(C)] = \{0\}$, hence h induces a homomorphism of the factor group $C' = \mathcal{Z}(C)/\mathcal{B}(C)$ into A'. The definition of g': $A' \to C'$ is more complicated. Let $a \in A'$; choose an element $b \in A$ such that $f(b) = a$. Then $g(b) \in \mathcal{Z}(C)$, and $g'(a)$ is defined to be the coset of $g(b)$ modulo $\mathcal{B}(C)$. It is easily verified that this definition is independent of the choice made of the element $b \in A$, and that g' is actually a homomorphism.

Of course, it is necessary to verify that the homomorphisms f', g', and h' satisfy the exactness condition of an exact couple. This verification is purely mechanical.

It is clear that this process of derivation can be applied to the derived exact couple $\langle A', C'; f', g', h'\rangle$ to obtain another exact couple $\langle A'', C''; f'', g'', h''\rangle$, called the *second derived exact couple*, and so on. In general, we shall denote the nth derived couple by $\langle A^{(n)}, C^{(n)}; f^{(n)}, g^{(n)}, h^{(n)}\rangle$, and $d^{(n)} = g^{(n)} \circ h^{(n)}$: $C^{(n)} \to C^{(n)}$ will denote the differential operator on $C^{(n)}$. Then the sequence of differential groups, $(C^{(n)}, d^{(n)})$, is readily seen to be a spectral sequence. It will be reerred to as *the spectral sequence associated with the exact couple* $\langle A, C\rangle$.

We can now indicate how the spectral sequence of a fibre space is defined. Let (B, p, X, F) be a fibre space as described above, and let $C^*(B)$ denote the group of cochains of B (we are omitting the coefficient group G from our notation). The fibre space structure on B defines in a rather natural way a nested sequence of subgroups[10] of the group $C^*(B)$:

$$C^*(B) = A^{-1} \supset A^0 \supset A^1 \supset A^2 \supset \cdots \supset A^p \supset \cdots .$$

Each of these subgroups is stable under the coboundary operator, δ:

$$\delta(A^p) \subset A^p.$$

Hence we can form the derived groups, and there is associated with each adjacent pair of groups in this sequence an exact triangle, as described above:

$$\begin{tikzcd}
\mathcal{H}(A^{p+1}) \arrow[rr, "i_p"] & & \mathcal{H}(A^p) \arrow[dl, "j_p"] \\
& \mathcal{H}(A^p/A^{p+1}) \arrow[ul, "\delta'_p"] &
\end{tikzcd}$$

Now define graded groups A and C as follows:

$$A = \sum_n \mathcal{H}(A^n), \qquad C = \sum_n \mathcal{H}(A^n/A^{n+1}).$$

Then the homomorphisms i_*, j_*, and δ'_* define homomorphisms $i: A \to A$, $j: A \to C$, and $\delta': C \to A$ respectively, and the necessary exactness conditions are fulfilled, so that $\langle A, C; i, j, \delta' \rangle$ is an exact couple. The spectral sequence associated with this exact couple is the desired spectral sequence of the fibre space (B, p, X, F).

It should be mentioned here that Leray originally obtained the spectral sequence of a fibre space directly from the nested sequence of subgroups, $A^0 \supset A^1 \supset \cdots$, without the use of the intermediate notion of an exact couple. However, it is probably easier to keep in mind the definition of an exact couple and its derived exact couple than it is to remember Leray's method. In addition, the method of exact couples applies to some situations to which Leray's method does not apply (see the examples below).

8. **Some examples of other applications of spectral sequences and exact couples.** For our first example, let X be a connected topological space, and let $\tilde{X}$ be a covering space of X. Let Π denote the fundamental group of X, and $\tilde{\Pi}$ the subgroup of Π to which $\tilde{X}$ corresponds. We shall assume that $\tilde{X}$ is a *regular* covering; this means that $\tilde{\Pi}$ is a normal subgroup of Π. Also, this condition implies that the quotient

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[10] For the benefit of the reader who has some familiarity with the concepts of singular cohomology theory, we indicate one way of defining this nested sequence of subgroups in Appendix II.

group $\Pi/\tilde{\Pi}$ operates on $\tilde{X}$ as a group of homeomorphisms, none of which have any fixed points (with the exception of the identity element, of course). One may consider that X is obtained from $\tilde{X}$ by identifying points which correspond under the operations of $\Pi/\tilde{\Pi}$.

It is a problem of long standing in topology to determine what relations must exist between the homology (or cohomology) groups of X and $\tilde{X}$. Any such relations will clearly have also to involve something about the groups Π and $\tilde{\Pi}$; for, it is known that given any subgroup $\tilde{\Pi}$ of Π, one can construct a covering $\tilde{X}$ of X which corresponds to the subgroup $\tilde{\Pi}$.

In 1948, H. Cartan[11] proved the following result: There exists a spectral sequence (E_n, d_n), $n=2, 3$, and a nested sequence of subgroups of $H^*(X)$,

$$H^*(X) = D^0 \supset D^1 \supset D^2 \supset \cdots \supset D^p \supset \cdots,$$

such that the following two facts are true:

(a) The first term of the spectral sequence, E_2, depends only on the factor group, $\Pi/\tilde{\Pi}$, the cohomology group $H^*(\tilde{X})$, and the manner in which $\Pi/\tilde{\Pi}$ operates on $H^*(\tilde{X})$. (Note: since $\Pi/\tilde{\Pi}$ operates on $\tilde{X}$, it also operates on $H^*(\tilde{X})$.) To be precise, E_2 is isomorphic to *the cohomology group of the group* $\Pi/\tilde{\Pi}$ *with coefficients* $H^*(\tilde{X})$,

$$E_2 = H^*(\Pi/\tilde{\Pi}, H^*(\tilde{X})).$$

(b) Let E_∞ denote the limit group of this spectral sequence. Then E_∞ is naturally isomorphic to the direct sum $\sum_p D^p/D^{p+1}$.

The remarks (c), (d), (e), and (f) that were made in §6 about the spectral sequence of a fibre bundle apply verbatim to this spectral sequence also.

In the statement of this result, there occurs a concept which we have not mentioned before: the cohomology group of an arbitrary abstract group Π (which need not be abelian) with coefficients in an abelian group G, denoted by $H^*(\Pi, G)$. This concept (which is purely algebraic) had its origin in some purely topological results of Hurewicz, but since then it was proven of use in algebraic number theory.

It should also be pointed out that although a covering space is a special case of a fibre space, this spectral sequence of H. Cartan is *not* the same as that obtained by applying the spectral sequence of Leray to the case of a covering space. The spectral sequence of Leray is of little interest in the case of a covering space.

Our second example refers to the homotopy groups of a space.

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[11] H. Cartan announced this result without proof in [1]; he has not published a complete proof as yet, except in some mimeographed notes of limited circulation.

The homotopy groups of a space X, denoted by $\pi_p(X)$ $(p \geqq 1)$, were mentioned in passing at the beginning of this address. They are topological invariants of X, as are the homology and cohomology groups of X. In some problems they give more complete information about a space, or give deeper insight into its topology. Unfortunately it is usually much more difficult to determine the homotopy groups of X than it is to determine the homology or cohomology groups. Any theorem which contributes to a better understanding of the structure of the homotopy groups of spaces is usually rather significant. Spectral sequences come into this picture as follows: Given any topological space X, there exists a spectral sequence[12] (E_n, d_n), $n = 2, 3, \cdots$, such that the limit group E_∞ bears the same relation to homotopy groups of X that the limit group of the spectral sequence of a fibre space bears to the cohomology groups of the total space (see §6). The first term, E_2, is not completely known. It is a bigraded group, and some of its homogeneous components are ordinary homology groups of X. Much of our meager knowledge about the homotopy groups of a general space can be derived from the existence of this spectral sequence. Leray's method for defining spectral sequences does not apply to this example.

It should be mentioned that an analogous result holds for the Borsuk-Spanier cohomotopy groups of a finite-dimensional space X. There exists a spectral sequence,[13] (E_n, d_n) such that the limit group, E_∞, is closely related to the cohomotopy groups of X, and the initial term, E_2, is closely related to the cohomology groups of X.

Our final example is of a purely algebraic nature. As mentioned above, if Π is an arbitrary group, then the cohomology group $H^*(\Pi, G)$ (with coefficients in the abelian group G) is defined. Suppose Π' is a normal subgroup of Π. Then the question arises, What relations exist between the three cohomology groups, $H^*(\Pi, G)$, $H^*(\Pi', G)$, and $H^*(\Pi/\Pi', G)$? This question is answered by a theorem[14] of Serre and Hochschild: There exists a spectral sequence (E_n, d_n) such that $E_2 = H^*(\Pi/\Pi', H^*(\Pi'))$, and the limit group E_∞ is related to $H^*(\Pi, G)$ in a way we have described before (cf. the discussion of the spectral sequence of a fibre bundle above).

9. **Conclusion.** The main point of this paper may be stated as follows: The algebraic notion of a spectral sequence is essential for the statement of many basic theorems of algebraic topology. These

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[12] This spectral sequence was introduced by the author in [4, part II].

[13] This spectral sequence was introduced by the author in [4, part III].

[14] This theorem is stated and proved in [6].

spectral sequences often arise as the associated spectral sequences of certain exact couples.

10. Appendix I: The precise definition of the limit groups of a spectral sequence. Let (A_n, d_n), $n=1, 2, \cdots$, be a spectral sequence, and let $\kappa_n: Z(A_n) \to A_{n+1}$ be as defined in §5. Define a homomorphism κ_n^p of a certain subgroup of A_n onto A_{n+p} by composition of the homomorphisms $\kappa_n, \kappa_{n+1}, \cdots, \kappa_{n+p-1}$:

$$\kappa_n^p = \kappa_{n+p-1} \circ \cdots \circ \kappa_{n+1} \circ \kappa_n.$$

Let $\bar{A}_n$ denote the subgroup of A_n consisting of those elements $a \in A_n$ such that $\kappa_n^p(a)$ is defined for all values of p. Define $\bar{\kappa}_n: \bar{A}_n \to \bar{A}_{n+1}$ to be the restriction of κ_n to the subgroup $\bar{A}_n$. Then the sequence of groups $\{\bar{A}_n\}$ and homomorphisms $\{\bar{\kappa}_n\}$ is a direct sequence of groups in the usual sense, and its limit group is defined to be the limit group of the given spectral sequence.

11. Appendix II: The definition of the nested sequence of subgroups of the group of cochains of a fibre space. Let (B, p, X, F) be a fibre space, as defined in §6, such that the base space X and the fibre F are finite polyhedra. Choose a simplicial decomposition of X which is fine enough to satisfy the following condition: For any simplex $\sigma \subset X$, there exists a homeomorphism ϕ of $\sigma \times F$ onto $p^{-1}(\sigma)$ such that

$$p[\phi(x, y)] = x$$

for any points $x \in \sigma$ and $y \in F$. It is readily seen that such simplicial decompositions must exist. Let X^n denote the n-dimensional skeleton of X with respect to this simplicial decomposition (i.e., the union of all simplexes of dimension $\leqq n$), and let $B^n = p^{-1}(X^n)$ for $n = 0, 1, 2, \cdots$.

Let $C^*(B, G)$ denote the group of singular cochains of B with coefficients in G. An element $f \in C^*(B, G)$ is a function which assigns to a singular simplex T in B an element $f(T) \in G$. Define A^n to be the subgroup of $C^*(B, G)$ consisting of those cochains which vanish on all singular simplexes contained in the subspace B^n of B. Then

$$C^*(B) \supset A^0 \supset A^1 \supset A^2 \supset \cdots \supset A^n \supset \cdots$$

and

$$\delta(A^n) \subset A^n$$

as required. For full details of this method, see [4, part V].

This method does not apply in case X is not a polyhedron. The method to be used in the general case, using "cubical" singular cochains, has been described by J. P. Serre [5].

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BROWN UNIVERSITY

THE ANNUAL MEETING IN BALTIMORE

The Sixtieth Annual Meeting of the American Mathematical Society was held at the Johns Hopkins University on Monday through Wednesday, December 28–30, 1953 in conjunction with the annual meeting of the Mathematical Association of America, which extended through Thursday, December 31.

About 600 registered for the meeting, including the following 500 members of the Society:

J. C. Abbott, Milton Abramowitz, J. W. Addison, Jr., M. I. Aissen, R. W. Allen, C. B. Allendoerfer, S. A. Amitsur, R. D. Anderson, M. M. Andrew, N. C. Ankeny, H. A. Antosiewicz, Hirotada Anzai, Richard Arens, K. J. Arnold, L. A. Aroian, Sholom Arzt, Maurice Auslander, W. L. Ayres, F. E. Baker, B. J. Ball, N. H. Ball, R. P. Bambah, Wallace E. Barnes, I. A. Barnett, R. G. Bartle, R. W. Bass, G. E. Bates, H. F. Beckafort, E. G. Begle, T. J. Benac, A. A. Bennett, F. D. Bennett, Stefan Bergman, Lipman Bers, E. E. Betz, R. H. Bing, Z. W. Birnbaum, I. E. Block, H. W. Bode, Raoul Bott, T. A. Botts, S. G. Bourne, J. W. Bower, J. W. Brace, Joseph Bram, L. D. Bram, H. W. Brinkmann, A. R. Brown, Jr., Leon Brown, F. H. Brownell, C. E. Burgess, R. S. Burington, J. H. Bushey, Jewell H. Bushey, G. H. Butcher, P. L. Butzer, S. S. Cairns, Eugenio Calabi, W. R. Callahan, E. A. Cameron, R. H. Cameron, H. H. Campaigne, J. F. Canu, L. V. Carlton, R. C. Carson, H. C. Carter, W. B. Carver, J. W. Cell, E. W. Cheney, P. L. Chessin, Joshua Chover, W. L. Chow, D. E. Christie, J. L. Chung, G. F. Clanton, C. E. Clark, F. E. Clark, F. M. Clarke, G. R. Clements, A. H. Clifford, E. W. Coffin, E. J. Cogan, I. S. Cohen, L. W. Cohen, Teresa Cohen, Harvey Cohn, Nancy Cole, A. H. Copeland, Sr., W. H. H. Cowles, V. F. Cowling, M. J. Cox, J. B. Crabtree, Frederic Cunningham, J. H. Curtiss, Chandler Davis, M. D. Davis, R. L. Davis, B. V. Dean, R. A. Dean, L. J. Deck, Allen Devinatz, S. F. Dice, R. C. DiPrima, J. L. Doob, F. G. Dressel, Nelson Dunford, W. L. Duren, Jr., William H. Durfee, W. F. Eberlein, R. P. Eddy, H. P. Edmundson, Gertrude Ehrlich, C. C. Elgot, Joanne Elliott, Arthur Erdélyi, Paul Erdös, D. H. Erkiletian, Jr., W. H. Fagerstrom, Michael Fekete, William Feller, F. G. Fender, F. A. Ficken, N. J. Fine, R. S. Finn, W. T. Fishback, Harley Flanders, B. A. Fleishman, W. H. Fleming, E. E. Floyd, S. A. Foote, L. R. Ford, L. R. Ford, Jr., H. A. Forrester, M. K. Fort, Jr., Tomlinson Fort, A. H. Fox, W. C. Fox, J. S. Frame, L. K. Frazer, Gerald Freilich, F. N. Frenkiel, C. H. Frick, R. E. Fullerton, A. S. Galbraith, H. L. Garabedian, I. E. Gaskill, L. D. Gates, H. M. Gehman, J. J. Gehrig, J. J. Gergen, Murray Gerstenhaber, J. H. Giese, Edwin Gillette, Leonard Gillman, Wallace Givens, A. M. Gleason, Casper Goffman, V. D. Gokhale, Michael Goldberg, Oscar Goldman, R. A. Good, A. W. Goodman, Basil Gordon, Saul Gorn, Morikuni Goto, W. H. Gottschalk, E. C. Gras, L. W. Green, Simon Green, F. L. Griffin, J. S. Griffin, Jr., Emil Grosswald, P. E. Guenther, V. H. Haag, Felix Haas, V. B. Haas, Sister M. Raphael Hafner, Franklin Haimo, E. E. Hammond, Jr., M. E. Hamstrom, L. S. Hart, Philip Hartman, H. H. Hartzler, Melvin Hausner, E. V. Haynsworth, E. A. Hedberg, G. A. Hedlund, Alex Heller, Melvin Henriksen, R. T. Herbst, Coleman Herpel, I. R. Hershner, Jr., I. N. Herstein, P. S. Herwitz, I. I. Hirschman, Jr., F. E. P. Hirzebruch, A. J. Hoffman, J. E. Hoffman, S. P. Hoffman, F. E. Hohn, B. W. Holz, J. G. Horne, Jr., E. M. Hove, Alfred Huber, G. B. Huff, Ralph Hull, D. W. Hullinghorst, L. S. Hunter, W. A. Hurwitz, M. A. Hyman, J. R. Isbell, R. F. Jackson, S. B. Jackson, H. G. Jacob, Jr., Arno Jaeger, R. E. Johnson, B. W. Jones, F. B. Jones, M. L. Juncosa, Mark Kac, Hyman Kamel, A. F. Kay, N. D. Kazarinoff, John Keesee, M. E. Kellar, L. M. Kells, J. B. Kelly, L. M. Kelly, J. H. B. Kemperman, M. R. Kenner, J. R. F. Kent, D. E. Kibbey, C. C. Kilby, Jr., George Klein, J. R. Kline, Fulton Koehler, T. L. Koehler, H. L. Krall, J. B. Kruskall, Jr., R. R. Kuebler, Jr., Harold W. Kuhn, S. H. Lachenbruch, Serge Lang, Harry Langman, Leo Lapidus, E. H. Larguier, C. G. Latimer, Patrick Leehey, Solomon Lefschetz, Marguerite Lehr, R. A. Leibler, Walter Leighton, R. B. Leipnik, W. W. Leontief, Marie Lesnick, J. J. Levin, Norman Levinson, A. L. Lewis, D. C. Lewis, Jr., S. D. Liao, H. M. Lieberstein, F. W. Light, Jr., T. P. G. Liverman, Charles Loewner, Lee Lorch, F. W. Lott, D. B. Lowdenslager, L. L. Lowenstein, C. I. Lubin, Eugene Lukacs, R. C. Lyndon, V. O. McBrien, B. H. McCandless, S. L. McDonald, E. A. McDougle, R. W. McKelvey, J. D. McKnight, Brockway McMillan, S. S. McNeary, E. J. McShane, C. C. MacDuffee, G. R. MacLane, Saunders MacLane, H. M. MacNeille, W. G. Madow, Wilhelm Magnus, Lawrence Markus, M. H. Martin, P. E. Martin, C. W. Mathews, A. P. Mattuck, K. O. May, F. M. Mears, A. E. Meder, Jr., Herman Meyer, L. F. Meyers, A. N. Milgram, Joseph Milkman, D. D. Miller, F. H. Miller, A. K. Mitchell, Don Mittleman, Deane Montgomery, A. H. Moore, T. W. Moore, W. E. Moore, W. K. Morrill, C. W. Munshower, F. D. Murnaghan, C. H. Murphy, Jr., F. H. Murray, G. G. Murray, W. R. Murray, Leopoldo Nachbin, L. W. Neustadt, August Newlander, Jr., Morris Newman, C. V. Newsom, R. Z. Norman, A. B. Novikoff, C. O. Oakley, G. G. O'Brien, R. E. O'Donnell, M. W. Oliphant, Gloria Olive, L. F. Ollman, E. M. Olson, Steven Orey, Morris Ostrofsky, J. C. Oxtoby, K. L. 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Vansant, A. H. Van Tuyl, M. C. Waddell, D. H. Wagner, R. J. Walker, A. D. Wallace, J. A. Ward, W. H. Warner, Gerard Washnitzer, Daniel Waterman, G C.

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The Twenty-seventh Josiah Willard Gibbs Lecture, entitled *Mathematics in economics*, was delivered by Professor Wassily Leontieff of Harvard University on Monday evening. Professor G. T. Whyburn, President of the Society, presided.

The Society was addressed by Professor A. D. Wallace of Tulane University on the *Structure of topological semi-groups* on Monday afternoon and by Professor D. V. Widder of Harvard University on *Convolution transforms* on Tuesday afternoon upon invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings. At the former address Professor Deane Montgomery presided and Professor Leon W. Cohen presided at the latter.

The membership and their guests were entertained at tea in Levering Hall on Monday afternoon by the ladies of the Department of Mathematics of the University. Arrangements were made for visits to Annapolis on Tuesday and to places of interest in Baltimore on Wednesday.

On Wednesday evening, at the banquet under the auspices of both the Society and the Association, an address was delivered by Dr. Alan T. Waterman, Director of the National Science Foundation, in which he discussed the program of the Foundation in the mathematical sciences. Professor D. C. Lewis of the Johns Hopkins University was the toastmaster. A resolution expressing the appreciation of the Society and the Association to the Johns Hopkins University and to all who contributed their personal efforts to the success of the meeting was moved by Professor Ralph Hull and adopted by a rising vote.

On Wednesday afternoon December 30, 1953 the Bôcher prize was awarded to Professor Norman Levinson for the contributions to the theory of linear, nonlinear, ordinary and partial differential equations contained in his papers of recent years. These include studies on asymptotic behavior of solutions, stability questions, existence of periodic solutions, and perturbation theory. Professor Levinson then delivered a short address on the work for which he had been awarded this prize.

The annual Business Meeting of the Society was held on Wednesday December 30, 1953. The Secretary reported on the affairs of the Society.

At the annual election in which 958 votes were cast the following officers were elected.

President Elect, Professor R. L. Wilder.

Vice Presidents, Professors L. V. Ahlfors and Antoni Zygmund.

Associate Secretaries, Professors J. W. Green, J. H. Roberts, and J. W. T. Youngs.

Member of the Editorial Committee of the Bulletin, Professor W. T. Martin.

Member of the Editorial Committee of the Proceedings, Professor Richard Brauer.

Member of the Editorial Committee of the Transactions and Memoirs, Professor Herbert Busemann.

Member of the Editorial Committee of the Colloquium Publications, Professor A. A. Albert.

Member of the Editorial Committee of Mathematical Reviews, Professor R. P. Boas.

Member of the Editorial Committee of Mathematical Surveys, Professor Leo Zippin.

Representative on the Board of Editors of the American Journal of Mathematics, Professor Reinhold Baer.

Member of the Committee on Printing and Publishing, Dean C. J. Rees.

Members-at-large of the Council, Professors R. P. Dilworth, R. D. James, C. C. MacDuffee, D. V. Widder, and Antoni Zygmund.

Members of the Board of Trustees, Professors Einar Hille and H. F. Bohnenblust.

The Secretary reported that as directed by the membership at a Business Meeting of the Society in Kingston, Ontario on September 1, 1953 a statement concerning proposed amendments to the By-laws had been included in the Programs for the November, 1953 meetings and had been sent to all members of the Society. In this statement the members were requested to write to the Secretary of the Society expressing opinions concerning these amendments. Thirty-two letters were received of which eighteen expressed approval and thirteen expressed disapproval. The Secretary reported that these letters had been considered by the Council and that the Council had voted to recommend again to the membership that these amendments to the By-laws be adopted and that the Council had requested the Secretary to read a statement concerning the attitude of the Council toward these amendments. The Secretary read the following statement:

In presenting these amendments, the Council is not asking for broader powers, nor that it be made easier for the Council to speak with respect to matters which affect the dignity and effective position of mathematics.

The Council believes, the world around us being what it is, that the Council will most probably have to consider such matters in the future.

The Council therefore wishes an orderly procedure to be established to ensure that action will be taken only after careful study and only with due deliberation.

The Council believes that these amendments guarantee such study and deliberation, and they are presented to you in this spirit.

After some discussion the members of the Society present at the Business Meeting voted 161 to 26 to adopt the following two amendments to the By-Laws:

1. To add to Article IV the following section:

Section 8: The Council shall also have power to speak in the name of the Society with respect to matters affecting the status of mathematics or mathematicians, such as proposed or enacted Federal or State legislation, conditions of employment in universities, colleges, or business, research or industrial organizations, regulations, policies or acts of governmental agencies or instrumentalities, and other items which tend to affect the dignity and effective position of mathematics.

With the exception noted in the next paragraph, a favorable vote of two-thirds of the entire membership of the Council, taken in accordance with the provisions of Section 4 of this Article, shall be necessary to authorize any statement in the name of the Society with respect to such matters. With the exception noted in the next paragraph, such a vote may be taken only if written notice shall have been given to the Secretary by the proposer of any such resolution not later than one month prior to the Council meeting at which the matter is to be presented, and the vote shall be taken not earlier than one month after the resolution has been discussed by the Council.

If, at a meeting of the Council, there are present members having a total of twelve votes, as defined in Section 4 of this Article, then the prior notification to the Secretary may be waived by unanimous consent. In such a case, a unanimous favorable vote by those present shall empower the Council to speak in the name of the Society.

The Council may also refer the matter to a referendum mail vote of the entire membership of the Society, and shall make such reference if a referendum is requested, prior to final action by the Council, by two hundred or more members. The taking of a referendum shall act as a stay upon Council action until the votes have been canvassed, and thereafter no action may be taken by the Council except in accordance with a plurality of the votes cast in the referendum.

2. To amend Section 1 of Article X by replacing the last sentence of this article, "No matter of general business shall be considered at any meeting of the Society except the Annual Meeting, without the recommendation of the Council" by the following:

There shall be a business meeting of the Society at the Annual Meeting and at the Summer Meeting. A business meeting of the Society shall take final action only on business accepted by unanimous consent, or business notified to the full membership of the Society in the call for the meeting. Such notification shall be made only when so directed by a previous business meeting of the Society or by the Council.

The Council met on Monday evening, December 29, 1953.

The Secretary announced the election of the following forty-two persons to ordinary membership in the Society:

Mr. Richard Logan Becker, Yale University;
Dr. Jacob Floyd Blackburn, U. S. Air Force, Institute for Advanced Study;
Mr. Thomas Roy Brahana, Dartmouth College;
Mrs. Ilse Brauer, Brandeis University;
Dr. William Egbert Briggs, University of Colorado;
Reverend Anthony Edmund Cahill, Belmont Abbey College;
Mr. George Charles Caldwell, University of North Carolina;
Mr. Alan Cobham, Navy Department, Washington, D. C.;
Dr. Margaret Frances Conroy, Purdue University;
Mr. Ralph Deutsch, Hughes Aircraft Company, Culver City, California;
Mr. Sylvan Howard Eisman, Frankford Arsenal, Philadelphia, Pennsylvania;
Dr. Pao-hsien Paul Fang, Catholic University of America;
Dr. Clemens Bernard Hanneken, Marquette University;
Mr. William Edward Hartnett, University of Kansas;
Mr. Walter Hoffman, Wayne University;
Professor George E. Hudson, College of Engineering, New York University;
Mr. Charles William Huff, University of Georgia;
Mr. Andrew McCampbell Jones, University of Kansas;
Professor Tosio Kato, University of Tokyo;
Mr. Clement Fisher Kent, Navy Department, Washington, D. C.;
Dr. David Carman Kleinecke, University of California, Berkeley;
Professor Joachim Lambek, McGill University;
Dr. Philip Joseph Leah, L'Immaculée Conception, Montreal, Quebec;
Professor Anthony Richard Lovaglia, San Jose State College;
Mr. Marvin David Marcus, University of California, Berkeley;
Mr. James Michael McLynn, Logistics Research Project, Washington, D. C.;
Mr. Robert Price Mitchell, University of Denver;
Mr. John Henry Monahan, Boston College;
Mr. Dewey Moore, Naval Ordnance Laboratory, White Oak, Maryland;
Dr. Paul Stallings Mostert, Tulane University;
Miss Mary Jane Palmerlee, Tulane University;
Mr. Jack Herbert Padgett, Armstrong College;
Mr. Lee Howard Pennington, Florida Agricultural and Mechanical College;
Mr. Robert George Pohrer, U. S. Armed Forces;
Professor Hanan Rubin, University of Tennessee;
Dr. William Richard Seugling, Allstates Engineering Company, Trenton, New Jersey;
Dr. Ray Orville Skinner, National Research Council, Ottawa, Ontario, Canada;
Miss Ruth Rebekka Struik, New York University;
Dr. Beat Andreas Troesch, New York University;
Mr. Charles Wyatt Williams, Retired Consulting Engineer;

Professor Carvel Stewart Wolfe, Shepherd College;
Mr. Bernd Zondek, I. B. M. Corporation, New York, New York;

It was reported that the following one hundred fifty-three persons had been elected to membership on nomination of institutional members as indicated:

University of Alabama: Professor Holland C. Filgo, Jr.

University of British Columbia: Miss Charlotte Foree and Mr. Rimhak Ree.

Brooklyn College: Mr. Donald Solitar.

Brown University: Professor Paul Germain, Mr. Frank Pantaleone Palermo, and Professor Ronald S. Rivlin.

University of California, Berkeley: Mr. Donato Anthony D'Esopo, Mr. Thomas Shelburne Ferguson, Mr. Marvin David Marcus, and Mr. Dana Stewart Scott.

University of California, Davis: Mr. Peter Jay Chrisler.

University of California, Los Angeles: Mr. Genovevo López, Mr. John L. Selfridge, Mr. Wayne Earl Smith, and Mr. Roy Takenaga.

Carnegie Institute of Technology: Professor Hirsh Gerald Cohen.

University of Chicago: Professor Florent Joseph Bureau, Mr. George Kolettis, Jr., Mr. Robert Harbison Mountjoy, Mr. Edward Charles Posner, Dr. Marcel Riesz, Mr. Alexandre Augusto Martins Rodriques, and Dr. Geoffrey Colin Shephard.

University of Cincinnati: Mr. Harvey M. Weitkamp.

City College of New York: Mr. Samuel Jacob Klein, and Mr. Menasha Joseph Tausner.

Columbia University: Mrs. Betty J. Flehinger, Mr. Bernard A. Fusaro, Mr. Bert Mendelson, Mr. Subramanian Sankaran, Mr. Harold R. Steele, and Mr. Bruce Travis Weber.

Cornell University: Mr. Robert McCallum Blumenthal, Mr. Kalman Joseph Cohen, Mr. George Edwin Collins, Mr. Leonard Evens, Mr. Angelo Margaris, Mr. Joel David Pincus, and Mr. Norman Stein.

Duke University: Mr. E. Warren Herron, Mr. John Herbert Hodges, and Mr. Robert Louis Padgett.

Haverford College: Mr. Robert T. Seeley.

Johns Hopkins University: Mr. Joseph Auslander, Mr. Robert I. Berg, Professor Carl F. Christ, Mr. Howard Charles Kerpelman, Mr. Marc L. Nerlove, Mr. Bernard Okun, Mr. Rolf R. Piekarz, Dr. Theodore J. Rivlin, Mr. Lee Suyemoto, and Mr. Richard Alan Swirnow.

University of Illinois: Mr. Peter George Braunfeld, Mr. William Carleton Healy, Jr., Mr. Charles Lewis Keller, Jr., Mr. Charles Neville Maxwell, Mr. Jack Unger Russell, Dr. Michio Suzuki, and Mr. John Eldon Whitesitt.

Indiana University: Mr. Norman George Meyers, and Mr. Walter Noll.

Institute for Advanced Study: Dr. Hirotada Anzai, Dr. Rene Deheuvels, Dr. Benoit B. Mandelbrot, Professor Kakutaro Morinaga, Dr. Gert Otto Sabidussi, Mr. John C. Shepherdson, and Dr. Kurt Strebel.

Iowa State College: Mr. William John Swartz.

University of Kansas: Mrs. Marta Mejía de Valle, and Mr. Elbert Abner Walker.

Lehigh University: Mr. Ervin Kenneth Dorff.

Massachusetts Institute of Technology: Mr. Fred Gunther Brauer, Mr. A. Jay Goldstein, Mr. John Evans Kimber, Jr., Mr. Robert Francis McIntosh, Mr. James Thomas Morse, and Mr. Roger Mayfield Simons.

University of Michigan: Mr. John Mitchell Gary, Mr. Samuel Roscoe Knox, Mr

John Paul Line, Mr. Robert Fred Lyjak, Mr. David Arne Storvick, Mr. Thomas Patrick Walker, Jr., and Mr. John Harris Walter.

Michigan State College: Mr. James Leland Bailey, Mr. William John Hardell, Mr. Bernard Jacobson, Dr. Delia Koo, Miss Mary Margaret Quinn, and Mr. Robert Earl Sechler.

University of Minnesota: Mr. Ernest Godfrey Klimme, Mr. Eugene James Putzer, and Mr. Jerome Z. Yeh.

University of New Hampshire: Miss Evelyn Jean Jones.

New York University: Mr. Walter Koppelman, Mr. Herbert C. Kranzer, Mr. Walter Littman, and Mr. Nathan Newman.

Northwestern University: Mr. John Martin Bachar, Jr., Mr. Mykola Marchenko, Jr., Mrs. Vera Stepen Pless, and Mrs. Elizabeth Maude Rowlinson.

Ohio State University: Mr. George Arthur Craft.

University of Oregon: Mr. LeRoy John Warren.

University of Pennsylvania: Mr. Hugh Norton Albright, Mr. Michael Rabin, and Dr. Karl Zeller.

The Pennsylvania State University: Professor Sidney Foler Mack.

Purdue University: Mr. Theodore S. Chihara.

Queens College: Mr. Arthur Kaufman.

The Rice Institute: Mr. Horace Perry Flatt.

University of Rochester: Mr. Edward T. Wong.

Rutgers University: Professor Orhan H. Alisbah and Mr. Sufian Y. Husseini.

University of Southern California: Mr. Joel M. Alperin, Mr. A. V. Balakrishnan, Dr. Julius Samuel Bendat, Mr. Howard Bryner Jenkins, Mr. H. Potter Kerfoot, Mr. Ching-Hwa Meng, Mr. Allen Ralph Sims, Mr. Alton Hutchison Smith, and Mr. Eugene Michael Zaustinsky.

Stanford University: Mr. Charles Ballantine, Mr. Joseph Omer Carter, Dr. James Joseph McMahon, Captain Ralph Hugh Pennington, and Mr. Peter William Shaw.

Syracuse University: Mr. Rafael Van Severen Chacon, Mr. David Chedell Hiser, Mrs. May E. Risch Kinsolving, Mr. William Eugene Kopka, and Mr. Luke Nicholas Zaccaro.

University of Toronto: Mr. Douglas West Allan, Dr. Paul Carl Gilmore, Mr. Albert Harold Lightstone, and Mr. Floyd Grant Robinson.

University of Utah: Mr. Louis Carl Barrett.

University of Virginia: Mr. Billey Joe Gilpin, Mr. William Lawrence Harkness, Mr. James Louis Hatfield, Mr. Powhatan Moncure, Jr., and Mr. Henry Warren Thwing.

University of Washington: Miss Thelma Mae Chaney and Professor Orval M. Klose.

Wellesley College: Miss Mary Louise Holton.

University of Wisconsin: Mr. John Edward Barger, Mr. William Robert Brittenham, Mr. Morton Brown, Mr. S. Ashby Foote, Mr. George Gioumousis, Mr. Robert Leroy Glass, Mr. Harold Irwin Ottoson, Miss Catherine Shelton Standerfer, Mr. A. D. Stewart and Mr. Richard E. Vesley.

Yale University: Mr. Alvin Norman Feldzamen, Mr. Bruno Harris, Mr. Stephen Elliott Puckette, and Mr. Earl Jay Taft.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: Edinburgh Mathematical Society: Miss Nancy Walls, Edinburgh Mathematical Society; Indian Mathematical Society: Miss Abha Mitra, Women's College, Patna, India; Islenzka Staerðfraeðafelagið: Mr. Gumnar Bodvarsson, State Electricity Authority, Iceland; Schweizerische Mathematische Gesellschaft: Dr. Peter Henrici, American University (under contract with National Bureau of Standards); Wiskundig Genootschap: Professor Nicolaas Govert deBruijn, University of Amsterdam.

The Secretary is pleased to report at this time that the ordinary membership of the Society is now 4485, including 436 nominees of institutional members and 38 life members. The membership of the Society continues to show a slow, but steady, growth. There are also 125 institutional members. The total attendance at all meetings in 1953 was 2748; the number of papers read was 790; there were 16 hour addresses, 4 Colloquium Lectures, 1 Gibbs Lecture, and 21 papers at the Applied Mathematics Symposium. The number of members attending at least one meeting was 1600.

The following appointments by the President were reported: as a committee to nominate a representative of the Society on the Policy Committee for Mathematics: Deane Montgomery, Chairman, A. A. Albert and E. R. Kolchin; as a committee to cooperate with a Committee of the Mathematical Association of America on a Placement Service: Leon W. Cohen, Chairman, and James A. Clarkson; as a committee to nominate the award of the Cole Prize in Algebra: Oscar Zariski, Chairman, Richard Brauer and C. C. MacDuffee; as a Committee on Arrangements for the Summer Meeting in Laramie, Wyoming, August 30–September 3, 1954: C. F. Barr, Chairman, H. M. Gehman, J. W. Green, Greta Neubauer, O. H. Rechard, Nathan Schwid, S. R. Smith, W. N. Smith, P. O. Steen and V. J. Varineau; as a member of the Committee on Places of Meetings for a term of three years beginning January 1, 1954: Professor W. M. Whyburn (Committee consists of Professors M. F. Smiley, Chairman, E. R. Lorch and W. M. Whyburn); as a member of the Committee on Visiting Lectureships for a term of three years beginning January 1, 1954: Professor Paul Halmos (Committee now consists of Professors Garrett Birkhoff, Chairman, Einar Hille and Paul Halmos); as Chairman of the Committee on Applied Mathematics: F. J. Murray; and as members for terms of three years beginning January 1, 1954: R. S. Burington and R. E. Bellman; as members of the Committee to Select Hour Speakers (terms to expire December 31, 1956): Summer and Annual Meetings: Wilfred Kaplan (Committee now consists of E. G. Begle, Chairman, Oscar Zariski and Wilfred Kaplan); Eastern Sectional Meetings: Nelson Dunford (Committee now consists of L. W. Cohen, Chairman, Nelson Dunford and A.

M. Gleason); Southeastern Sectional Meetings: W. V. Parker (Committee now consists of J. H. Roberts, Chairman, J. J. Gergen and W. V. Parker); Western Sectional Meetings: M. E. Shanks (Committee now consists of J. W. T. Youngs, Chairman, E. E. Moise and M. E. Shanks); Far Western Sectional Meetings: J. L. Kelley (Committee now consists of J. W. Green, Chairman, R. D. James and J. L. Kelley); as tellers for the 1953 annual election: W. S. Massey and F. M. Stewart.

The following appointments to represent the Society were reported: at the inauguration of Perry Epler Gresham as President of Bethany College on October 23, 1953: Professor Bradford Tye; at the inauguration of Clarence B. Hilberry as President of Wayne University on Monday, November 9, 1953: Professor T. H. Hildebrandt; at the Seventy-fifth Anniversary of the founding of Duquesne University, November 11 and 12: Professor J. S. Taylor.

The Secretary reported that Professor Paul White had accepted an invitation to deliver an hour address at the May 1954 meeting at Yosemite National Park and that Professor Enrico Fermi had accepted an invitation to deliver the Gibbs Lecture in 1954.

The Council approved the substitution of Professor E. R. Kolchin as a representative of the Society on the Editorial Board of the American Journal of Mathematics for the academic year 1953–54.

The Council voted to establish a new periodical to be called Notices of the American Mathematical Society to be under the editorial supervision of the Editorial Committee for the Bulletin and to contain programs of meetings, the notes and the list of new publications which now appear in the Bulletin as well as special notices to members of the Society.

The Council approved the following times and places of sectional meetings in 1954: February 27 at Hunter College, New York City; April 30 and May 1 at the University of Chicago; May 1 at Yosemite National Park; October 30 at Massachusetts Institute of Technology.

The Council voted to accept an invitation to hold the 1954 annual meeting December 27–29 at the University of Pittsburgh.

The Bulletin Editorial Committee reported that the journals used 622 pages in 1953. The Council voted to recommend to the Board of Trustees that the Bulletin be authorized for 1954 to print 625 pages less the number of pages of notes and lists of new periodicals which would be transferred to the Notices.

The Transactions and Memoirs Editorial Committee reported that the interval between receipt of a manuscript and publication is approximately one year. The Council voted to recommend to the

Trustees that 1100 pages be published in the Transactions for 1954.

The Mathematical Reviews Editorial Committee reported that 1150 pages had been published in 1953. The subscription list as of November, 1953 was 2287.

The Proceedings Editorial Committee reported that the interval between receipt of a manuscript and publication was approximately eight months. The Council voted to recommend to the Board of Trustees that 1000 pages be authorized for the 1954 Proceedings. Professors Bernard Friedman, W. H. Gottschalk, Marshall Hall, and J. A. Jenkins were reported as new assistant editors for the Proceedings.

On recommendation of the Colloquium Editorial Committee the Council voted to invite Professor Nathan Jacobson to deliver the 1955 Colloquium Lecture.

The Council voted to co-sponsor with the Office of Ordnance Research a Second Symposium in Applied Mathematics at the University of Chicago on April 29 and 30, 1954.

The Council voted with the cooperation of the Mathematical Association of America to establish an Employment Register. This Register will consist of notices from academic institutions, industrial firms, and government agencies of vacant positions. These notices will be available for inspection at the Summer and Annual Meetings and at the April Meetings.

The Council adopted the following resolution:

> The Council of the American Mathematical Society here takes cognizance of the resignation this past September of Dr. Mina Rees as Head of the Mathematics Section of the Office of Naval Research. She has accepted a position as Dean of the Faculty at Hunter College. We congratulate Hunter College on this wise selection and can only say that our heavy loss as mathematicians is the gain of Hunter College.
>
> The very striking and brilliant contributions made by pure (non-military, non-applied) science, not least of these by mathematics, to the winning of World War II is well known. It was clearly seen by the government and those responsible for the armed services that a large scale fostering by the U. S. government of fundamental research, the basis of all research, was unavoidable. Only thus could we hope to hold our own in years to come, and incidentally build up a suitable reserve of talented men for emergencies. This was actually acted upon by the Navy who thus took the lead by some years with the creation of the Office of Naval Research. Needless to say as the purest of all sciences, mathematical research might well have lagged behind in such an undertaking. That nothing of the sort happened is beyond any doubt traceable to one person—Mina Rees. Under her guidance, basic research in general, and especially in mathematics, received the most intelligent and wholehearted support. No greater wisdom and foresight could have been displayed and the whole postwar development of mathematical research in the United States owes an immeasurable debt to the pioneer work of the Office of Naval Research and to the alert, vigorous and far-sighted policy conducted by Miss Rees. The influence of these policies has been such

that it vitally affected later developments: the activities of Air Force and Ordnance research, the National Science Foundation itself. It is well known that in these more recent organizations Mina Rees was constantly appealed to for counsel and guidance.

As Miss Rees leaves her task, the Council of the American Mathematical Society desires to express to her in the name of the whole mathematical community its warmest feelings of appreciation for her past performance and extends to her its best wishes for the future.

There were fifteen sessions for contributed papers, including two sessions for late papers, presided over by Professors I. A. Barnett, V. F. Cowling, L. R. Ford, R. E. Fullerton, Casper Goffman, G. B. Huff, W. A. Hurwitz, R. E. Johnson, Saunders MacLane, A. N. Milgrim, P. C. Rosenbloom, Ernst Snapper, E. W. Titt, A. D. Wallace, W. M. Whyburn. The abstracts of the papers are appended. Those having the letter "*t*" following the abstract number were read by title. Where a paper, presented in person, has more than one author, the symbol (p) follows the name of the author who presented it. Professor Hanan Rubin was introduced by Professor J. J. Stoker, Dr. Shimada and Dr. Uehara by Professor W. S. Massey, Mr. S. I. Plotnik by Professor Orrin Frink, Dr. P. S. Mostert by Professor S. T. Hu, Professor G. E. Uhlenbeck and Mr. G. W. Ford by Professor Frank Harary, Mr. J. H. Monahan by Professor H. G. Haefeli, Professor Hao Wang by Mr. Joel Pitcairn, Dr. W. B. Jurkat by Professor C. N. Moore, Mr. W. J. Feeney by Professor O. J. Ramler, and Professor John Nash, Dr. Schwartz, Captain R. H. Pennington and Mr. J. H. Case by Professor Leon W. Cohen.

ALGEBRA AND THEORY OF NUMBERS

177*t*. S. A. Amitsur: *Differential polynomials and division algebras. I. The resultant of differential polynomials.*

Let $F[t]$ be the ring of differential polynomials in t with multiplication defined by $at = ta + a'$, where $a \to a'$ is a derivation in a field F of characteristic zero. Using a correspondence between differential operators and differential polynomials (Jacobson, Ann. of Math. vol. 38 (1937) pp. 484–507), make correspond to any two polynomials $p(t)$, $q(t)$ of degrees n and m a polynomial $p(t) \times q(t)$ of degree nm. This new operation $\times$ is proved to be associative, commutative, and distributive with respect to multiplication and decomposition between classes of similar polynomials Let $p(t) = \sum t^{\nu} a_{\nu}$, define the adjoint polynomial $p^*(t) = \sum a_{\nu}(-t)^{\nu}$, and the nullity of $p(t)$ to be the maximal number of independent solutions in F (over the field of constants) of the linear differential equation $\sum a_{\nu} z^{(\nu)} = 0$. With these definitions it is shown: (1) A necessary and sufficient condition that there exists a factorization $p(t) = p_1(t)p_2(t)$, $q(t) = q_1(t)q_2(t)$ such that $p_1(t)$ is a non-constant polynomial similar to $q_2(t)$ is that the nullity of $p(t) \times q^*(t)$ be >0. (In a special case this can be shown to yield the well known result on the resultant of commutative polynomials.) Refer to $p(t) \times q^*(t)$ as the resultant of $p(t)$ and $q(t)$. (2) The nullity of $p(t) \times p^*(t)$ is exactly the dimension of the invariant ring of $p(t)$, that is: the ring of all classes of $F[t]/p(t)F[t]$ whose representatives $q(t)$ satisfy $q(t)p(t) \equiv 0 \bmod p(t)F[t]$. (The finiteness of this dimension was shown by Ore, see also Jacobson, Ibid. p. 507.) (Received November 12, 1953.)

178. S. A. Amitsur: *Differential polynomials and division algebras. II. Central simple algebras.*

Let C be a field of characteristic zero algebraically closed in an extension field F. Define in F a derivation: $a \to a'$ with C as the field of constants. Using the notation of the preceding abstract, a polynomial $p(t)$ in $F[t]$ of degree n is called an A-polynomial if the nullity of $p(t) \times p^*(t)$ is n^2. The main results obtained are: (1) The invariant ring of an A-polynomial $p(t)$ of degree n is a central simple algebra A_p of order n^2 over C and is split by F; conversely, every central simple algebra of order n^2 over C which is split by F is isomorphic to some invariant ring A_p of an A-polynomial $p(t)$ of degree n. In particular, the algebra is a division algebra if and only if the corresponding A-polynomial is an irreducible polynomial. (2) $A_p \times A_q \simeq A_{p \times q}$. (3) If $p(t) = \sum t^\nu a_\nu$ is an A-polynomial of degree n then the set of *all* solutions of the differential equation $\sum a_\nu z^{(\nu)} = 0$ has a basis of the form $x_i y^{1/m}$, $i = 1, \cdots, n$, where x_i, $y \in F(D)$ with D any splitting field of A_p, and where m is the exponent of A_p. As an application of the preceding results it is shown that: if D is a normal extension of C with a Galois group G, then the Brauer group of all division algebras split both by F and D is isomorphic with the first cohomology group of G with coefficients in the multiplicative group $[F(D)]^*/D^*$ where $(^*)$ indicates the multiplicative group of all nonzero elements. The last result is obtained also by methods of the cohomology theory of groups. (Received November 12, 1953.)

179. N. C. Ankeny: *Quadratic forms in quadratic real fields.*

One defines certain fundamental domains for Hilbert modular groups associated with a quadratic real extension of the rationals. One shows that certain group extensions are necessary if the class number of the field is greater than 1. One then applies these results to representations of numbers by quadratic forms. This is a generalization of certain results of Maass. (Received October 21, 1953.)

180*t*. H. W. Becker: *Maverick and parametric solutions of* $\epsilon^2\zeta^2 - \eta^2\theta^2$, $\epsilon^2\eta^2 - \zeta^2\theta^2$, $\epsilon^2\theta^2 - \zeta^2\eta^2 = \boxed{s}$.

Euler proposed this problem, Dickson's *History* II, p. 661–662, and came up with one solution: 25, 12, 39, 20. But a solution is deducible from any triad of Petrus vectors with common index. Thus from $(59^4 - 58^4)\{(9^4 - 7^4)$ or $(11^4 - 2^4)\} = \boxed{s}$ one gets the solutions: 185, 12, 39, 148 and 925, 117, 60, 444. Triadizing with $137^4 - 7^4$ (also of index 65) one gets 97, 48, 780, 388; 485, 468, 300, 291; and 3589, 468, 2220, 291. From $(31^4 - 27^4)\{(7^4 - 3^4)$ or $(17^4 - 1)\} = \boxed{s}$ one gets the solutions 65, 29, 24, 78 and 654, 264, 319, 545. Triadizing with $271^4 - 213^4$ (also of index 145) one gets the solutions 545, 319, 218, 88 and 1308, 638, 1090, 528. The foregoing are all mavericks, as to this problem. But as by-products of P.V. forms there are the parametric solutions: $[\epsilon = \eta,\ \zeta,\ \theta] = [r^4 + s^4,\ 2rs(r^2 \pm s^2)]$; $[r^8 + 20r^6s^2 - 26r^4s^4 + 20r^2s^6 + s^8,\ 2(r^4 - 6r^2s^2 + s^4)\{4rs(r^2 - s^2),\ (r^2 + s^2)^2\}]$; $[(r^2 - s^2)(r^8 - 4r^6s^2 - 6r^4s^4 - 4r^2s^6 + s^8),\ (r^2 \pm s^2)(r^8 + 4r^6s^2 - 6r^4s^4 + 4r^2s^6 + s^8)]$; $[\Pi(9r^4 \pm 6r^3s + 2r^2s^2 \pm 2rs^3 + s^4),\ 4rs(r^2 - s^2)(9r^2 - s^2)(3r^2 + s^2),\ (9r^4 \pm 6r^3s + 2r^2s^2 \pm 2rs^3 + s^4)]$. The latter is the Hillyer case of any Carmichael vector in which $2ab(a^2 - b^2) = 2cd(c^2 - d^2) = y^{1/2}$, $(a^2 + b^2)\ (c^2 + d^2) = t_1t_2 = t$. In terms of such C.V., $[\epsilon = \eta,\ \zeta,\ \theta] = [t,\ y,\ t_1^2$ or $t_2^2]$; this is equivalent to a Petrus transform, with

$y(\zeta_2^2 \pm \zeta_1^2) = 2uv$, $2xz$; these also constitute a Hero △ of 2 sq. sides ζ_1^2, ζ_2^2 and alt. y. (Received November 16, 1953.)

181*t*. H. W. Becker: *O'Riordan conjugates of rational cuboids.*

Let $x^2+y^2=u^2$, $y^2+z^2=v^2$, $x^2+z^2=w^2$. Utilize O'Riordan's forms, Dickson's *History* II, p. 504, under C, α or δ, B, β, γ, A or $D \to x, y, z, u, v, w$. The resulting quadratics are $e/f = [3ab(c^2-d^2)+cd(a^2-b^2) \pm 2\Delta_e^{1/2}]/(ad-bc)^2$, $\Delta_e = 2ab(c^2-d^2)(ac-bd)(ad+bc)$; same, under $e, f \leftrightarrow a, b$; $c/d = [(ae-bf)(af+be) \pm 2\Delta_c^{1/2}]/[(af-be)^2-4abef]$, $\Delta_c = 2abef\{(ae+bf)^2-(af-be)^2\}$. The forms $\Delta = \square$ are a new approach to rational cuboids. Moreover, from a known cuboid K, if reducible to O'Riordan's forms via $v \pm z$, $u \pm x$, etc., 3 new cuboids may be found, the conjugates of K under change of sign of $\Delta_{a,c,e}^{1/2}$. From each of these 2 more may be found, etc. ad inf. Thus in the Rignaux-Crussol parametric solution, $a, b; c, d; e, f = (2r-s)(r+2s)(r-s)^2, rs(r-3s)^2; r(2r-s), s(2r+s); r(r+2s)(r^2+s^2), 2s(2r-s)(r^2-rs-s^2)$, the first set of conjugates is: $a', b'; c', d'; e', f' = (2r-s)(r+2s)(r^3+r^2s+4rs^2+2s^3)^2, rs(7r^3+r^2s+2s^3)^2$; (c, d) (12th degree polynomials); $e(2r^2-6rs+5s^2)^2, fs^2(2r-s)^2$. Here the 1st and 3d conjugates are equivalent, and both cover nos. 1, 19 and 20 of Kraitchik's table, Scripta Mathematica vol. 11 (1945) p. 326, in which 41 mavericks now remain. In the classic Saunderson solution, $a, b; c, d; e, f = (r^2+s^2)^2, 8rs(r^2-s^2); r, s; r, s$, the 1st set of conjugates being: (degenerate); $r(r^8+52r^6s^2+118r^4s^4-176r^2s^6+17s^8)$, same under $r \leftrightarrow s$; $r(r^4 10r^2s^2-7s^4)^2$, same under $r \leftrightarrow s$. (Received November 16, 1953.)

182*t*. Leonard Carlitz: *Congruence properties of special elliptic functions.*

In a previous paper: *The coefficients of singular elliptic functions* (to appear in Math. Ann.) the writer derived certain divisibility properties of the coefficients of $\operatorname{sn} x$ and $x/\operatorname{sn} x$ when the function admits of complex multiplication. In the present paper it is assumed that the modulus k^2 satisfies $W_p(k^2)=0$, where $W_p(u) = \sum_{s=0}^{m} \binom{m}{s}^2 u^s$ and $p=2m+1$ is a prime. Results similar to but somewhat less precise than those in the singular case are obtained. For example if $\operatorname{sn} x = \sum \alpha_m x^m/m!$, then $\alpha_m \equiv 0 \pmod{p^r}$ for $m \geqq pr$; if $x/\operatorname{sn} x = \sum \beta_m x^m/m!$, then $\beta_m \equiv 0 \pmod{p^{r+t}}$ for $m > pr$, $p-1 \nmid m$, $p^t \mid m$, while if $p-1 \mid m$ then $\beta_m \equiv 0 \pmod{p^{r-1}}$. The coefficients of $(x/\operatorname{sn} x)^k$, $1 \leqq k < p-1$, are also discussed. The polynomial $W_p(u)$ is related to the Legendre polynomial; a number of results on the factorization of $W_p(u) \pmod{p}$ are obtained. (Received November 14, 1953.)

183*t*. Leonard Carlitz: *The number of solutions of a special quadratic congruence.*

Let p be an odd prime and $m, r, t \geqq 1$. Let $A=(a_{ij})$ be a square matrix of order m, where the a_{ij} are integers such that $\det A \not\equiv 0 \pmod{p}$. Making use of some results of Siegel on the matric congruence $X'AX \equiv B \pmod{p^r}$ (Ann. of Math. (2) vol. 36 (1935) pp. 527–606), one determines the number of solutions X of the congruence $\left|\begin{smallmatrix} A & X \\ X' & 0 \end{smallmatrix}\right| \equiv b \pmod{p^r}$, where $b \not\equiv 0 \pmod{p^r}$ and X is an $m \times t$ matrix. (Received November 14, 1953.)

184*t*. Leonard Carlitz: *The number of solutions of some equations in a finite field.*

The types of equations discussed are (1) $Q(\xi_1, \cdots, \xi_{2r+1}) = g(\eta_1, \cdots, \eta_s)$ where Q denotes a quadratic form with coefficients in $GF(p^n)$ and the polynomial $g(\eta)$ is a product of quadratic, cubic, or quartic polynomials of certain kinds. Application is made of results on Jacobsthal sums. (2) As in (1), with $g(\eta) = \prod_{i=1}^{s} (\eta_i^p - \eta_i + \alpha_i)$. (3) $\sum_{i=1}^{s} \alpha_i(\eta_i^p - \eta_i) = \alpha$ and $Q(\xi) = \sum_{i=1}^{s} \alpha_i(\eta_i^p - \eta_i) + \alpha$. (4) $\xi^f = \eta^p - \eta + \alpha$, where $f \mid p^n - 1$. (Received November 14, 1953.)

185*t*. A. H. Clifford: *Bands of semigroups.*

A semigroup S is called a *band of semigroups of type* $\mathfrak{T}$ if it is the class sum of a set of mutually disjoint subsemigroups S_α $(\alpha \in I)$, each of type $\mathfrak{T}$, such that the product $S_\alpha S_\beta$ of any two of them is wholly contained in a third: $S_\alpha S_\beta \subseteq S_\gamma$. S is called a *semilattice of semigroups of type* $\mathfrak{T}$ if $S_\alpha S_\beta$ and $S_\beta S_\alpha$ are both contained in the same S_γ. S is called a *matrix of semigroups of type* $\mathfrak{T}$ if the index class I is a direct product $J \times K$ of two classes J and K, and if $S_{i\kappa} S_{j\lambda} \subseteq S_{i\lambda}$; for all $i, j \in J$ and all $\kappa, \lambda \in K$. It is shown that a band of semigroups of type $\mathfrak{T}$ is a semilattice of semigroups each of which is a matrix of semigroups of type $\mathfrak{T}$. Conditions are given on a semigroup in order that it be a band or semilattice of (1) simple semigroups, (2) completely simple semigroups, and (3) groups. (Received November 12, 1953.)

186. A. H. Clifford: *Naturally totally ordered commutative semigroups.*

A commutative semigroup S is called *naturally totally ordered* (n.t.o.) if the division relation in S is a total (i.e. "linear" or "simple") ordering of S. O. Hölder [Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math.-Nat. Kl. vol. 53 (1901) pp. 1–64] showed that every archimedean n.t.o. semigroup S in which the cancellation law holds can be embedded in the additive semigroup P of all positive real numbers, and that P is to within isomorphism the only such semigroup which is complete in the sense that every bounded subset of P has a LUB in P. In the present paper, analogous results are shown for archimedean n.t.o. semigroups in which the cancellation law does not hold. In this case, there are exactly two non-isomorphic complete such semigroups, in which all others can be embedded. Commutative n.t.o. semigroups in general have been investigated by F. Klein-Barmen [Math. Zeit. vol. 48 (1942–43) pp. 715–734; Math. Zeit. vol. 51 (1947–49) pp. 355–366], who calls them "linear holoids. A reduction theorem is given which generalizes the main result of the first of these two papers. (Received November 12, 1953.)

187. W. F. Eberlein: *A note on Fourier-Stieltjes transforms.*

Given a locally compact Abelian group G and a bounded Radon measure μ on the character group G^*, the Fourier-Stieltjes transform $\hat{\mu}(x) = \int_{G^*} \overline{(x, y)} d\mu(y)$ is a weakly almost periodic function on G. Then $M(|\hat{\mu}|^2) = \sum_{G^*} |\mu\{y\}|^2$. Previous results of this type have presupposed the existence of a sequential limit representation for the mean value M. (Received November 16, 1953.)

188. Gertrude Ehrlich: *Structure theorem for continuous rings.* Preliminary report.

A regular ring whose lattice of right (or left) ideals is isomorphic to the lattice of subspaces of a continuous geometry is called a continuous ring (see von Neumann, Proc. Nat. Acad. Sci. U.S.A. vol. 22 (1936) pp. 92–100, 101–108 and 707–713; vol. 23 (1937) pp. 16–22 and 341–349). It is shown here that, for a continuous ring which is not of characteristic 2, the multiplicative group of nonsingular elements determines the ring to within isomorphism or anti-isomorphism, and hence determines the associated continuous geometry to within isomorphism or dual isomorphism. This is an extension to the continuous case of the structure theorem known for finite- and infinite-dimensional discrete projective geometries. The methods employed are based on Baer's treatment of the discrete case (R. Baer, *Linear algebra and projective geometry*, Academic Press, 1952, Chap. VI) in which systems of involutions having a common plus or minus invariant space play a decisive role. (The results presented in this paper constitute part of a doctor's dissertation written at the University of Tennessee under the direction of Professor Wallace Givens). (Received November 4, 1953.)

189. Harley Flanders: *The norm function of an algebraic field extension.* II.

Let K be a finite extension of a field k and let $F(X)$ be in the ring $K[X_1, \cdots, X_n]$. It is shown that if F is irreducible, then $N_{K/k}(F)$ is a power of an irreducible in $k[X]$. From this it follows that the general norm is a prime power; the author's previous result (Pacific Journal of Mathematics vol. 3 (1953) p. 103) characterizing the norm function is a consequence. It is shown that standard properties of the norm and trace follow from this, for example, their transitivity. (Received November 4, 1953.)

190. Franklin Haimo: *Semi-direct products with ample homomorphisms.* Preliminary report.

Let G be the semi-direct product of groups H and K with a homomorphism ϕ on K into the automorphism group of H [A. Malcev, *On linear Lie groups*, C. R. Acad. Sci. URSS vol. 40 (1943) pp. 87–89]. Let ϕ be *ample* in that $\phi(K) \supset J(H)$, the set of inner automorphisms of H. The G/Z_i are found, where $Z_i(G) = Z_i$ is the ith member of the ascending central series of G. In particular, a simple formulation is given for the ascending central series of the holomorph of H. An equivalent condition is found for each $Z_i(G)$ to be the set of all (h, k), $h \in Z_i(H)$, $k \in Z_i(K)$. This work has been supported, in part, by the OSR, USAF. (Received November 13, 1953.)

191*t*. Marshall Hall and H. J. Ryser: *Normal completions of incidence matrices.*

Let A_1 be an r by v incidence matrix of zeros and ones, which satisfies the equation $A_1 A_1^T = B_1$. Here A_1^T denotes the transpose of A_1 and B_1 is the r by r matrix with k in the main diagonal and $\lambda = k(k-1)/(v-1)$ in all other positions. Let B be the v by v matrix with k in the main diagonal and λ in all other positions. Suppose that B is rationally congruent to the identity. Then there exists a v by v rational and normal matrix A having A_1 as its first r rows such that $AA^T = A^T A = B$. Integral solutions of the matric equation $AA^T = B$ are also studied. In particular, an integral solution of this equation is exhibited for the geometry case of $N = 10$. The solution is not normal, and, of course, not a geometry. (Received October 19, 1953.)

192. Emilie V. Haynsworth: *Quasi-stochastic matrices.*

Given a matrix, $A = (a_{ij})$, let $R_i = \sum_{j=1}^{k} a_{ij} + p \sum_{j=k+1}^{n} a_{ij}$ $(i = 1, \cdots, n)$, where p is an arbitrary complex number, and k is an integer, $1 \leq k \leq n$. A is here defined as a generalized quasi-stochastic matrix if $R_i = S$ $(i = 1, \cdots, k)$, and $R_i = pS$ $(i = k+1, \cdots, n)$. Obviously, if A has non-negative elements, and $S = p = 1$, A is stochastic. This paper generalizes some of the theorems on stochastic matrices for quasi-stochastic matrices. For instance, one root is always S, and, when $p > 0$, $a_{ij} \geq 0$, S is the maximum root. Also, the new bounds for the roots of stochastic matrices obtained by A. Brauer [Duke Math. J. vol. 19 (1952) pp. 75–91] are applied to quasi-stochastic matrices. In addition to the above generalizations on stochastic matrices, it is shown that certain types of matrices are generalized quasi-stochastic for two different values of p, and thus two roots can be found by inspection. Other results concerning the location of the roots of these matrices are given, and group properties for the set are discussed. (Received November 13, 1953.)

193. Arno Jaeger: *On rings of differential operators over fields of prime number characteristic.* II.

Using the definitions and the notations of the first part of the paper (Bull. Amer. Math. Soc. Abstract 59-2-163) the ring R_t of differential operators is now shown to be (isomorphic to) the ring of *all* endomorphisms of F^+ over F_t. If A denotes the matrix (with elements in F_t) corresponding to the differential operator α relative to the basis $1, x, \cdots, x^{p^t-1}$ of F^+ over F_t the matrix A^* corresponding to the adjoint operator $\alpha^* = \sum_{n=0}^{p^t-1} (-1)^n D^n \cdot a_n$ is obtained from A by reflection of the elements about the matrix diagonal from the upper right to the lower left. The methods established are applied to find the solutions and to discuss the solvability of differential equations in the usual sense of this theory and of differential equations $\sum_{n=0}^{\infty} a_n D^n y = 0$ *of infinite order*. (Received December 11, 1953.)

194t. W. E. Jenner: *Arithmetics of Lie algebras.* Preliminary report.

Let $\mathfrak{o}$ be a Dedekind ring of characteristic zero with quotient field k and let $\mathfrak{L}$ be a semisimple Lie algebra of finite dimension over k. A subring $\mathfrak{O}$ of $\mathfrak{L}$ is said to be *integral* if it is a finitely-generated $\mathfrak{o}$-module and contains a k-basis for $\mathfrak{L}$. A discriminant ideal $\mathfrak{D}$ is associated with $\mathfrak{O}$ in analogy with the definition for associative algebras but using the discriminant of E. Cartan. If $\mathfrak{O}'$ is an integral subring with discriminant $\mathfrak{D}'$ and such that $\mathfrak{O}' \supseteq \mathfrak{O}$, then $\mathfrak{O}' \supset \mathfrak{O}$ if and only if $\mathfrak{D}' \supset \mathfrak{D}$. Every integral subring can be embedded in a maximal integral subring. If $\mathfrak{p}$ is a prime of k, then $\mathfrak{O} - \mathfrak{p}\mathfrak{O}$, as an algebra over $\mathfrak{o}/\mathfrak{p}$, is a direct sum of simple nonabelian algebras if $\mathfrak{p}$ does not divide $\mathfrak{D}$. The proof depends on a result of J. Dieudonné. For any prime $\mathfrak{p}$ of k let $\mathfrak{o}_\mathfrak{p}$ denote the ring of local integers at $\mathfrak{p}$ and set $\mathfrak{O}_\mathfrak{p} = \mathfrak{o}_\mathfrak{p}\mathfrak{O}$. Then $\mathfrak{O}_\mathfrak{p}$ is a multiplicative unit operator on its ideals for almost all $\mathfrak{p}$. A direct-intersection ideal theory is obtained for Lie rings which are multiplicative unit operators on their ideals. (Received November 14, 1953.)

195. J. B. Kelly: *A characteristic property of quadratic residues.*

Let p be a prime of the form $4k+1$. Let $r_1, r_2, \cdots, r_{2k}$ denote the quadratic residues (mod p), exclusive of 0, and let $n_1, n_2, \cdots, n_{2k}$ denote the quadratic nonresidues (mod p). The following theorem is well known. Theorem I: The set of all residues $r_i + n_j$ where i is arbitrary but fixed and j runs from 1 to $2k$ contains exactly as many quadratic residues as nonresidues; so does the set $r_i + n_j$ where j is arbitrary but fixed and i runs from 1 to $2k$. We have established the following converse of Theorem I. Theorem II: Let m be a positive integer of the form $4k+1$. Let the nonzero residues (mod m) be divided into two mutually exclusive classes of $2k$ elements each: $A\{1=a_1, a_2, \cdots, a_{2k}\}$ and $B\{b_1, b_2, \cdots, b_{2k}\}$ such that the set of all residues $a_i + b_j$, where i is arbitrary but fixed and j runs from 1 to $2k$, contains exactly as many elements of A as of B and such that the set $a_i + b_j$, where j is arbitrary but fixed and i runs from 1 to $2k$, has the same property. Then (1) m is a prime, p. (2) A consists of the quadratic residues (mod p) and B consists of the quadratic nonresidues (mod p). Theorems similar to I and II hold for positive integers of the form $4k-1$. The analogue of Theorem I is true in any finite field; this is not the case for Theorem

II. The principal tool employed in the proof of Theorem II is cyclotomy. (Received November 16, 1953.)

196*t*. M. S. Klamkin: *On vector sums and products.*

In most treatments of vectors it is usual to introduce the definitions of vector sums and products ad hoc, and then to develop their properties. In this paper the usual expressions for the vector sum, scalar product, and vector product of two vectors will be derived from postulated properties which have direct physical significance. The vector sum has been derived previously by Poisson in this manner. However, the proof presented here is a modified version. (Received November 14, 1953.)

197*t*. Serge Lang: *A new proof of Harnack's theorem.*

Harnack (Math. Ann. (1876)) proved that there are at most $g+1$ real branches on a real curve of genus g. We give a new proof based on the Riemann-Roch theorem. Remark first that if x is a real function on the curve, then x cannot have a single zero of order 1 on a branch B. (Indeed, B is homeomorphic to a circle. If x has a single zero p on B, then x changes sign in a neighborhood of p, and gives a continuous image of $B-\{p\}$. If x has no other zero or pole on B, this is impossible.) Now if there are $g+2$ branches, select a place on each branch, and let $p_1, \cdots, p_{g+2}$ be these places. Let q be a complex place and its conjugate, so q has degree 2. The divisor $p_1+\cdots+p_{g+2}-(g+1)q$ has degree $-g$. By Riemann-Roch, there exists a function x, multiple of this divisor, which will have at most g more real zeros on the branches. Hence on at least two branches x will have a single zero, a contradiction. (Received November 12, 1953.)

198*t*. B. H. Neumann: *Groups covered by permutable subsets.*

Using his results on unitary representations (Duke Math. J. vol. 17 (1950) pp. 437–441), F. I. Mautner has proved (unpublished): If the group G has a finite subgroup K whose double cosets in G permute, and if H denotes the subgroup consisting of all elements with finitely many conjugates in G, then H has finite index in G. An elementary proof of the sharper result $G=KH$ is now given, and extended as follows: The group G can be covered by permutable boundedly finite subsets if, and only if, G has a subgroup of finite index whose derived group is finite. It is also shown that every finitely generated free group, but no nondenumerable free group, can be covered by permutable finite (but unbounded) subsets. (Received November 13, 1953.)

199*t*. L. J. Paige: *Power associative loop algebras.*

Let L be a finite loop, written multiplicatively, and F an arbitrary field. Construct the *loop algebra* A of L over F by considering the elements of L as linear independent basis elements for a vector space A over F and define multiplication in A by the use of both destributive laws and the definition of multiplication in L. If L is not associative (not a group), A will be a linear nonassociative algebra of finite order over F. The author proves: (i) For fields F of characteristic not 2, the assumption that A is commutative and power associative implies that A is associative. (ii) The existence of non-commutative, power associative (not associative) loop algebras. (Received November 12, 1953).

200*t*. M. A. Rosenlicht: *Automorphisms of function fields.*

This concerns the theorem that the group of automorphisms (leaving the elements of the ground field fixed) of an algebraic function field of one variable of genus greater than one over an algebraically closed ground field is finite. An easy proof is given in which one of the classical proofs for function fields over the complex numbers is generalized by replacing integration by the jacobian variety of the function field. The remainder of this paper deals with the extension of this result to the case in which the ground field is not algebraically closed. (Received November 14, 1953.)

201*t*. E. V. Schenkman: *The structure of the group of automorphisms of a nilpotent group.*

Let G be a nilpotent group; that is, let $G^{\omega}=E$, the identity, where $G^{\omega}=\bigcap_{i=1}^{\infty}G^{i}$ and $G^{i}=[G^{i-1}, G]$, $G^{1}=G$. Let A be the group of automorphisms of G and let A_i be the subgroup of automorphisms of A which induce the identity transformation in G/G^{i}. Then some of the results obtained are as follows: A_1 is a normal nilpotent subgroup of A of class at most one less than that of G; $[A_k, A_l]\subseteq A_{k+l-1}$. If H is the holomorph of G, then the subgroup of H generated by G and A_1 is normal in H and nilpotent of the same class as G. (Received November 16, 1953.)

202*t*. Alfred Tarski: *On equationally complete rings and relation algebras.*

For terminology see Kalicki-Scott, Bull. Amer. Math. Soc. Abstract 58-6-582 and 59-1-116. The following two theorems provide a mathematical characterization for the class of equationally complete rings and for that of equationally complete relation algebras (thus presenting solutions for two problems formulated by Kalicki). Theorem I: *A ring* $\mathfrak{R}$ *with more than one element is equationally complete if and only if, for some prime* p, *either the equations* $px=0$ *and* $x^{p}=x$, *or else the equations* $px=0$ *and* $x\cdot y=0$, *identically hold in* $\mathfrak{R}$. Theorem II: *A relation algebra* $\mathfrak{R}$ (in the sense of Jónsson-Tarski, Amer. J. Math. vol. 74, p. 127 ff.) *with more than one element is equationally complete if and only if one of the three equations* $0';0'=0$, $0';0'=1'$, *and* $0';0'=1^{\circ}$ *as well as the equation* $x;1;\bar{x};1;(x\cdot 1'+\bar{x}\cdot 0');1;(x\ 0'+\bar{x}\cdot 1')=0$, *identically holds in* $\mathfrak{R}$. Partial results related to Theorem I are stated in Abstract 59-1-116 [Theorems (4) and (5)]. Also, a result related to Theorem II can be found there [Theorem (9)]; it is simpler than Theorem II, but applies only to relation algebras in which 1' is not included among the fundamental notions. (Received November 4, 1953.)

203. Morris Weisfeld: *Derivations in division rings.* Preliminary report.

The structure of a division ring Δ with center Φ is studied in relation to the existence of a restricted p-Lie algebra over Φ of derivations in Δ. If $\mathfrak{D}$ is such an algebra, the elements a in Δ such that $d(a)=0$ for all d in $\mathfrak{D}$ form a division subring Γ of Δ. It is known that such a situation occurs when Δ is commutative and is a purely inseparable extension of exponent one of Γ. (Cf. Jacobson, Amer. J. Math. vol. 66 (1944) pp. 645–648.) One can determine conditions for the existence of a Galois correspondence between division subrings of Δ containing Γ and certain Lie subalgebras of $\mathfrak{D}$. The ordinary Galois theory does not apply because one can show that all automorphisms of Δ leaving the elements of Γ fixed are inner automorphisms. Some other results concern the extension of a derivation defined in a division subring of Δ to a derivation in Δ. (Received November 13, 1953.)

204*t*. Edwin Weiss: *Bounded topological rings.*

A class of topological rings which includes both the compact and the discrete classes is studied without recourse to the structure theory of topological groups. By elementary means, a theorem on semisimple rings, which includes the standard Jacobson theorem and the Kaplansky theorem for the compact case, is proved. The unified proof clarifies the relation between the descending chain condition and compactness. (Received November 14, 1953.)

205. George Whaples (p) and R. E. MacKenzie: *Artin-Schreier equations in characteristic zero*. Preliminary report.

Let k be a regular local field with residue class field of characteristic p and consider the extensions C/k which are cyclic of degree p. Known result: If k has characteristic p, then every such C is generated by a root of an Artin-Schreier equation $x^p-x-\lambda=0$, $|\lambda|=1$, and there is a simple explicit formula for the norm residue symbol. New result: If k has characteristic 0, then with certain exceptions every C is generated by a root of an Artin-Schreier equation. A simple formula for the norm residue symbol can at present be defined only for elements of a specially chosen basis for $k^{\cdot}$ modulo $N(C/k)$. The exceptional C occur only when k contains primitive pth roots of unity; they are the fields generated by a pth root of a prime element. (Received November 16, 1953.)

ANALYSIS

206*t*. R. M. Baer: *Some partial ordering relations for self-adjoint operators*,.

Let A and B be self-adjoint and everywhere defined on the hilbert space H. Define $A \leqq B$ (strongly) if $E_B(\lambda) \leqq E_A(\lambda)$, all λ, where we write $E_A(\lambda)$ for the spectral resolution of A, and projections are ordered by inclusion of range spaces; define $A \leqq B$ (weakly) if $(Ax, x) \leqq (Bx, x)$, all vectors x in H; and $A \leqq B$ (normally) if $Ax \leqq Bx$, all x in H. Then if A and B are strongly comparable, they are weakly comparable; and strong and weak comparability are equivalent for commuting A and B. Consequently, for positive commuting A and B, all three forms of comparability are equivalent. All three partial orderings are unitary invariant and induce dualities under the mapping of nonsingular operators to their inverses. The relation between (strong) comparability and algebraic operations over the self-adjoint operator space is studied. (Received November 14, 1953.)

207*t*. R. M. Baer: *Lattice structure in the self-adjoint operator space.*

Let H be a separable hilbert space, and let $R(H)$ be the (real) linear space of all everywhere-defined self-adjoint operators on H. We define a *quasi-vector lattice* to be a real linear space L which is also a lattice with respect to a partial ordering $x \leqq y$, such that if $x \leqq y$ (for x and y in L), then $x+s \leqq y+s$ for all s in some subspace Z (called the vector lattice center) of L. Then, with respect to the strong partial ordering (see preceding abstract) for self-adjoint operators, $R(H)$ is a quasivector lattice with vector lattice center consisting of the scalar operators. $R(H)$ exhibits a number of the lattice characteristics of the abstract (L) and (M) spaces, provided the strong (operator) topology is substituted for the uniform (norm) topology. The results extend directly to the $R(H)$ intersection with weakly closed operator algebras. (Received November 14, 1953.)

208. Frederick Bagemihl and Leonard Gillman (p): *Generalized dissimilarity of ordered sets.*

An ordered set is $\mathfrak{c}$-*homogeneous* if every one of its intervals has power $\mathfrak{c}=2^{\aleph_0}$. Let M, $N\neq 0$ be ordered sets. M is defined to be *pseudo-similar* to N if there exists a function f, with $f(M)=N$, an ordered set T without gaps, and a set $\{M_t\}_{t\in T}$ of mutually exclusive intervals of M, such that $\sum_{t\in T} M_t=M$ and $f|M_t$ is a similarity or anti-similarity transformation of M_t $(t\in T)$. M is defined to be *essentially identical* to N if there is a pseudo-similarity f, with $f(M)=N$, having an associated set of intervals $\{M_t\}_{t\in T}$ such that $f|M_t$ is the identity on M_t $(t\in T)$ minus any border elements that it may have. If an ordered set M, with $|M|=\mathfrak{c}$, contains a $\mathfrak{c}$-homogeneous subset D that can be imbedded in the continuum, then M is the union of $\mathfrak{c}$ mutually exclusive $\mathfrak{c}$-homogeneous sets E_τ having the following properties: There is associated with every E_τ a certain $u_\tau\in E_\tau$ such that $F_\tau=E_\tau-\{u_\tau\}$ is a $\mathfrak{c}$-homogeneous subset of D dense in D. Let U be the set of all these elements u_τ. For any τ, let I_τ be any unbordered interval of F_τ, $A\subset M$, $B\subset M$, and $|B\cap U|<\mathfrak{c}$. Then if $|I_\tau-A|<\mathfrak{c}$, we have not $A\simeq B^*$; if, in addition, $A\cup B\subset D$ and $I_\tau\cap A\neq I_\tau\cap B$, then A not $\simeq B$; if, furthermore, $A\subset I_\tau$ and A is not essentially identical to B, then A is not pseudo-similar to B. Some extensions, applications, and analogous results are also obtained. (Received November 16, 1953.)

209*t*. Stefan Bergman: *On solutions of linear partial differential equations in three variables.*

Using integral operators (see Trans. Amer. Math. Soc. vol. 68, pp. 468 and 500) which transform functions of two complex variables into harmonic functions $H(x, y, z)$ and solutions $\phi(x, y, z)$ of $\Delta\phi+C\phi=0$, where C is an entire function of $x^2+y^2+z^2$, the author investigates single and multivalued solutions H and ϕ and their singularities. He shows that various theorems about harmonic functions of two variables, which one obtains using analytic functions of one complex variable, can be generalized to the case of solutions ϕ. In particular he determines a complete set of particular solutions ϕ_n, $n=1, 2, \cdots$. This result leads to a new method for solving boundary and eigenvalue problems. Further he shows that between the properties of a subsequence of the coefficients $\{A_{mnp}\}$ of a series development $\phi=\sum A_{mnp}x^m y^n z^p$ on one side and the location and properties of line singularities of ϕ on the other side, there exist simple relations. They are independent of the coefficient C of the equation. They represent a generalization of theorems by Hadamard, Mandelbrojt and others about the relations between the coefficients of a series development of a meromorphic function $f(Z)$ and location of its poles. (Received December 28, 1953.)

210*t*. F. E. Browder: *On eigenfunction expansions for singular elliptic partial differential operators.* II. *Uniqueness theorems.*

By an eigenfunction expansion for an elliptic differential operator K of arbitrary order on a domain D is meant a (possibly infinite) sequence of eigenfunctions $s_r(x, \lambda)$ of $(K-\lambda)$ on D and a sequence of Lebesgue-Stieltjes measures ρ_r on the real line such that (1) setting $(S_rf)(\lambda)=\int_D s_r(x, \lambda)f(x)dx$, S_rf is defined in the square mean of ρ_r; (2) S_r is a partially isometric mapping of $L^2(D)$ onto $L^2(\rho_r)$ isometric on a subspace H_r; (3) $L^2(D)=\sum_r \oplus H_r$. To every self-adjoint realization K_1 of a formally self-adjoint elliptic K, there corresponds an eigenfunction expansion for which, if $\{E_\lambda\}$ is the resolution of the identity of K_1, $(E_\lambda-E_\mu)f(x)=\sum_r \int_\mu^\lambda s_r(x, \lambda)\{\int_D s_r(y, \lambda)f(y)dy\}d\rho_r(\lambda)$. The correspondence between eigenfunction expansions and self-adjoint realizations of K is one-one and both are uniquely determined by the sequence of eigenfunctions $\{s_r(x, \lambda)\}$. The usual uniqueness result for the "limit point" case for ordinary differential operators is a corollary of this general theorem. (Received November 16, 1953.)

211. R. H. Cameron: *The translation pathology of Wiener space.*

In this paper it is shown that almost no translations in Wiener space preserve measurability. It is also shown that Wiener measure is not expressible as an indefinite integral over a σ-finite translation invariant measure on a σ-algebra which includes all Wiener measurable sets. Finally, it is shown that an indefinite integral with respect to Wiener measure cannot be translation invariant unless it vanishes identically. (Received October 29, 1953.)

212. R. C. Carson: *Analytic functions in an annulus.*

The following problem is investigated. Let $\mathcal{J}$ be the family of all functions $f(z)$ regular and single-valued for $q<|z|<1$ which satisfy $|f(z)|\leq 1$, $|z|=1$ and $|f(z)| \leq p$, $|z|=q$. Given an arbitrary point z_0, $q<|z_0|<1$, find a function $F(z)$ in $\mathcal{J}$ such that $|F'(z_0)|\geq|f'(z_0)|$ for all $f(z)$ in $\mathcal{J}$. The function $F(z)$ is shown to belong to a certain one-parameter subfamily of $\mathcal{J}$ and the function is exhibited for certain classes of values of p. The methods used are related to those of R. M. Robinson in *Analytic functions in circular rings* (Duke Math. J. vol. 10 (1943) pp. 341–353). (Received November 16, 1953.)

213*t*. Mischa Cotlar: *A general interpolation theorem.* I.

Let $Tf=Tf(x)$ be a linear operator, defined on the set D of step functions of R^n, such that: (1) $\|Tf\|_2\leq C_2\|f\|_2$ for all $f\in D$. By a theorem of Riesz, (1) and (2) $\|Tf\|_1 \leq C_1\|f\|_1$ imply $\|Tf\|_p\leq C_p\|f\|_p$ $(1<p<2)$. Let $S(f)=\{x;\ |f(x)|>0\}$, $m(f)=\inf |f(x)|$ for $x\in S(f)$, Λf a bounded operator on $L^p\cap D$, and $S_\Lambda(f)=\{x;\ |\Lambda f(x)|\geq m(f)\}$. Consider all operators Uf on D such that $|Uf(x)|\leq \text{cst.}\ m(f)$, $Uf(x)=0$ if $x\in R^n-S_\Lambda(f)$, and define $\|Tf\|_\Lambda=\inf_U \int_{R^n-S_\Lambda(f)}|T(f-Uf)(x)|\,dx$. Theorem: Riesz's theorem still holds if one replaces $\|Tf\|_1$ by $\|Tf\|_\Lambda$ $(\leq\|Tf\|_1)$ in (2). (Received October 26, 1953.)

214*t*. Mischa Cotlar: *A general interpolation theorem.* II.

For any set $E\subset R^n$ define $\|T\|_{\Lambda,E}=\sup\{\|Tf\|_\Lambda/\|f\|_1\}$ [for all f such that $\|f\|_1\leq 1$ and $f(x)=0$ on R^n-E. For any cube Q define $\|T\|_{\Lambda,Q}=\text{SUP}_E\ \|T\|_{\Lambda,E}$ for $Q/2\subset E\subset Q$, and for any $x\in R^n$ define $\|T\|_{\Lambda,x}=\inf_Q\|T\|_{\Lambda,Q}$ for all cubes Q with center at x. Theorem: If, for all $x\in R^n$, $\|T\|_{\Lambda,x}\leq C$, then also $\|T\|_\Lambda\leq C$. For any cube Q let $\mu(f, Q)(x)=$ mean value of f on Q if $x\in Q$ and zero otherwise, and define $\|Tf\|_*=\sup_{Q\supset S(f)}\int_{R^n-2Q}|T(f-\mu(f, Q))(x)|\,dx$. Theorem: If $\|Tf\|_*\leq C_*\|f\|_1$ for all $f\in D$, then $\|Tf\|_{\Lambda_0}\leq C_{\Lambda_0}\|f\|_1$, where $\Lambda_0 f$ is the Hardy-Littlewood maximal function. (Received October 26, 1953.)

215*t*. Mischa Cotlar: *A general maximal theorem.*

Let $D=\{f(x)\}$ be the set of step functions of R^n, $g_{y,\epsilon}(x)$ the characteristic function of the sphere $|x-y|<\epsilon$. For any linear operator $Tf=Tf(x)$ defined on D define: $T_\epsilon f(x)=Tf(x)-T(g_{x,\epsilon})(x)$ $(\epsilon>0)$, $MTf(x)=\sup_\epsilon |T_\epsilon f(x)|$. $\Delta Tf(x) =\sup_{\epsilon>0}[\sup_{|x-x'|<\epsilon/2}|T_\epsilon f(x)-\{Tf(x')-T(g_{x,\epsilon}f)(x')\}|]$. (i) If Tf and ΔTf are bounded operators on D, considered as a subspace of $L^p(R^n)$, then MTf is also bounded on $L^p\cap D$. (ii) If T' and T'' are operators in R^n and in R^m satisfying the hypothesis of Theorem (i), and if $T=T'\times T''$ is the product operator in $R^n\times R^m$, then MT is bounded on $L^p(R^n\times R^m)$. (Received October 26, 1953.)

216t. Mischa Cotlar: *On general Hilbert transforms.*

Let $R^2=\{z\}$, $z=|z|e^{it}$, $k(z)=\varpi(t)|z|^{-2}$, $\varpi(t)$ a smooth function with $\int_0^{2\pi}\varpi(t)dt=0$, $k_\epsilon(z)=k(z)$ if $|z|>\epsilon$ and zero otherwise, and $T_\epsilon f=f*k_\epsilon$ ($*$ convolution). In a recent paper (Acta Math. vol. 88) Zygmund and Calderón prove the pointwise and mean convergence of $T_\epsilon f$ and the "boundedness" of the limit operator Tf. Similar results hold in R^n; for $n=1$ this reduces to classical theorems on the ordinary Hilbert transform. These results can be easily derived using the general maximal and interpolation theorems given in the above abstracts. Moreover, the author's methods give more complete results and the generalization for double transforms. In fact, the operator $Tf=f*k$ is well defined for any step function f and has the properties: $\Delta Tf(x) \leq \text{cst.}\ \Delta_0 f(x)=$ Hardy-Littlewood's maximal function, and $\|Tf\|_p \leq C_p\|f\|_1$. (Received October 26, 1953.)

217. Frederic Cunningham, Jr.: *L^1-structure in Banach spaces.* Preliminary report.

An L^p-projection in a Banach space is a linear idempotent operator E satisfying the identity $\|Ex\|^p+\|x-Ex\|^p=\|x\|^p$. Any commutative family of L^p-projections can be enlarged to a complete Boolean algebra. In particular the family $\mathcal{E}_X$ of all L^1-projections in X is such an algebra. This leads by methods well known in Hilbert space to the following: a Banach space X is isometric isomorphic to an L^1 if and only if for all $x\in X$ the set $\{Ex;\ E\in\mathcal{E}_X\}$ spans a subspace dense in the smallest L^1-subspace (range of an L^1-projection) containing x; in any case X has an essentially unique representation as an L^1 continuous direct sum on the Stone space of $\mathcal{E}_X$. For any abstract σ-ring $\mathcal{E}$ let $\mathcal{L}(\mathcal{E})$ denote the space of bounded σ-additive functions on $\mathcal{E}$ (a notion equivalent to L^1). Then there always exists a unique σ-ring $\mathcal{E}_*$ having a proper σ-finite measure and such that $X=\mathcal{L}(\mathcal{E})$ and $\mathcal{L}(\mathcal{E}_*)$ are identical, namely the σ-ideal in $\mathcal{E}_X$ of those elements whose corresponding principal ideals are measure algebras. The method leads also to a description of all isometric automorphisms of L^1. (Received November 14, 1953.)

218. Chandler Davis: *Theorems on projections in Hilbert space.*

Here is the central theorem: Let the operators $A(\lambda)$, $0\leq\lambda\leq 1^+$, on the Hilbert space $\mathcal{E}$, satisfy (a) $A(0)=0$, $A(1^+)=1$; (b) for $\lambda\leq\mu$, $A(\lambda)\leq A(\mu)$; (c) $A(\lambda)=A(\lambda^-)$. Then one constructs a Hilbert space $\mathcal{H}$ in which $\mathcal{E}$ is imbedded, and a resolution of the identity $E(\lambda)$ in $\mathcal{H}$ (for $0\leq\lambda\leq 1^+$; normalized $E(\lambda)=E(\lambda^-)$) such that $A(\lambda)=EE(\lambda)E$ (where E is the projection on $\mathcal{E}$). In a special case (roughly: no continuous spectra), this follows easily from a construction of Hadwiger; but the method used in the present paper is new. It is related to ideas of Dixmier and others in the case where $E(\lambda)$ is replaced by a single projection. (Received November 16, 1953.)

219t. R. B. Davis: *The normal derivative problem for $\partial(\Delta u)/\partial x\doteq 0$.* Preliminary report.

A boundary value problem existence theorem is proved for the normal derivative problem for $\partial(\Delta u)/\partial x=0$. The regions considered are of rather special type, bounded in part by a straight line. The present paper differs from the writer's earlier work on the normal derivative problem in the following way: free boundary problems, or else problems requiring analytic data, which entered into previous methods of proof, have been eliminated. These have been replaced by a Dirichlet problem for $\Delta u=0$, with a consequent alteration of hypothesis, both as regards prescribed boundary values, and also regarding regions considered. Symmetry is again used to reduce to a problem of Sjöstrand. Uniqueness is proved, as well as existence. (Received November 14, 1953.)

220. Nelson Dunford (p) and Jacob Schwartz: *Finitely additive Lebesgue integral.*

Let μ denote an additive (but not necessarily bounded, finite, or countably additive) function defined on a field of subsets of a set S. The theory of measure and integration with respect to μ can be carried out with suitable modifications, so as to yield many of the standard principal theorems of measure theory. The μ-measurable functions $f(s)$ form an algebra closed with respect to limits in μ-measure. The standard Lebesgue spaces L_p can be defined, and it can be shown that convergence in measure plus uniform integrability is equivalent to convergence in norm. This gives a form of the dominated convergence theorem. Consequently, the case of a countably additive measure μ is distinguished by two principal facts: (i) convergence almost everywhere implies convergence in measure, and (ii) L_p is complete. (Received November 13, 1953.)

221*t*. Albert Edrei: *On the zeros of successive derivatives.*

The following analogue of a classical theorem of Landau seems to be new. Let $R(>0)$ be the radius of convergence of the series $1+2+z^2/2+\cdots=f(z)$ and assume $\alpha\neq 1$. Then there exists an absolute constant C such that $R\leqq R_1=\max\{C, [C/|\alpha-1|]\log(1/|\alpha-1|)\}$, or else $f(z)f'(z)f''(z)$ vanishes at some point of the disk $|z|\leqq R_1$. This theorem easily yields some results of Pólya and Saxer [Math. Zeit. vol. 17, pp. 206–227] and also a number of new propositions relative to the limit-points of zeros of successive derivatives. (Received November 16, 1953.)

222. Joanne Elliott: *Expansion theorems for singular differential operators.*

This paper contains theorems connected with pairs of adjoint operators typified by (1) $d^2/dx^2+b(x)d/dx$ and (2) $(d/dx)(d/dx-b(x))$ in $-\infty\leqq r_1\leqq x\leqq r_2\leqq\infty$. Here $b(x)$ is assumed continuous, but not necessarily bounded, in (r_1, r_2). Define $B(x)=\int_{x_0}^{x}b(s)ds$. If at least one of the integrals $\int_{x_0}^{x}e^{B(s)}ds\int_{x_0}^{s}e^{-B(t)}dt$ or $\int_{x_0}^{x}e^{-B(s)}ds\int_{x_0}^{s}e^{B(t)}dt$ converges as $x\to r_i$, then it is shown that, under certain boundary conditions, the eigenfunctions of (1) form a set which is orthonormal with respect to e^B and which is complete both in the weighted L_2 space and in $C[r_1, r_2]$. The eigenfunctions $\{\phi_n\}$ and the eigenvalues $\{\lambda_n\}$ are common to both spaces. It is shown that any $f\in C[r_1, r_2]$ can be expanded as follows: $\sum_{n=0}^{\infty}e^{\lambda_n t}\phi_n(x)\int_{r_1}^{r_2}\phi_n(s)f(s)e^{B(s)}ds\to f(x)$ as $t\to 0$ in the sense of bounded pointwise convergence. The subspace of $C[r_1, r_2]$ on which the convergence is uniform is specified. A similar expansion is proved for (2) on the space $L[r_1, r_2]$, but the limit as $t\to 0$ is a limit in the mean of L. The proofs employ primarily the theory of semigroups and the theory of integral equations in a Hilbert space. (Received November 14, 1953.)

223*t*. Paul Erdös and Michael Golomb: *Functions which are symmetric about several points.*

Real functions $f(t)$ are constructed that are bounded, not equivalent to a constant, and satisfy the equation $f(x+t)+f(x-t)-2f(t)=0$ for all t and a nonmeasurable set of x's that is everywhere dense and of power c. This solves a problem proposed by R. P. Boas. It is also shown that there is a bounded complex-valued function $f(u)$, not equivalent to a constant, which satisfies the equation (*) $\sum_{k=1}^{n} f(s+c_k u)-f(s)=0$ for all complex u and a set of s's that is everywhere dense and of power c provided $\sum_{k=1}^{n} \gamma_k(c_k/|c_k|)^p|c_k|^{iq}=0$ for some real p, q. This latter condition is also proved to be necessary for the existence of an essentially bounded $f(u)$ not equivalent to 0 that satisfies (*) for almost all u and at least one number s. (Received November 18, 1953.)

224. Michael Fekete: *Generalized transfinite diameter and minimum of the energy integral.*

Call $g(r)$ a generator function if (a) $g(r)$ is continuous for $0<r<\infty$; (b) $g(r)$ is strictly decreasing with r^{-1}; (c) $\lim g(r)=\infty$ for $r\downarrow 0$. Let S be any compact point set of a Euclidean space E_q of q dimensions, $q\geq 1$, finite. Let $\mu(e)$ be a Borel measure on E_q such that $\mu(S)=1$. Then the Stieltjes-Radon integrals $u(P)=\int_S g(PQ)\mu(e_Q)=\lim_{N\to\infty}\int_S g_N(PQ)\mu(e_Q)$, $I(\mu)=\int\int_S g(PQ)\mu(e_P)\mu(e_Q)=\lim_{N\to\infty}\int\int_S g_N(PQ)\mu(e_P)\mu(e_Q)$ respectively yield the potential at $P\in E_q$ and the energy integral of the unit mass distributed on S in accordance with $\mu(e)$ when the "law of attraction (repulsion)" is given by $g(r)$; here $g_N(r)=g(r)$ as long as $g(r)<N$ while $g_N(r)=N$ whenever $g(r)\geq N$; furthermore e_P, e_Q denote vicinities of P and Q in S. These concepts of Frostman (Thesis, 1935, Lund) and the generalized transfinite diameter $\delta=\delta(S, g)$ of S with respect to $g(r)$ (cf. Fekete, Bull. Amer. Math. Soc. Abstract 53-7-288) have the following connections: $I(\mu)$ has a finite minimum if and only if $\delta>0$, being attained for the "equilibrium distribution" $\mu^*(e)=\lim_{n\to\infty}\mu_n(e)$ and equal to $g(\delta)$; the corresponding potential: $u^*(P)=\lim_{n\to\infty} n^{-1}\sum_{\nu=1}^{n} g(PP_\nu^{(n)})$ whenever $P\in T$, the complement of S with respect to E_q. Here $P_{\nu}^{(n)}$, $1\leq\nu\leq n$, means any extremal system of order $n\geq 2$ of S with respect to $g(r)$, minimizing $\sum_{1\leq a<b\leq n} g(P_aP_b)$ when $P_1, \cdots, P_n$ range over S, while $\mu_n(e)=n^{-1}\times$the number of all points $P_\nu^{(n)}$ contained in e. (Received November 20, 1953.)

225. F. A. Ficken: *Periodic solutions of a nonlinear wave equation.*

Let $Lu=u_{xx}-u_{tt}-2\kappa u_t-\alpha u$, with κ and α positive constants. Let $b(x, t)$ be such that $b(0, t)=0=b(\pi, t)$ and $b(x, t+p)=b(x, t)$. The initial value problem $Lu=b+\epsilon u^3$ (ϵ constant), $u(0, t)=0=u(\pi, t)$, $u(x, 0)=f(x)$, $u_t(x, 0)=g(x)$ was shown by B. A. Fleishman (Bull. Amer. Math. Soc. Abstract 59-4-365) to have a solution $u(f, g; x, t)$. He also proved that f and g can be so chosen that u is p-periodic in t: $u(f, g; x, t+p)=u(f, g; x, t)$ under certain conditions including requirements that $|\alpha-\kappa^2|$ and ϵ be small. Similar results are established here for $\kappa>0$, $\alpha>0$, and sufficiently small ϵ without restriction on $|\alpha-\kappa^2|$. Instead of an integral operator inverting $u_{xx}-u_{tt}$, we use here an integral operator M inverting Lu; M is obtained by solving a nonhomogeneous telegraph equation. The solution for $f(x)$ and $g(x)$ of the functional equations $u(f, g; x, p)=f(x)$, $u_t(f, g; x, p)=g(x)$ expressing the periodicity conditions is also simplified in some respects. (Received November 12, 1953.)

226t. Y. Fourès and I. E. Segal: *Causality and analyticity.*

A real vector group G is *causally oriented* if there is distinguished in G a proper convex cone with interior. An operator in a space of functions on G is *homogeneous* if it commutes with translations by elements of G, and a homogeneous operator is *causal* if it maps those elements of its domain that vanish outside C into functions vanishing outside C. A closed homogeneous densely defined operator T on the space $L_2(G, K)$ of square-integrable functions on G with values in the complex Hilbert space K is characterized by its gain, which is the measurable function on the dual G^* of G to the closed operators on K multiplication by which, acting on $L_2(G^*, K)$, is the Plancherel transform of T. Various results are obtained essentially to the effect that T is causal if and only if its gain admits an analytic extension to the tube over the dual of C which does not grow too rapidly, a simple necessary and sufficient conditions for causality being obtained for bounded operators and for Green's operators of partial differential equations with constant coefficients. (Received December 2, 1953.)

227t. E. E. Floyd and V. L. Klee: *A characterisation of reflexivity by the lattice of closed subspaces.*

Šmulian proved that a closed convex subset of a normed linear space fails to be weakly compact if and only if it contains a decreasing sequence of closed convex sets whose intersection is empty. The authors establish here a similar result: *a bounded closed convex subset C of a normed linear space fails to be weakly compact if and only if there is a decreasing sequence of closed linear manifolds whose intersection is empty and each of which intersects C.* Applied to the unit cell of the space, this yields a new characterization of reflexivity, and from it follows the principal result of the paper: *A normed linear space is reflexive if and only if the lattice of all its closed linear subspaces is, in the order topology of G. Birkhoff, a Hausdorff space.* (Contrast this with Theorem 13 on page 60 of *Lattice theory*.) (Received October 30, 1953.)

228. A. W. Goodman: *Almost bounded functions.*

Let $G=\{L_1(w), L_2(w), \cdots, L_{2n}(w)\}$ be a group of linear transformations of even order. A function $f(z)$ is said to be almost bounded with respect to the group G in a region R if it is meromorphic in R and if for each w, $f(z)$ assumes in R at most n values from the set $\{L_1(w), L_2(w), \cdots, L_{2n}(w)\}$. The simple case $G=\{w, 1/w\}$ has been studied extensively. In this paper all of the groups of linear transformations of finite order are examined, and certain standard forms are selected to represent the equivalence class. Let $\mathfrak{G}$ denote any group in its standard form. It is proved that if $f(z)=a_1z+a_2z^2+\cdots$ is univalent and almost bounded for $\mathfrak{G}$ in $|z|<1$, then $|a_1|\leq 1$ with equality if and only if $f(z)=\eta z$, $|\eta|=1$. (Received November 14, 1953.)

229t. W. T. Guy, Jr.: *Jacobian extensors and fractional differentiation.*

The concept of Jacobian extensor is extended from integral order to nonintegral order by use of one of the fractional differentiation operators. A contrast is made with the generalization due to Hiroyoshi Sasayama. (Received November 16, 1953.)

230. Violet B. Haas: *Singular perturbations of solutions of a nonlinear differential equation.*

The differential equation, $\epsilon\ddot{x}+f(t, x, \dot{x}; \epsilon)=0$ is considered, where ϵ is a small positive parameter. $f(t, x, y; \epsilon)$ belongs to class $C^2(t, x, y)$ in the domain D: $[T_0\leq t<\infty, |x|\leq X, |y|<\infty]$, f and its partial derivatives being bounded here. f_{yy} is not identically zero in D, and f is continuous in ϵ for $0\leq\epsilon\leq\epsilon_0$. Let $x=x(t)$ be a solution of the differential equation under discussion, and let $s=s(t)$ be a solution of the corresponding degenerate equation ($\epsilon=0$). Let $t_0>T_0$, and let $|x(t_0)-s(t_0)|\leq\delta_1$, $|\dot{x}(t_0)-\dot{s}(t_0)|\leq\delta_2/\epsilon$. If t_1 and q are such that $f(t_1, s(t_1), q; 0)=0$, $f_y(t_1, s(t_1), q; 0)=0$, $t_1>t_0$, then under certain added conditions if τ_1, τ_2 are any two numbers satisfying $t_0<\tau_1<\tau_2<t_1$, $|x(t)-s(t)|\to 0$ uniformly for $t_0\leq t\leq\tau_2$ and $|\dot{x}(t)-\dot{s}(t)|\to 0$ uniformly for $\tau_1\leq t\leq\tau_2$ as ϵ, δ_1, $\delta_2\to 0$. Furthermore, $x(t)$ has rapid oscillatory behavior for $t>t_1$, the amplitude and period of these oscillations being not constant for fixed ϵ and approaching $|s(t_1)|$

and $O(\epsilon^{1/2})$ respectively as $\epsilon\to 0$. The proof depends partially on some results of N. Levinson [Acta Math. vol. 82 (1950) pp. 71–106] and partially on certain qualitative estimates on the behavior of solutions in phase space. (Received November 14, 1953.)

231t. M. H. Heins: *The set of asymptotic values of an entire function.*

The set of asymptotic values of a meromorphic function is an analytic set (Mazurkiewicz, Kierst). An example of a meromorphic function in the unit circle whose set of asymptotic values is an assigned analytic set of the extended plane has been constructed by Kierst. The problem arises to characterize the set of asymptotic values of a nonconstant entire function. It is shown that a set of the extended plane is the set of asymptotic values of a nonconstant entire function if and only if it is analytic and contains the point at infinity. The proof is based upon the use of the representation theorem of Lusin and Sierpinski (as in the work of Kierst) and a direct geometric construction of an entire function which has the desired properties. (Received October 19, 1953.)

232t. M. H. Heins: *Universal Blaschke products.*

Let $\mathfrak{B}$ denote the class of functions which are analytic and of modulus at most one in $|z|<1$. An example is constructed of a convergent infinite Blaschke product b such that (1) for each $f\in\mathfrak{B}$ there exists a monotone increasing sequence of positive numbers $\{x_n\}_0^\infty$ with $\lim x_n=1$ such that the sequence $\{b((z+x_n)/(1+x_nz))\}$ tends uniformly to f in $|z|<1$, (2) the zeros of b cluster only at $z=1$. The proof is based upon the fact that there exists a sequence $\{b_n\}_0$ of finite Blaschke products which satisfy: (1) $b_n(-1)=b_n(1)=1$ for $n=0, 1, \cdots$, (2) for each $f\in\mathfrak{B}$ there exists a subsequence $\{b_{n(k)}\}_0^\infty$ which tends uniformly to f in $|z|<1$. By this example an affirmative answer is given to the following question proposed to the author by Professor J. L. Walsh: Does there exist a bounded analytic function g in $|z|<1$ such that for each $f\in\mathfrak{B}$ there exists a sequence $\{T_n\}$ of non-euclidean motions with the property that $\{g\circ T_n\}$ tends uniformly to f in $|z|<1$? With the aid of the indicated example one can construct a convergent infinite Blaschke product b^* such that given $f\in\mathfrak{B}$ and ζ on the unit circumference there exists a sequence of non-euclidean motions $\{T_n\}$ with $\lim T_n=\zeta$ and $\lim b^*\circ T_n=f$. (Received October 19, 1953.)

233. Einar Hille and George Klein (p): *On Riemann's localisation theorem for Fourier series.*

Let $S_n(x;f)$ be the nth partial sum of the Fourier series of the L-integrable, nonconstant function $f(x)$ of period 2π, and let R be the difference $R(x, \delta, n, f)=S_n(x;f)-\pi^{-1}\int_{-\delta}^{\delta}f(x+t)t^{-1}\sin nt\,dt$. The theorem of the title states that for each δ, $0<\delta<\pi$, $R\to 0$ uniformly in x as $n\to+\infty$. This result is sharpened by establishing that $|R|\leq A\delta^{-1}(\|f\|_1+1)\omega_1(n^{-1};f)$, where A is an absolute constant, $\|f\|_1=(2\pi)^{-1}\int_{-\pi}^{\pi}|f(x)|dx$, and $\omega_1(h;f)=\sup_{|t|\leq|h|}\|f(x+t)-f(x)\|_1$. A result of independent interest obtained in the course of the proof is that for all x, $\int_0^h|f(x+t)|dt\leq A(\|f\|_1+1)\omega_1(h;f)$. (Received November 10, 1953.)

234. I. I. Hirschman, Jr.: *On a conjecture of Hewitt.*

Let G with elements $x, y, \cdots$ be a locally compact Abelian group, and let $\hat{G}$ with elements $\hat{x}, \hat{y}, \cdots$ be its dual. Haar measure in G is denoted by dx, and in $\hat{G}$ by $d\hat{x}$. We suppose that these measures are so adjusted that if $\hat{f}(\hat{x})=\int_G f(x)\overline{(x, \hat{x})}dx$ then $f(x)=\int_{\hat{G}}\hat{f}(\hat{x})(x, \hat{x})d\hat{x}$. The Weil-Young-Hausdorff theorem asserts that if $f(x) \in L^p(G)$, $1 \leqq p \leqq 2$, and if $p^{-1}\ q^{-1} = 1$ then $\|f^\frown(x^\frown)\|_q \leqq \|f(x)\|_p$. The cases of equality for $1 < p < 2$ have been investigated by E. Hewitt, who conjectured that $\|f^\frown(x^\frown)\|_q = \|f(x)\|_p$ if, and only if, $f(x)$ is of the form $c\chi_{yA}(x)(x, y^\frown)$, where c is a complex constant, y an element of G, A an open and compact subgroup of G, $\chi_{yA}(x)$ the characteristic function of yA, and $y^\frown$ an element of $G^\frown$. Hewitt verified this result in the cases when $1 < p < 4/3$, and when G is compact or discrete. The present paper is devoted to establishing this conjecture without restriction. The method employed depends upon an examination of the conditions for equality in the Riesz-Thorin convexity theorem. (Received November 9, 1953.)

235t. A. J. Hoffman: *On an inequality of Hardy, Littlewood, and Pólya.*

A well known inequality of Hardy, Littlewood, and Pólya (*Inequalities*, p. 49) is strengthened as follows: let $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n$ be given. Then in order that $\beta_1, \beta_2, \cdots, \beta_n$ satisfy $\beta_1 \geqq \beta_2 \geqq \cdots \geqq \beta_n$ and $\beta_1 \leqq \alpha_1, \cdots, \beta_1 + \cdots + \beta_{n-1} \leqq \alpha_1 + \cdots + \alpha_{n-1}$, $\beta_1 + \cdots + \beta_n = \alpha_1 + \cdots + \alpha_n$, it is necessary and sufficient that there exist n^2 numbers a_{ij} such that $\beta_i = \sum a_{ij}\alpha_j$, where the matrix (a_{ij}) is stochastic, symmetric, and satisfies $i \leqq j$, $i' \leqq j'$, $i \geqq i'$, $j \leqq j' \rightarrow a_{ij} \geqq a_{i'j'}$. The proof follows readily from an examination of the graphs of $A_t = \sum_{i=1}^{t} \alpha_i$, and $B_t = \sum_{i=1}^{t} \beta_i$. (This work was supported (in part) by the Office of Scientific Research, USAF.) (Received November 14, 1953.)

236t. A. J. Hoffman: *On the singular values of the product of two operators.*

Let A, B, C be $n \times n$ matrices with singular values $\alpha_1 \geqq \cdots \geqq \alpha_n$, $\beta_1 \geqq \cdots \geqq \beta_n$, $\gamma_1 \geqq \cdots \geqq \gamma_n$ respectively, and let $C = AB$. Let $0 < i_1 < i_2 < \cdots < i_m \leqq n$. Then $\gamma_{i_1}\gamma_{i_2} \cdots \gamma_{i_m}\alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_m}\beta_1\beta_2 \cdots \beta_m$. This result is equivalent to a theorem of Lidskiĭ on the eigenvalues of the product of two positive definite matrices. The new proof, a modification of ideas used by H. Wielandt in proving a similar theorem for eigenvalues of the sum of Hermitian matrices, is based on a minimax characterization of the product $\lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_m}$, where $\lambda_1 \geqq \cdots \lambda_n \geqq 0$ are the eigenvalues of a Hermitian positive semi-definite matrix. The main theorem can also be shown to hold if A, B, C are completely continuous operators in Hilbert space. (This work was supported (in part) by the Office of Scientific Research, USAF.) (Received November 14, 1953.)

237t. J. C. Holladay: *On conjugacy of some transformations of the interval.*

Two continuous transformations $f(x)$ and $g(x)$ mapping the interval $0 \leqq x \leqq 1$ into itself are called conjugate if there exists a homeomorphism, $h(x)$, of this interval such that $f(x) = hgh^{-1}(x)$. A result of S. Ulam and J. von Neumann (*On combination of stochastic and deterministic processes*, Bull. Amer. Math. Soc. Abstract 53-11-403) establishes the conjugacy of the functions $f(x) = 4x(1-x)$ and $g(x) = 2x$ for $0 \leqq x \leqq 1/2$ and $g(x) = 2 - 2x$ for $1/2 \leqq x \leqq 1$. A general study of this notion is undertaken for certain types of transformations. Necessary and sufficient conditions are given for conjugacy between more general transformations. This study involves the investigation of measures, invariant under the inverses of the transformations, and of their sometimes ergodic and mixing behavior. One of the results is that for a transformation of a certain type, including the above-mentioned transformation, there exists a non-denumerably infinite number of linearly independent measures of the interval, assigning positive values to each subinterval, for which the transformation is mixing. (Received November 9, 1953.)

238t. M. A. Hyman: *The order of the remainder term in Taylor expansions.*

A well known corollary of Taylor's Theorem is "Given that $F(x)$, F', F'', $\cdots$, $F^{(n)}$ exist and are continuous at all points of an interval I, then for two points x_0, x_0+h in I, $F(x_0+h)=F(x_0)+hF'(x_0)+(h^2/2)F''(x_0)+\cdots+(h^n/n!)F^{(n)}(x_0)+o(h^n)$." The small o has its usual order-of-magnitude meaning. However, the assumed continuity of the last derivative can be dropped; the relationship holds in the neighborhood of any point x_0 for which $F^{(n)}(x_0)$ exists. Without difficulty this result can be extended to more general forms of Taylor's Theorem, in particular to cases involving more than one independent variable. (Received November 3, 1953.)

239. J. R. Isbell: *Pathology of monotone set functions.*

Let f be a monotone real function, continuous in the order topology, on a measure algebra which is either complete as a Boolean algebra or complete as a measure algebra; the range of f is compact. However, there exist two such functions, f_1, f_2, such that both the range of f_1-f_2 and the set of all pairs $(f_1(x), f_1(y))$, $x\cap y=0$, are not compact. (Received November 12, 1953.)

240. M. L. Juncosa (p) and D. M. Young: *On the Crank-Nicolson procedure for parabolic partial differential equations.*

In Proc. Cambridge Philos. Soc. (1947) J. Crank and P. Nicolson proposed replacing u_t and u_{xx} by $[u(x, t+k)-u(x, t)]/2k$ and $[u(x+h, t+k)+u(x-h, t+k)-2u(x, t+k)+u(x+h, t)+u(x-h, t)-2u(x, t)]/2h^2$ respectively for numerical solution of quasi-linear parabolic partial differential equations. For the case of $u_t=u_{xx}$ with homogeneous boundary conditions and fairly general conditions on the initial function $f(x)$, convergence proofs of the Crank-Nicolson procedure are given. Bounds on the order of convergence are also obtained for different classes of the initial function. When $f(x)$ is a step function or of bounded variation, the bounds are $O(h^{-1})$. When $f(x)$ has a derivative of bounded variation, has a continuous graph composed only of straight-line segments, or is a step function and the mesh points are preserved through the limiting process, the bounds are $O(h^{-2})$. If $f(x)$ or its derivative satisfy a Lipschitz condition of order $\alpha\leqq 1$, the bounds are $O(h^{-\alpha})$ or $O(h^{-1-\alpha})$ respectively. If $f(x)$ has a modulus of continuity $\omega(\delta)$, then the bounds are max $\{O(\omega(h)), O(h^2)\}$ and if $f'(x)$ exists and has a modulus of continuity $\omega_1(\delta)$, then the bounds are max $\{O(h\omega(h)), O(h^2)\}$. Since no better bounds than $O(h^2)$ are given and bilinear interpolations on functions of two variables are good to $O(h^2)$, the results are also valid for extensions of the approximate solutions of the differential equation by bilinear interpolation. (Received December 11, 1953.)

241. W. B. Jurkat: *Some new inclusion theorems for Nörlund means.*

For the inclusion $N(p_n)\subseteq N(q_n)$ sufficient monotony conditions are obtained similar to the well known conditions of Hardy. Let $N(q_n)$ be regular and $p_0=q_0=1$. Then a typical pair of sufficient conditions is $0<p_n\searrow$, $q_n-p_n\nearrow$, including the special theorem $N(p_n)\subseteq C_1$ for decreasing positive p_n. There are some more general results using the differences of higher order of p_n and q_n. They contain e.g. a theorem for Nörlund means analogous to the familiar second theorem of consistency for Riesz

means. The proof of these results depends on some theorems on the distribution of signs in a power series formerly obtained. (Received November 14, 1953.)

242t. W. B. Jurkat: *The Faltung of Nörlund means.*

A Nörlund method $N(p_n)$ is given by the linear transformation $\sigma_n=(1/P_n)\cdot\sum_{\nu=0}^{n} p_{n-\nu}s_\nu$ where p_n is real and $P_n=\sum_{\nu=0}^{n} p_\nu\neq 0$. A method $N(r_n)$ is said to be the Faltung of $N(p_n)$ and $N(q_n)$, in signs $N(r_n)=N(p_n)*N(q_n)$, if $r_n=\sum_{\nu=0}^{n} p_\nu q_{n-\nu}$. Every inclusion theorem for Nörlund means can be stated in the form $N(p_n)*N(q_n)\supseteq N(p_n)$ or $\supseteq N(q_n)$, and for these inclusions general sufficient conditions are obtained extending the well known theorem of Nörlund. A typical theorem is that both inclusions hold if $N(p_n)$ and $N(q_n)$ are regular and one of the methods is positive, for instance $p_n\geqq 0$. As a special result one has $N(p_n)\subseteq M(P_n)$ for all regular methods $N(p_n)$. Furthermore some equivalence theorems are obtained, for instance $N(p_n)*N(q_n)$ is equivalent to the regular method $N(q_n)$ if and only if $N(p_n)$ is equivalent to convergence. (Received December 10, 1953.)

243. N. D. Kazarinoff: *Asymptotic expansions for the Whittaker functions of large complex order m.*

The investigation of the Whittaker functions $M_{k,m}(x)$ and $W_{k,m}(x)$ which the writer announced previously (Bull. Amer. Math. Soc. Abstract 59-4-468) has been completed. The dominant terms in the asymptotic expansions of these functions for large $|m|$ have been obtained explicitly. These dominant terms are seen to be simple rational and exponential functions of x except when $|x/2m|$ is close to one. When $|x/2m|$ is close to or equal to one the behavior of $W_{k,m}(x)$ and $M_{k,m}(x)$ is given in terms of that of the Bessel functions $J_{1/3}(\xi)$ and $J_{-1/3}(\xi)$, $\xi\sim m[(x/2im)^2-1]^{3/2}/3$. (Received November 12, 1953)

244. J. H. B. Kemperman: *Generalizations of the Euler-Maclaurin formula.*

Let $Q(x)$ be integrable in $[0, 1]$ and $\int_0^1 Q(x)dx=1$. Define the generalized Bernoulli numbers B_k $(k=0, 1, \cdots)$ recursively by $(B+1)^m-B^m=m\int_0^1(1-x)^{m-1}(1-Q(x))dx$ $(m=2, 3, \cdots; B_0=1$; B^k to be interpreted as $B_k)$. Let $k!P_k(x)=(x+B)^k-k\cdot\int_0^x(x-y)^{k-1}(1-Qdy))dy$ $(0\leqq x\leqq 1; k=1, 2, \cdots)$. Finally, $Q(x+k)=Q(x)$, $P_k(x+k)=P_k(x)$ for $k=0, 1, \cdots$. Let Γ be a smooth curve connecting ξ_0 with ξ_n and intersecting Re $(x)=m$ in exactly one point $\xi_m=m+i\eta_m$ $(m=0, 1, \cdots, n)$. Let $f(x)$ be a function on Γ which has an integrable kth derivative. Consider the formula $\sum_{m=1}^{n-1}f_m=\int_\Gamma Q(x)f(x)dx+\sum_{j=1}^{k}(-)^jP_j(\xi_n-0)f^{(j-1)}(\xi_n)-\sum_{j=1}^{k}(-)^jP_j(\xi_0+0)f^{(j-1)}(\xi_0)+R_k$, where $f_m=\sum_{h=0}^{k-1}f^{(h)}(m+i\eta_m)(-i\eta_m)^h/h!$ and $R_k=(-)^{k+1}\int_\Gamma f^{(k)}(x)P_k(x)dx$. This formula holds: (i) If $Q(x)$ is a trigonometric polynomial, in which case $P_k(x)$ is analytic for $0<\text{Re }(x)<1$. (ii) If Γ is the interval $[0, n]$ on the real axis, in which case $f_m=f(m)$, $P_j(n-0)=P_j(+0)=B_j/j!$ $(j>1)$, $P_1(n-0)=1+B_1$, $P_1(+0)=B_1$. Let $x=x_1+ix_2$. Suppose that for $0\leqq x_1\leqq n$, $f(x)$ is analytic except for a finite number of poles and that $f(x)=O(e^{2\pi p|x_2|})$ for large $|x_2|$ with $p<1$. Then, in both cases, $R_k=S-I_0+I_n$, where S is the finite sum of certain residues and $I_n=\int_{-\infty}^{\infty}\phi_k(v)f^{(k)}(x+iv)dv$, $\phi_k(v)$ only depending upon $Q(x)$. This result generalizes the so-called complex form of the Euler formula. (Received November 14, 1953.)

245. Fulton Koehler: *The maximum value of an arbitrary real linear combination of the coefficients of a bounded power series.*

Let $f(z)=\sum_{\nu=0}^{\infty}a_\nu z^\nu$, $|f(z)|<1$ for $|z|<1$. For given real numbers $\lambda_0, \lambda_1, \cdots, \lambda_n$, not all zero, it is shown that the maximum of $\left| \sum_{\nu=0}^{n} \lambda_\nu a_\nu \right|$ is attained when $f(z)$ is of the form $Q_k(z)/z^k Q_k(1/z)$ $(k \leq n)$, where $Q_k(z)$ is a polynomial of degree k whose zeros all lie in $|z| < 1$. The actual value of the maximum is given implicitly from the solution of a system of algebraic equations. Szász [Math. Zeit. vol. 1 (1918) pp. 163–183] found an upper bound for $\left| \sum_{\nu=0}^{n} \lambda_\nu a_\nu \right|$ for arbitrary complex λ_ν, but this upper bound is the actual maximum only in certain special cases, including the cases $\lambda_\nu = 1$ and $\lambda_\nu = 1 - \sigma(n+1)$ which had been solved previously by Landau [Archiv der Math. und Physik (3) vol. 21 (1913) pp. 42–50 and pp. 250–255; vol. 24 (1916) pp. 250–260] and Fejér [Rend. Circ. Mat. Palermo vol. 38 (1914) pp. 79–97] respectively. Other special cases have been solved by Schur and Szegö [Preussischen Akad. Wiss. Sitzungsber. (1925) pp. 545–560] and by Rogosinski and Szegö [Math. Zeit. vol. 28 (1928) pp. 73–94]. The idea of the present method is to choose a subset of admissible functions $f(z)$ on which the maximum can be found by elementary methods and then to show that such a maximum holds for all admissible $f(z)$. (Received November 4, 1953.)

246t. R. E. Lane: *A condition necessary and sufficient for a continued fraction expansion of a function to be equal to the function.*

Suppose that f is a function and that a continued fraction expansion for f is obtained from a sequence of identities $f = a_1/(b_1 - u_2)$, $u_p = a_p/(b_p - u_{p+1})$, where $a_{p-1} \neq 0$, $p = 2, 3, 4, \cdots$. We use the notation of a recent paper (Proc. Amer. Math. Soc. vol. 3 (1952) p. 914); if the continued fraction converges, then there is a sequence $\{R_p\}_{p=1}^{\infty}$ which is in B_f. In the present paper we show that if $\{R_p\}_{p=1}^{\infty}$ is in B_f, then for the continued fraction to be equal to f it is necessary and sufficient that there be an infinite sequence P of positive integers such that if p is in P then u_{p+1} is in $t_p^{-1}(R_p)$, which is a circle plus its exterior. Moreover, if there is such a sequence P and if p is a positive integer, then the value of f and the pth approximant of the continued fraction are in R_p. (Received November 13, 1953.)

247t. J. P. LaSalle: *Basic principle of the "bang-bang" servo.*

The differential equation of a simple servo system consisting of a power source, a feedback circuit, and a motor can be written in the form $\ddot{x} + g(x, \dot{x}) = f(x, \dot{x})$, where x is the error in positioning the motor and $g(x, \dot{x})$ is determined by the motor, and $f(x, \dot{x})$ depends on the feedback circuit. The power available to the feedback circuit is limited and $f(x, \dot{x})$ is limited to values between $\pm F$. In the past it has been assumed on intuitive grounds (see: D. McDonald *Nonlinear techniques for improving servo performance*, Cook Research Laboratories Bulletin S-2, Chicago, 1950, and D. W. Bushaw, Ph.D. Thesis, Department of Mathematics, Princeton University, 1952) that errors can be reduced to zero in the shortest time by always using the full strength of the power source to apply full acceleration or deceleration (i.e., f is restricted to the values $\pm F$ and such a system is called a "bang-bang" servo). This assumption is given conditional justification. If for a given function g, there is a unique "bang-bang" system which is the best of all "bang-bang" systems, then it is shown that it is the best of all possible systems operated from the same power source. In a number of cases Bushaw (loc. cit.) has shown the existence of such unique "bang-bang" systems and in these cases the "bang-bang" system is therefore shown to be the best of all possible systems. (Received November 16, 1953.)

248. R. B. Leipnik: *Embedding a semi-group of linear operators into a group.* Preliminary report.

If $\{P^a, a>0\}$ is a semi-group of linear operators and A is a linear operator such that $AP^{a+1}=P^a$ for $a>0$, sufficient conditions are derived for the existence of a group $\{Q^a, -\infty<a<\infty\}$ of operators such that $Q^a=P^a$ for $a>0$, $Q^0=I$ the identity operator, and $Q^{-n}=A^n$ for $n=1, 2, \cdots$. The most interesting condition is that for each $a>0$, there is an integer n_a such that $A^n(P^aA-AP^a)=0$ for $n\geqq n_a$. The abstract analogue of this weak kind of commutativity has curious properties in an associative ring. (Received December 14, 1953.)

249. J. J. Levin: *First order partial differential equations containing a small parameter.*

The behavior, as $\epsilon\to 0+$, of the solution of (1) $\epsilon[a_1(x_1,x_2,u)\partial u/\partial x_1+a_2(x_1, x_2, u)\partial u/\partial x_2]=b(x_1, x_2, u)$ passing through a fixed initial curve, C, is investigated. The existence of a constant $\mu>0$ such that $\partial b[x_1, x_2, h(x_1, x_2)]/\partial u\leqq -\mu$ on the projection of C in the (x_1, x_2) plane, where $b[x_1, x_2, h(x_1, x_2)]=0$, is hypothesized. It is shown (under the preceding hypothesis and certain standard conditions) that there exists a nonvacuous closed subregion, D, of the (x_1, x_2) plane and positive constants δ and ϵ_0, all independent of ϵ, such that if C is within δ of the surface $u=h(x_1, x_2)$, then (1) has a unique solution $u=u(x_1, x_2, \epsilon)$ defined over D for $0<\epsilon\leqq\epsilon_0$ which passes through C. Furthermore, $u(x_1, x_2, \epsilon)\to h(x_1, x_2)$ as $\epsilon\to 0+$ uniformly over every subset of D which is a positive distance from the projection of C in the (x_1, x_2) plane. The analysis is similar to that employed in a recently announced result of Levinson and the author. (Received November 12, 1953.)

250t. E. R. Lorch: *The projection of a linear functional on the manifold of measures.*

Given an abstract set $\mathcal{E}$, let $\mathfrak{B}$ be a (real or complex) Banach space consisting of a set of bounded functions $f(x)$, $x\in\mathcal{E}$, with norm $\|f\|=\text{l.u.b.}\ |f(x)|$; $\mathfrak{B}$ is also a lattice and hence contains $|f(x)|$ along with $f(x)$. By a positive measure is meant a bounded linear functional F over $\mathfrak{B}$ with the properties: (i) $Ff\geqq 0$ if $f(x)\geqq 0$ and (ii) if $f_n(x)\geqq 0$ and $f_n(x)\uparrow f(x)$, then $Ff_n\to Ff$. The theorem of Lebesgue and the work of Daniell are justification of this use of the term measure. An arbitrary real measure is the difference of positive measures; a complex measure is of the form F_1+iF_2 where F_1, F_2 are real. It is shown that there exists a projection P on the space $\mathfrak{B}^*$ of linear functionals over $\mathfrak{B}$ such that the range of P is the totality of measures. Thus each functional F can be represented uniquely as the sum of a measure and a "nonmeasure." The existence of P is established by transfinite arguments. Application is made to the case in which $\mathfrak{B}$ is a commutative Banach algebra. If F is the functional associated with a maximal ideal, either $PF=F$ or $PF=0$. The conditions determining the measure theoretic behavior of F are topological in nature. (Received October 13, 1953.)

251. R. W. McKelvey: *The solution of second order linear ordinary differential equations about a turning point of order two.*

This paper is concerned with the behavior, for large values of a complex parameter λ, of the solutions of differential equations having the form $du/dx-[p_0(x)+p_1(x)/\lambda+p_2(x)/\lambda^2+\cdots]u=0$, where the $p_j(x)$'s are suitably restricted bounded functions. x is to vary over a real interval which contains a second order turning point of the equation—that is, a second order zero of $p_0(x)$. An algorithm is given, involving only the functions $p_j(x)$, which leads simply to the determination of expressions in negative powers of λ which formally satisfy the differential equation. These formal solutions are then shown to represent true solutions in an asymptotic sense. The expressions seem to be of much simpler form than has been obtained previously. The algorithm resembles one employed by R. E. Langer in his theory of the simple turning point, but with certain marked differences. The central role of the Bessel function $H_{1/3}$ in the first order theory is taken here by the Whittaker function $W_{k,1/4}$, with index k dependent upon λ. (Received December 29, 1953.)

252. G. R. MacLane: *Meromorphic functions with specified asymptotic behavior.*

The following theorem is proved: Let $\{D_n\}$ be a finite or infinite sequence of domains in the z-plane such that: (1) the boundary of D_n is a simple curve Γ_n with both ends at ∞, (2) $(D_m \cup \Gamma_m) \cap (D_n \cup \Gamma_n) = 0$ for $m \neq n$, and (3) $(D_n \cup \Gamma_n) \cap (|z| \leq R) = 0$ for $n > n_0(R)$. Let $\{g_n(z)\}$ be any sequence of functions such that $g_n(z)$ is meromorphic in $D_n \cup \Gamma_n$. Let a and p be any positive numbers. Then there exists a function $f(z)$, meromorphic in $|z| < \infty$, holomorphic in the complement of $\bigcup(D_n \cup \Gamma_n)$, such that $f(z) - g_n(z)$ is holomorphic in D_n and such that $|f(z) - g_n(z)| < a(1+|z|)^{-p}$ for $z \in D_n$. (Received November 14, 1953.)

253. H. M. MacNeille: *The approximation to surfaces by polyhedrons.* Preliminary report.

Inscription as the basis for approximating surfaces by polyhedrons is well known not to provide a satisfactory approach to the problem of defining and determining the areas of surfaces. The present paper proposes an alternative approach to this problem suggested by an earlier paper of the author (Proc. Nat. Acad. Sci. U.S.A. vol. 27 (1941) pp. 71–76). (Received November 14, 1953.)

254. Lawrence Markus: *Quadratic ordinary differential equations and their algebras.*

The quadratic differential equations $\dot{x}^i = a^i_{jk} x^j x^k$, $i = 1, 2, \cdots, n$ $(a^i_{jk} = a^i_{kj}$ real or complex numbers), are determined up to affine equivalence by the tensor a^i_{jk}, which can also be realized as the multiplication table of a commutative linear algebra $u_j u_k = a^i_{jk} u_i$ for the basis $(u_1, \cdots, u_n)$. By obtaining all commutative algebras one determines all global quadratic differential systems first up to affine, then up to topological, equivalence. This program is carried out explicitly for low dimensions n. (Received November 12, 1953.)

255. Joseph Milkman: *Kolmogorov's axiom of continuity for rectangular cylinder sets in Hilbert space.*

An n-dimensional rectangle is the Cartesian product of n sets of real numbers $A_1, A_2, \cdots, A_n$. Let E_n denote an n-dimensional Euclidean space which is a subspace of the real Hilbert space H. The n-dimensional cylinder set $C(A)$ consists of all elements $x+h$ of H such that $h \in A \subset E_n$. $C(R)$ is a Borel rectangular cylinder set if R is an n-dimensional rectangle and a Borel set in E_n for some n. Let $P(S) = \int_R f_1(x_1) f_2(x_2) \cdots f_n(x_n) dx_1 dx_2 \cdots dx_n$ where $f_i(x_i)$ is a non-negative Lebesgue integrable function with the properties (1) $\int_{-\infty}^{\infty} f_i(t)dt = 1$ and (2) There is an $m > 0$ such that $|x| < m$ implies $\int_{-\infty}^{\infty} f_i(t)dt > x$. The following theorem is proved: If $S_1, S_2, \cdots, S_n, \cdots$ is a monotone decreasing sequence of Borel rectangular cylinder sets whose product is empty, then $\lim_{n\to\infty} P(S_n) = 0$. (Received November 4, 1953.)

256t. J. H. Monahan: *Topological mappings by analytic quaternion functions.*

The differential of a right analytic quaternion function w in the sense of Fueter is given by $dw = R\,dz i_1 + S\,dz i_2 + T\,dz i_3$. The functional determinant for such a function is then given by $\Delta = R(\overline{R}R\overline{R} + \overline{S}R\overline{S} + \overline{T}R\overline{T}) + S(\overline{R}S\overline{R} + \overline{S}S\overline{S} + \overline{T}S\overline{T}) + T(\overline{R}T\overline{R} + \overline{S}T\overline{S} + \overline{T}T\overline{T})$ and it may vanish in a regular point. Hence, unlike the case of the complex variable, the corresponding mapping is not always locally topological. If $R = 0$, w is a complex-analytic quaternion function $w = f_1(z_1, z_2) + i_3 f_2(z_1, z_2)$ and the determinant reduces to $\Delta = (SS - TT)^2 + (ST + TS)^2$ and can be zero only if $|S| = |T|$ and S is perpendicular to T. In general $R = M + N$, where M lies in the plane of S and T and N is perpendicular to it. If $N = 0$, $R = \beta S + \gamma T$ and Δ is always positive. If $M = 0$, $\Delta = 0$ when $NN = (S\overline{S} + T\overline{T}) \pm (-(S\overline{T} - T\overline{S})^2)^{1/2}$. For M, $N \neq 0$, $\Delta = 0$ when $4S\overline{S}M\overline{M} \cos^2(\phi - \alpha) + 4T\overline{T}M\overline{M} \cos^2 \alpha \leqq -(S\overline{T} - T\overline{S})^2$ which is the equation of an ellipse. If $M\overline{M}$ is outside the ellipse, Δ is always positive; if $M\overline{M}$ is on the ellipse there is one value of $N\overline{N}$ for which $\Delta = 0$ and if $M\overline{M}$ is inside the ellipse there are two values of $N\overline{N}$ for which $\Delta = 0$. Over this ellipse we have a closed surface which bounds a three-dimensional region. Thus Δ is positive outside this surface, $\Delta = 0$ on the surface, and Δ is negative inside the surface. (Received November 13, 1953.)

257t. C. N. Moore: *On relationships between Nörlund means for double series.*

In this paper the theorems of Nörlund and Marcel Riesz concerning the consistency, inclusion, and equivalence of Nörlund means are generalized to double series. The results are analogous to those for single series, except that certain supplementary conditions enter into the discussion. (Received November 12, 1953.)

258t. John Nash: *C^1 isometric imbeddings of closed Riemannian n-manifolds in E^{2n}.*

The manifold should be differentiable and the components of the metric tensor should be continuous. First put a C^∞ structure on the manifold and use Whitney's results to have a C^∞ imbedding in E^{2n} which is shrunk proportionally until distances under the induced metric are everywhere less than under the intrinsic [given] metric. Now proceed to "stretch" the imbedding locally and directionally by adding [at each stage of this process] a spiral-like perturbation. The spiraling perturbation twists in the direction of the desired increase of the induced metric. By decreasing the amplitude but increasing the number of twists per unit distance we can indefinitely increase the accuracy and selectivity of the effect. So an infinite number of perturbations are added successively, the induced distances always being increased until the induced metric becomes the intrinsic metric in the limit. The process is such that first derivatives converge but not second derivatives, and its nature is such that it is easy to avoid introducing multiple points in the imbedding. To obtain imbeddings with more derivatives one expects to use a higher-dimensioned imbedding space. The author hopes to have results on the C^∞ case before long. (Received December 21, 1953.)

259t. A. B. Novikoff: *On infinitely differentiable functions.*

A. Rosenthal (Proc. Amer. Math. Soc. vol. 4 (1953)) has given a simple construction for an infinitely differentiable function satisfying $f^{(n)}(0) = b_n$, $n = 0, 1, 2, \cdots$, where the b_n's are an arbitrary prescribed sequence of real numbers. The author exhibits infinitely differentiable functions satisfying $f^{(n)}(a_\nu)=b_{\nu,n}$, $n=0, 1, 2, \cdots$; $\nu=1, 2, \cdots, N$. The case of interpolating through a sequence a_ν, $\nu=1, 2, 3, \cdots$, is solved if the a_ν's have no finite limit point. This is the most that can be hoped for without restricting the prescribed values $b_{\nu,n}$. (Received October 19, 1953.)

260. L. E. Payne (p) and H. F. Weinberger: *Upper and lower bounds for capacity and other functionals.*

Let $D(u)$ be the Dirichlet integral of a harmonic function u in an N-dimensional domain D with boundary B. Let n be the interior normal on B and r the distance from an arbitrary origin; let $u=O(r^{-1})$ if D extends to infinity. Upper and lower bounds for $D(u)$ are obtained by using the Green's identity $(N-2)D(u)=\int_B[2r(\partial u/\partial r)(\partial u/\partial n)-r\partial r/\partial n\sum_{i=1}^{N}(\partial u/\partial x_i)^2]d\sigma$, already mentioned in another connection by Rellich (Math. Zeit. vol. 46 (1940) p. 635). If u is the capacity function for B, the right-hand side reduces to $\int_B r(\partial r/\partial n)(\partial u/\partial n)^2d\sigma$ while the left-hand side with $N=3$ is 4π times the capacity C. If B is star-shaped so that $r\partial r/\partial n>0$ this permits one to obtain upper and lower bounds for C by a method like that proposed by Wolf Gross (Rendiconti Lincei vol. 12 (1952) p. 496). Namely, if ϕ is any harmonic function of order r^{-1} at ∞, then $(C-(1/4\pi)\int(\partial\phi/\partial n)d\sigma)^2\leq(1/16\pi^2)\int_B r(\partial r/\partial n)(\partial u/\partial n)^2d\sigma\int_B((1-\phi)^2/r(\partial r/\partial n))d\sigma=(1/4\pi)C\int_B((1-\phi)^2/r(\partial r/\partial n))d\sigma$. Thus, if ϕ is chosen to make the integral on the right small, close bounds for C are obtained. In particular, it is shown that Gross's estimate of $\int_B(\partial u/\partial n)^2d\sigma$ and hence his bounds for the capacity of the unit cube are incorrect. However, it is shown that his method gives the rigorous upper and lower bounds 0.668 and 0.627. The former is an improvement of the best upper bound given in the recent book by G. Pólya and G. Szegö. The method can be extended to general Dirichlet and Neumann problems in two or more dimensions. It has the computational advantage that even for exterior problems all calculations are to be performed on the boundary. (Received November 14, 1953.)

261*t*. R. S. Phillips: *The adjoint semi-group.*

Associated with each semi-group of linear bounded operators $T(s)$ defined on a Banach space X to itself and possessing suitable continuity properties, there is an adjoint semi-group with like continuity properties defined on an "adjoint" Banach space X^+ which is, in general, a proper subspace of the full adjoint space X^*. The choice of the subspace X^+ is decisive; it is the largest domain over which the ordinary adjoint semi-group $T^*(s)$ has suitable continuity properties. Moreover, if A denotes the infinitesimal generator of $T(s)$ and A^* denotes the adjoint to A, then X^+ can also be characterized as the closure of the domain of A^*. The infinitesimal generator A^+ of the adjoint semi-group defined on X^+ turns out to be the maximal contraction of A^* with domain and range in X^+. Both A and A^+ have the same spectral sets. As in the ordinary theory of adjoint spaces, it is possible to define an entire hierarchy of "adjoint" spaces corresponding to a given semi-group of operators. Likewise it can happen that the second "adjoint" space is equal to the original space (under the natural mapping). This situation occurs not only when X is reflexive in the usual sense but, more generally, when the resolvent of A is weakly compact. (Received October 7, 1953.)

262. M. H. Protter: *Boundary value problems for equations of mixed type.*

Uniqueness theorems are established for boundary value problems for an equation of the form (*) $K(z)(u_{xx}+u_{yy})+u_{zz}=0$ where $K(z)$ is a monotone increasing function of z with $K(0)=0$. Let S be a simple surface in the half-space $z>0$ with boundary curve C on the plane $z=0$. Let T be the set of points in the half-space $z<0$ with the property that the retrograde cone with vertex at a point of T intersects the plane $z=0$ in a circle lying entirely in the interior of C. Consider a point $P(x, y, 0)$ interior to C. Construct the direct cone K with vertex at P; denote its boundary by L. Designate by Γ the intersection of L with the boundary of T. Then the following problem is considered: prescribe boundary values on S and the boundary of T between $z=0$ and Γ. For this problem a uniqueness theorem is established for (*). The condition on $K(z)$ is the same as that required in the corresponding two-dimensional case. Other problems in which the boundary of T is not a characteristic surface can be treated by the same method. (Received November 6, 1953.)

263. R. K. Ritt: *Some remarks on the theory of summation.*

Banach, in his treatise, derived the Toeplitz theorem on the basis of the theory of linear functionals on a Banach space. In this paper, using methods due to Dieudonné and Schwartz, these results are extended. Consider the space of infinite sequences, (S), and choose: (a) a locally convex topology for (S); (b) a subspace (E) of (S) which includes the space of finite sequences, (S_f). The linear functionals on (E) which are continuous with respect to the topology chosen for (S), and which, when restricted to (S_f), coincide with usual summation are then studied. For different choices of (a) and (b), classes of linear functionals, which correspond, in special cases, to classical summation techniques are secured. (Received November 14, 1953.)

264. L. B. Robinson: *A functional equation with quasi analytical solution.*

Let ω, ω^2, $\omega^3=1$. Write $u(\rho)=u_0+\lambda\int_0^\rho(1/(1+\rho_1)^2)u(\rho_1^{2p})d\rho_1$. The solution is a meromorphic function whose poles are given by $\Delta_1(\lambda)=0$. Write $u(\omega\rho)=u_0+\lambda\int_0^{\omega\rho}(1/(1+\rho_1)^2)u(\rho_1^{2p})d\rho_1$, $u(\omega^2\rho)=u_0+\lambda\int_0^{\omega^2\rho}(1/(1+\rho_1)^2)u(\rho_1^{2p})d\rho_1$. The solution is a meromorphic function whose poles are given by $\Delta_2(\lambda)=0$. Write the cyclic groups $\omega_1^{6p}=1$, $\omega_2^{12p}=1$, $\omega_3^{24p}=1, \cdots$. $u(\omega_i\rho)$ also has poles determined by $\Delta_2(\lambda)=0$. Now $\omega_i\rho\rightarrow\rho$ as i grows greater. You would think that the determinant of $u(\omega_i\rho)$ would then $\rightarrow\Delta_1(\lambda)$ but it does not. It remains $\Delta_2(\lambda)$. This curious result is due to this fact: $u(x)$ admits the circumference of the unit circle as a singular line. And so the solution of our equation, continuous within the unit circle, ceases to be continuous outside. $u(x)$ may be called a quasi-analytic function. (Received November 10, 1953.)

265. P. C. Rosenbloom: *Approximation of eigenvalues for operators of the trace class.*

The following lemma is proved: if f is analytic and $|f|\leq 1$ for $|z|<1$, $|f(0)|=a>0$, $|f(z_0)|<\delta=\delta(a, r_0, \epsilon)$, $|z_0|=r_0$, $\epsilon>0$, then f has a zero in the circle $|z-z_0|<\epsilon$. The function $\delta(a, r_0, \epsilon)$ is estimated explicitly. We apply this to show that there is a function $\eta=\eta(M, r, \epsilon)$ such that if K is an operator of the trace class on a Banach space X into itself with trace norm $\leq M$, $\|x\|=1$, $|\lambda_0|\leq r$, and $\|x-\lambda_0 Kx\|<\eta$, then there is an eigenvalue of K in the circle $|\lambda-\lambda_0|<\epsilon$. (Received November 14, 1953.)

266. Hanan Rubin: *The dock of finite extent.*

The scattering on a finite dock of a prescribed progressive wave in an ocean of infinite depth is considered. The required velocity potential $\Phi(\xi, \eta; \tau)$ is to satisfy Laplace's equation $\Phi_{\xi\xi}+\Phi_{\eta\eta}=0$ for $\eta<0$ and the boundary conditions $\Phi_{\tau\tau}(\xi, 0; \tau)$

$+g\Phi_\eta(\xi, 0; \tau) = 0$ for $|\xi| > \alpha$ and $\Phi_\eta(\xi, 0; \tau) = 0$ for $|\xi| < \alpha$. Moreover, $\Phi(\xi, \eta; \tau)$ is to have a logarithmic singularity of prescribed strength for $(\xi, \eta) \sim (\alpha, 0)$, and $\Phi(\xi, \eta; \tau) - Ae^{(\sigma^2/g)\eta} \cos((\sigma^2/g)\xi + \sigma\tau - \gamma)$ is to satisfy a "radiation condition" for $\xi \sim \pm\infty$ with arbitrarily prescribed A, γ, and σ. Explicit formulas are given expressing $\Phi(\xi, \eta; \tau)$ in terms of the solution to a minimum problem for the functional $F[w(x, y)]$ defined by $F[w] = \int_{y<0}\int (w_x^2 + w_y^2)dxdy + \int_{-a}^{a} [w_x(x, 0)]^2 dx$. The direct methods of the calculus of variations are employed to establish the existence and uniqueness of the solution to this minimum problem. The admissibility conditions require piecewise continuous $w_x(x, 0)$ for $|x| < a$. Consequently, after a uniformly convergent minimizing sequence of harmonic functions is obtained, estimates for the x-difference quotient near the boundary are derived from the "variational condition" in order to verify this admissibility condition for the limit function. The dependence of the reflection coefficient on the strength of the logarithmic singularity is noted. (Received November 12, 1953.)

267. Walter Rudin: *Natural boundaries of bounded analytic functions.*

Definitions: $B(D)$ is the set of all single-valued bounded analytic functions on D, a domain on the Riemann sphere. A point x of K (the complement of D) is removable if for every $f \in B(D)$ there exists a domain D^* containing x and D and a function $f^* \in B(D^*)$ such that $f^*(z) = f(z)$ $(z \in D)$. The other points of K are essential. If all boundary points of D are essential, D is maximal. Results: (1) x is removable if and only if there is some $r > 0$ such that the intersection of K with the closed disc of radius r and center x is a Painlevé null-set (in the terminology of Ahlfors and Beurling, Acta Math. vol. 83 (1950) p. 107). (2) Every domain D is contained in a smallest maximal domain Δ; every $f \in B(D)$ can be extended to Δ. (3) If x is an essential boundary point of D, then there exists a function $f \in B(D)$ whose cluster set at x consists of the entire unit disc, while $|f(z)| < 1$ for all $z \in D$. (4) If D is maximal (and only in this case), then some $f \in B(D)$ has the boundary of D as its natural boundary. (Received November 12, 1953.)

268*t*. Walter Rudin: *Rings of bounded analytic functions.*

Let D_1 and D_2 be two maximal domains (see the preceding abstract for definitions) in the finite plane and suppose that there exist an isomorphic mapping ψ of the ring $B(D_1)$ onto the ring $B(D_2)$. Then either $\psi(i) = i$ or $\psi(i) = -i$. In the former case it is shown that there exists a one-to-one conformal mapping ϕ of D_2 onto D_1 such that $f(\phi(w)) = f^*(w)$ for $w \in D_2$, where $f^* = \psi(f)$ (if $\psi(i) = -i$, conjugates have to be introduced). It follows that D_1 and D_2 are conformally equivalent if and only if the rings $B(D_1)$ and $B(D_2)$ are algebraically isomorphic; this is a theorem of Kakutani and Chevalley whose proof is unpublished. The author's proof makes no use of the theory of normed rings: the identity function $I(z) = z$ is a member of the quotient field $F(D_1)$ of $B(D_1)$; extending ψ to $F(D_1)$ and setting $\phi = \psi(I)$, it is shown directly that ϕ has the desired properties. (Received November 12, 1953.)

269. I. J. Schoenberg: *On multiply positive sequences and functions.*

Let k be an integer ≥ 2 and let P_k (the class of k-times positive functions) denote the class of functions $f(x)$, $-\infty < x < \infty$, satisfying the following two conditions: 1. $f(x)$ is non-negative and $0 < \int_{-\infty}^{\infty} f(x)dx < +\infty$. 2. If $x_1 < x_2 < \cdots < x_n$, $y_1 < y_2 < \cdots < y_n$, then $\det \|f(x_i - y_k)\|_{1,n} \geq 0$ and this should hold for every $n = 1, 2, \cdots, k$. Theorem 1. If $f(x) \in P_k$ and $f(x) = 0$ outside the interval $(-\pi/2, \pi/2)$, then its Laplace transform

$F(s)=\int_{-\pi/2}^{\pi/2}\exp(-sx)f(x)dx$ has no zeros in the strip $|Is|<k$. Here k is the best constant. Indeed the function $f_k(x)=(\cos x)^{k-1}$ for $-\pi/2\leq x\leq \pi/2$, $f_k(x)=0$ outside this interval, is shown to be in P_k while having a transform vanishing for $s=\pm ki$. This theorem is derived from an analogous theorem for zero-free sectors for polynomials whose coefficients form a k-times positive sequence in the sense of Fekete (Rendiconti di Palermo vol. 34 (1912) p. 3). The proof of this last theorem uses results on variation-diminishing transformations and Gaussian binomial coefficients. (Received November 14, 1953.)

270. Jacob Schwartz: *Essential spectra.*

The essential spectrum $\sigma_e(T)$ of an operator T is the complement in its spectrum of the set of all points of finite multiplicity. The open complement of $\sigma_e(T)$ is the essential resolvent $\rho_e(T)$. Let $\rho_e'(T)$ be that component of $\rho_e(T)$ containing the point at infinity. Then, if K is a compact (i.e., completely continuous) operator, $\rho_e'(T+K)=\rho_e'(T)$. An example of Yood (Duke Math. J. vol. 18 (1951) p. 599) shows that $\rho_e(T+K)\neq\rho_e(T)$ is possible. As a special case it follows that $\rho_e(T+K)=\rho_e(T)$ if T is a bounded or semibounded self-adjoint operator. The proof is based on the perturbation theory of spectra. (Received November 13, 1953.)

271. J. B. Serrin: *On the Phragmén-Lindelöf principle for elliptic partial differential equations.*

Consider the equation $L(u)=a_{ij}(x)\partial^2u/\partial x_i\partial x_j+b_i(x)\partial u/\partial x_i=0$, where the coefficients are defined in a region D in the half-space $x_n\geq 0$, $x=(x_1\cdots x_n)$. Suppose that the boundary $\overline{D}$ of D lies in a strip $0\leq x_n\leq\sigma$, that L is uniformly elliptic in D, and that there exists a decreasing function $p(r)$ such that $(\sum a_{ii})^{-1}(\sum b_i^2)^{1/2}\leq p(r)$, $r^2=\sum x_i^2$, and $\int_0^\infty p(r)dr<\infty$. Let v be any function with the properties (1) $L(v)\leq 0$, (2) $v\geq 0$ on D, (3) $\lim_{x_n\to\infty} v/x_n=1$, uniformly in $x_1,\cdots,x_{n-1}$. (It is shown that functions v always exist; in particular if all $b_i=0$ we may take $v=x_n$.) *Theorem.* Suppose $u=u(x)$ satisfies $L(u)\geq 0$ in D, $u\leq 0$ on $\overline{D}$. Then the limit $\alpha=\lim_{R\to\infty}M(R)/R$, where $M(R)=\max_{r=R}u(x)$, always exists (finite or infinite); moreover α is nonnegative and $(*)$ $u<\alpha v$ throughout D. Equality cannot hold in $(*)$ unless $v=0$ on $\overline{D}$, $L(v)=0$, and in this case equality at any point in D implies equality everywhere. This result is a strict analogue for elliptic equations of the classical Phragmén-Lindelöf theorem for harmonic functions; the proof makes essential use of the work of Gilbarg and Hopf (Journal of Rational Mechanics and Analysis vol. 1, pp. 411–424). (Received November 13, 1953.)

272. A. L. Shields: *On the measure of a sum set.*

Let G be a compact, connected, abelian, 2nd countable topological group (written additively) with Haar measure m, $m(G)=1$. Let $A, B\subset G$ be measurable and $m(A)+m(B)\leq 1$. Then, $m_i(A+B)\geq m(A)+m(B)$, where m_i is interior measure, and $A+B$ is the set of all $a+b$, $a\in A$, $b\in B$. The idea of the proof is to reduce the problem to a form of the $\alpha+\beta$ hypothesis of additive number theory. The following lemma is proven; let G satisfy the hypotheses above and let T be the transformation defined by $T(g)=g+g$. Let $A\subset G$ be measurable. Then $T^{-1}(A)$ is measurable, and $m(T^{-1}A)=m(A)$. (Received November 9, 1953.)

273. George Springer (p) and Daniel Resch: *On sums of schlicht functions.*

Let Σ be the class of functions $f(z)=z+a_1/z+a_2/z^2+\cdots$ which are regular and schlicht in $|z|>1$ except for a simple pole at $z=\infty$. Let $h(z)=\alpha f(z)+\beta g(z)$ ($f, g\in\Sigma$, $\alpha\geqq 0$, $\beta\geqq 0$, $\alpha+\beta=1$) be schlicht in $|z|>R(h)$ but not in $|z|>r$, $r<R(h)$. Define $R=\max R(h)$ for all such h. Then it is proved that $1.381\leqq R\leqq 2^{1/2}$. The mappings f and g for which $R(h)=R$ are mappings of $|z|>1$ onto slit regions of the w-plane. (Received November 18, 1953.)

274. J. M. Stark: *On value distribution of functions of two complex variables.*

The author considers analytic functions $f(z_1, z_2)$ of two complex variables in domains M^4 whose boundary b^3 consists of finitely many segments j_k^3, $k=1, 2, \cdots, n$, of analytic hypersurfaces where $j_k^3=\sum_\lambda J_k^2(\lambda)$ are one-parameter families of segments $J_k^2(\lambda)$ of analytic surfaces. The sum of intersections $j_m^3\cap j_n^3$, $m\neq n$, forms the distinguished boundary surface D^2 of M^4. The author shows that certain information about the behavior of f in $c^3=b^3-D^2$ leads to bounds for $|f(t_1, t_2)|$ where (t_1, t_2) belongs to a subdomain of M^4. Using and generalizing results by Bergman (Mathematica vol. 14 (1938) p. 116 and Math. Zeit. vol. 39 (1934) p. 78), the author shows that if (I) in every boundary segment $j_k^3=\sum_\lambda J_k^2(\lambda)$ of M^4, $|f|\leqq c_k(\lambda)$ and $|\partial f/\partial z_\sigma|\leqq A_k(\lambda)$, $\sigma=1, 2$, on a curve $z_\nu=Z_\nu(\lambda)$, $\nu=1, 2$, and (II) in every lamina $J_k^2(\lambda)$, f assumes no value more than once, then Koebe's theorem yields an upper bound for $|f(t_1, t_2)|$ for (t_1, t_2) belonging to a certain subdomain of M^4. This bound is a function of t_1, t_2, $c_k(\lambda)$, $A_k(\lambda)$, and some quantities which depend only upon M^4. (Received December 29, 1953.)

275t. Gabor Szegö and Antoni Zygmund: *On certain mean values of polynomials.*

Let $f(z)$ be a polynomial of degree n and let $\|f\|_p$ denote the magnitude of $f(z)$ in L^p, $|z|=1$. Then $\|f\|_q/\|f\|_p<An^{1/p-1/q}$ where A depends on p and q, $0<p<q$. (In the case $p\geqq 1$ the inequality holds with an absolute constant A.) This inequality is discussed by two different methods and analogous inequalities are established for the case of an interval. For a wide class of curves the corresponding quotient as well as $\|f'\|_p/\|f\|_p$ are estimated. Finally A. Markov's theorem is proved for a square S in the following sharper form: Let $|f|\leqq 1$ on S. Then $|f'|\leqq O(1)\cdot\min\{n^{3/2}, n\delta^{-1/2}\}$ on S; here we assume that S has the side 1 and δ is the distance of z from the nearest vertex of S. (Received November 12, 1953.)

276t. C. T. Taam: *An extension of Osgood's oscillation theorem for a nonlinear differential equation.*

Consider (A): $(m(t)x')'+\sum_{i=1}^{k} f_i(t)x^{2i-1}=0$, where m^{-1} and f_i are non-negative, Lebesgue-integrable in $[T, R]$ for every $R>T$. It is shown that on $T\leqq t<\infty$ if $m<m_0$, $f_1>f_0$, $mf_i>r$ for some i, m_0, f_0 and r being positive constants, and if either all mf_i are nondecreasing or all nonincreasing, then every solution of (A) is oscillatory and the amplitudes of the oscillations remain finite. If, furthermore, all mf_i remain finite, the amplitudes of the oscillations have a positive lower bound. Moreover, the amplitudes vary monotonically, increasing when all mf_i decreases monotonically, and decreasing in the opposite case. This generalizes an oscillation theorem of Osgood. Also given is a sufficient condition for the boundedness of the solutions of (A). Further generalizations are indicated. (Received November 12, 1953.)

277. C. T. Taam: *Asymptotic relations between systems of differential equations.*

Consider (A): $y' = A(x)y + f(x, y)$, where $A(x) = (a_{ij}(x))$ is a $n \times n$ matrix and y and $f(x, y)$ are vectors with components y_i and f_i respectively, $a_{ij}(x)$ and $f_i(x, y)$ (for each y) being complex-valued functions of the real variable x and belonging to $L(0, R)$ for every $R > 0$. Furthermore $f_i(x, y)$ is a continuous function of y and $\|f(x, y) - f(x, Y)\| \leq g(x)\|y - Y\|$ for all y and Y, $g(x)$ being of $L(0, R)$ for every $R > 0$. Let Y be the solution of $Y' = A(x)Y$, $Y(0) = I$. It is shown that if (a) $\|Y(x)\| = O(k(x))$, $k(x)$ being measurable on $0 \leq x < \infty$, (b) $\|f(x, 0)\|(k(x))^{n-1}T(x)$ and $g(x)(k(x))^{n}T(x)$ belong to $L(0, \infty)$, $T(x) = \exp\left[-\mathrm{Re} \int_0^x \mathrm{tr}\, A(t)dt\right]$, then, for any solution $y(x)$ of (A), $y(x) = Y(x)(c + \epsilon(x))$ for some constant c and $\epsilon(x) \to 0$ as $x \to \infty$. Moreover if (b) still holds after n is replaced by $n+1$ and if $k(x)$ is nondecreasing, then $y(x) = Y(x)c + \epsilon(x)$. By Phragmén-Lindelöf arguments, the results are extended to the complex domain. (Received November 12, 1953.)

278t. C. T. Taam: *The boundedness of the solutions of an almost linear differential equation.*

Let $1/r(x)$, $q(x)$, and $f(x, y)$ (for each fixed y) be real and of the class $L(0, R)$ for every $R > 0$. For each $x \geq 0$ let also $f(x, y)$ be a continuous function of y satisfying the Lipschitz condition: $|f(x, y_1) - f(x, y_2)| \leq g(x)|y_1 - y_2|$, where $g(x)$ and $f(x, 0)$ belong to $L(0, \infty)$. It is shown that every solution of $(ry')y + qy = f$ is bounded if there exists a real-valued function $p = p(x)$ such that (1) p belongs to $L(0, R)$ for every $R > 0$; (2) $rp > 0$ and absolutely continuous on $0 \leq x < \infty$; (3) $p - q$ and $(rp)^{-1}(rp)'_-$ belong to $L(0, \infty)$, $(rp)'_- = \min((rp)', 0)$. A similar result holds when r, q, and f assume complex values. (Received November 12, 1953.

279. J. L. Ullman: *On Faber expansions.*

Let C be a simple analytic curve, $w = f(s)$ its normalized exterior mapping function, mapping C onto Σ_ρ, $|w| = \rho$, and let C_λ be the image of Σ_λ, $|w| = \lambda$. (a) If $F(s)$ is analytic inside and on C, then it has a unique expansion in the Faber polynomials $\{F_n(s)\}$ associated with C, say $\sum a_n F_n(s)$. This expansion converges inside C_μ, the level curve first encountering a singularity of $F(s)$ as the index increases. (b) Besides proving (a), Faber (Math. Ann. vol. 57 (1903) pp. 389–408) correlated the analytic character of $\phi(t) = \sum a_n t^n$ with that of $F(s)$. In particular he showed that points on Σ_μ and C_μ which correspond under the mapping $w = f(s)$ are both regular or both singular. (c) In this direction, the author adds the theorem that if $\phi(t)$ is rational, then $F(s)$ is also. (d) As an application, consider a Faber expansion associated with $C = C_\rho$. If the series has coefficients assuming finitely many values, then $\phi(t)$ is either rational or has $|t| = 1$ as a natural barrier, a result due to Szegö. Hence if $\rho < 1$, by (b) and (c), $F(s)$ is rational or has C as a barrier. (e) These remarks provide an essential simplification to L. Iliev, *Series of Faber polynomials whose coefficients assume a finite number of values*, Doklady Akad. Nauk SSSR vol. 40 (4) (1953) pp. 499–502. Furthermore, they apply with equal force to Faber expansions whose coefficients are integers. (Received November 14, 1953.)

280. Daniel Waterman: *Functions analytic in a half-plane.*

If Ω is a domain in the unit circle contained in the union of a Stolz angle η and a circle $|z| < R < 1$, and Ω_θ is this region rotated about $z = 0$ until 1 coincides with $e^{i\theta}$, Lusin defined the function $s(\theta) = \{\int_\Omega\int_\theta |\phi'|^2 dxdy\}^{1/2}$; Zygmund and Marcinkiewicz showed that $A_{p,\eta}\|\phi\|_p \leq \|s\|_p \leq A_{p,\eta}\|\phi\|_p$ ($A_{\alpha,\beta}\dots$ denoting positive constants depending on the indicated parameters) provided that $\phi \in H_p$, for $p > 0$. Littlewood and Paley defined $g^*(\theta) = \{(1/\pi)\int_0^1(1-\rho)\partial\rho\int_0^{2\pi}|\phi'(\rho e^{i(\theta+t)})|^2 P(\rho, t)\}^{1/2}$, where $P(\rho, t)$ is the Poisson kernel. It has been shown (Zygmund, Littlewood, and Paley) that $A_p\|g^*\|_p \geq \|\phi\|_p$ for $p>0$, and $A_p\|\phi\|_p \geq \|g^*\|_p$ for $p>1$. If one considers the Hille-Tamarkin class $\mathcal{H}_p$ in the half-plane $\Re(\sigma+i\tau)>0$ and defines $s(\tau) = \{\int_{\Omega}\int_\tau |\phi'(\sigma+i\beta)|^2 d\sigma d\beta\}^{1/2}$ where Ω_τ is a region contained between $\sigma = \pm k\beta+\tau$, and $g^*(\tau) = (1/\pi)\int_0^\infty \sigma d\sigma \int_{-\infty}^{\infty} |\phi'(\sigma+i\beta)|^2 P(\sigma, \tau-\beta) d\beta\}^{1/2}$ where $P(\sigma, \tau) = \sigma/(\sigma^2+\tau^2)$, then exact analogues of the cited results can be demonstrated. (Received November 13, 1953.)

281. John Wermer: *Maximal subalgebras.*

Let B be a commutative Banach algebra, A a closed subalgebra. Say that A is *maximal* if there are no closed subalgebras of B lying between A and B. When B is the algebra of all continuous functions on the circle $|\lambda| = 1$, the author has shown that the closed subalgebra of B generated by $\phi(\lambda) = \lambda$ is maximal. (*On algebras of continuous functions*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 866–869.) In this note a different method is used to show that the algebra of absolutely convergent Taylor series is a maximal subalgebra of the algebra of absolutely convergent Fourier series. The method used is applicable to a larger class of algebras. (Received November 14, 1953.)

282. F. M. Wright: *Some results on the backward extension of S-sequences.*

Using an S-fraction transformation theorem previously presented but as yet unpublished, the author first proves a result concerning the linear dependence of the $(k+1)$ terminating S-fractions involved in the first k backward extensions of an S-sequence $\{\mu_n\}$ such that the S-fraction expansion of the formal power series $Q(w) = \sum_{n=0}^{\infty} \mu_n w^{n+1}$ contains $2k$ partial quotients. The transformation theorem mentioned above is then used to examine somewhat more closely than Wall (Trans. Amer. Math. Soc. vol. 31 (1929) pp. 91–116) has done the problem of extending backward an indeterminate Stieltjes moment sequence. Finally, this theorem is used to discuss the first backward extension of a Hausdorff moment sequence. (Received November 10, 1953.)

APPLIED MATHEMATICS

283. W. C. Carter: *On Taylor-MacColl flow.*

The methods previously used to prove the existence of Taylor-MacColl flow may be improved by the introduction of more accurate approximations to the solution of the differential equations. These approximations allow the extension of the existence proof to include semi-cone angles ranging from 0 to 52.5 degrees and free stream Mach numbers from 1 to greater than 10. By the use of constants derived from Kopal's tables for supersonic conical flow these methods give new simple approximations in closed analytic form for supersonic right conical flow. In contrast to the usual linearized flow theory these approximations are more accurate for large cone angles and high free stream Mach numbers. (Received November 12, 1953.)

284. R. C. DiPrima: *A note on the asymptotic solutions of the equation of hydrodynamic stability.*

The equation of hydrodynamic stability is $\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi + i\alpha R[(w-c)(\phi''-\alpha^2\phi) - w''\phi] = 0$, where the disturbance is given by the stream function $\Psi = \phi(y) \exp [i\alpha(x-ct)]$. Tollmien (Z. Angew. Math. Mech. vols. 25 and 27 (1947) pp.

33–50, 70–83), considering only real y and real parameters α, R, c, gave a system of asymptotic solutions of this equation. Recently, Wasow (Ann. of Math. vol. 58 (1953) pp. 222–253), considering general values of y and the parameters, has given a complete theory of the manner in which the asymptotic solutions represent the fundamental solutions of the above equation. However, Wasow's error estimate for the derivative of the solution χ_3 corresponding to the logarithmically singular solution ϕ_2 of the reduced equation $(w-c)(\phi''-\alpha^2\phi)-w''\phi=0$ is not as good as that given by Tollmien for his solution ϕ_2^*. By more detailed analysis of the work of each author it is shown that in the case of real parameters the derivative of the Wasow solution and the Tollmien solution agree to within the order of error given by Tollmien. (Received November 14, 1953.)

285. M. A. Hyman: *Numerical determination of nonisentropic supersonic gas flows.*

The case of stationary, inviscid flow past an axially-symmetric body is considered. A new "computation-by-characteristics" method is described, which seems advantageous for use with automatic computing machines. Certain new variables are introduced, in terms of which all required physical and engineering quantities can be simply expressed. The equations to be satisfied become simple algebraic relations between the differentials of the new variables. The absence of transcendental and other complicated operations makes "table look-ups" and special programming unnecessary. By a simple device the "three-characteristic" problem is reduced (with a gain in accuracy over standard "three-characteristic" methods) to a "two-characteristic" problem; as a result, programming time and computing time are saved, and space in the machine's memory is conserved. This method, developed in 1950–51, has been used in calculations undertaken since at the U. S. Naval Ordnance Laboratory, with good results. Though developed for axially-symmetric flow with shock-waves, the basic ideas of this new numerical method seem to carry over to many other "three-characteristic" problems. (Received November 3, 1953.)

286*t*. Imanuel Marx: *On the structure of recurrence relations.*

Let $y_{n-1}^{(1)}$, $y_n^{(1)}$ be the characteristic solutions of $d/dx[p(x)dy/dx]+[q(x)+\lambda]y=0$ corresponding to λ_{n-1}, λ_n, respectively, let $y_{n-1}^{(2)}$, $y_n^{(2)}$ be a pair linearly independent of the first, and let z_{n-1}, z_n be an arbitrary pair of solutions. All recurrence relations between z_{n-1} and z_n are deducible from the pair $pA_n(x)dz_n/dx+B_n(x)z_n=z_{n-1}$, $pA_n(x)dz_{n-1}/dx+C_n(x)z_{n-1}=s_nz_n$, s_n a constant. The coefficients A_n, B_n, C_n satisfy a system of first-order nonlinear differential equations which combine to form a fourth-order linear equation for A_n. This is the same as the fourth-order equation satisfied by the product z_nz_{n-1}, so that $A_n=\sum_{j,k=1}^{2} a_n^{(jk)}y_n^{(j)}y_{n-1}^{(k)}$, and $B_n C_n$ are obtained from A_n by quadratures while s_n is easily computed. The constants $a_n^{(jk)}$ depend on the conditions that determine the characteristic values λ_n. A special case (Mathieu functions) was treated by E. T. Whittaker [J. London Math. Soc. (4) vol. 88 (1929)]. The factorization method of L. Infeld and E. T. Hull [Review of Modern Physics vol. 23] for recurrence relations corresponds to the simpler case $A_n=$ constant, but in the differential equation $p(x)=1$ while $q(x)+\lambda$ is replaced by a more general function $r(x, \lambda)$. (Received November 12, 1953.)

287*t*. John Nash: *Results on continuation and uniqueness of fluid flow.*

One can look at a gas in terms of the kinetic theory so that for each point and time there is a molecular velocity distribution. And a collision probability function enables the computation of the time derivative of the velocity distribution. One shows that initial conditions can be continued uniquely into the future so long as gross singularities of specified types do not occur. And one shows that the continuation from a C^∞ initial pattern will remain C^∞. The gross singularities are infinities and zeros of temperature and density. These results are relevant to the problem of turbulence which some have thought involved the nonexistence of higher derivatives of the flow quantities. It is hoped that an extension of these results to the continuum description [partial differential equations] of compressible viscous flow will be ready before long. (Received December 21, 1953.)

288t. R. H. Pennington: *Surface instability in pulsating gas bubbles.* I. Preliminary report.

The motion of an incompressible infinite fluid outside of a bubble of ideal gas is studied. The differential equations of motion and the correct boundary conditions are established for an ideal fluid, a viscous fluid, and for the case of surface tension. In the case of a nearly spherical bubble, a perturbation method is used which starts from the known radially symmetric solutions. The velocity field outside of the gas bubble can be obtained by linear superposition of velocity fields $\dot{r}_0(r, t)+\epsilon f(r, t)S_n(\theta, \phi)$, $\epsilon g(r, t) \csc^2 \theta(\partial S_n(\theta, \phi)/\partial\phi)$, $\epsilon g(r, t)(\partial S_n(\theta, \phi)/\partial\theta)$, where the three functions determine the rate of change of the Eulerian polar coordinates, r, ϕ, θ, and where the S_n are the nth order spherical harmonics. $\dot{r}_0(r, t)$ is a known radial velocity field of the unperturbed radially symmetric solution. $f(r, t)$ and $g(r, t)$ are connected by the relation $\partial(r^2f)/\partial r=n(n+1)r^2g$. The function $u(r, t)=(1/n(n+1))\{\partial^2[r^2f(r, t)]/\partial r^2-f(r, t)\}$ is closely related to the vorticity of the motion and satisfies the differential equation $u_t+\partial(\dot{r}_0u)/\partial r =(\mu/\rho)[u_{rr}-n(n+1)/r^2)u]$, where μ is a coefficient of viscosity and ρ is the density of the fluid. (Received December 21, 1953.)

289t. R. H. Pennington: *Surface instability in pulsating gas bubbles.* II. Preliminary report.

The solutions in the problem of nearly spherical gas bubbles are developed into spherical harmonics and the time dependence of each harmonic component is studied. The differential equation for the amplitude is given. Both irrotational and rotational flows are admitted. If the surface of the bubble is described by the equation $r=R_0(t)+\epsilon A_n(t)S_n(\theta, \phi)$, then $A_n(t)$ is determined by the two differential equations $-\ddot{A}_nR_0-3\dot{A}_n\dot{R}_0+(n-1)A_n\ddot{R}_0-(1/\rho)(n-1)(n+1)(n+2)A_nTR_0^{-2}+n(n+1)R_0^{n-1}\dot{R}_0[R_0^2\int_{R_0}^{\infty}\zeta^{-n-2}u(\zeta, t)d\zeta-\int_{R_0}^{\infty}\zeta^{-n}u(\zeta, t)d\zeta]=(n+1)(\mu/\rho)[-nR_0^{-1}u(R_0, t)+2(n+2)R_0^{-1}\dot{A}_n+4(n-1)A_nR_0^{-2}\dot{R}_0+2n(n+1)R_0^{n-2}\int_{R_0}^{\infty}\zeta^{-n}u(\zeta, t)d\zeta]$ and $\mu[2(n+2)\dot{A}_n-2(n-1)A_nR_0^{-1}\dot{R}_0+(n+1)u(R_0, t)+2(n+1)R_0^{n-1}\int_{R_0}^{\infty}\zeta^{-n}u(\zeta, t)d\zeta]=0$ where T is the surface tension and μ, ρ, and $u(r, t)$ are defined as in the preceding abstract, and $R_0(t)$ is known from the radially symmetric solution. The stability of the various modes is discussed and numerous special cases are treated. In the case of an irrotational nonviscous flow our differential equation for the $A_n(t)$ reduces to $\ddot{A}_nR_0+3\dot{A}_n\dot{R}_0-(n-1)A_n\ddot{R}_0+(1/\rho)(n-1)(n+1)(n+2)R_0^{-2}A_nT=0$, an equation obtained recently by Plesset (*On the stability of fluid flows with spherical symmetry*, The Rand Corporation, Santa Monica, California). (Received December 21, 1953.)

290. S. I. Plotnick: *Evaluation of constants in conformal representations.*

The Schwarz-Christoffel transformation, $dz = k\prod_{k=1}^{n} (\zeta - \xi_k)^{\alpha_k/\pi - 1} d\zeta = kf(\zeta)d\zeta$, maps the upper half ζ-plane into a simple connected polygon. If ξ_k in the ζ-plane corresponds to two points P_k, Q_k in the z-plane and $\zeta = \xi_k$ is a simple pole of $f(\zeta)$ (z-plane contains a strip), then it is proved that $k = \text{dist}\ (P_k, Q_k)/\pi i R(\zeta = \xi_k)$, R denoting residue and dist (P_k, Q_k) denoting the vector distance between the two points P_k and Q_k. (Received November 2, 1953.)

291. Domina E. Spencer: *Maxwell's equations and the new electrodynamics.*

A previous paper (D. E. Spencer, *Field theory and the new electrodynamics*, Bull. Amer. Soc. Abstract 59-6-672) has presented a single equation for the force between moving charges. The purpose of the present paper is to clarify the relationship between the Maxwell equations and the new electrodynamics. It is shown that the Maxwell equations can be obtained as a special case if two assumptions are made: I. The Ampere equation for the force between current elements is replaced by the Grassmann equation. II. A moving charge is interchangeable with a current element if both represent the same rate of transport of charge. The first assumption is rigorously correct only for closed circuits. The second assumption cannot be made if the theory is to be expressed in terms of relative velocities. Once this assumption is introduced, invariance under Galilean transformations is lost. For most engineering problems Assumptions I and II are satisfactory approximations. From the point of view of the new electrodynamics, the Maxwell equations are a useful approximation, but they constitute a shaky foundation on which to base all of electricity and magnetism. (Received November 14, 1953.)

292. A. H. Van Tuyl: *Some exact solutions in electrostatics and hydrodynamics.*

The following result of Ernst Neumann (J. Reine Angew. Math. vol. 120 (1899) pp. 277–304) is studied in special cases and extended: Given a sphere S of radius R, and a system of conductors B which is transformed into itself by inversion with respect to S, let $U(P)$ be the potential function of B at a point P (equal to unity on B and regular outside), let C be the capacity of B with respect to the infinite sphere, and let r be measured from the center of S; then the potential function of the composite conductor $B+S$ is $U(P)+R/r-(R/r)U(P')$, where $P'=(R/r)^2P$, and its capacity is $C+R-RU(0)$. This result gives the potential function of certain conductors which do not seem to have been considered explicitly in the past. They include: (1) a torus and a sphere intersecting orthogonally; (2) a torus and two spheres, all mutually orthogonal; (3) special configurations of four and five spheres. Using the generalization of Neumann's result to n-dimensional Euclidean space, and results of Payne and Weinstein (L. E. Payne, Quarterly of Applied Mathematics vol. 10 (1952) pp. 197–204), solutions in hydrodynamics and elasticity are obtained in the axially symmetric cases. (Received November 18, 1953.)

GEOMETRY

293*t*. Eugenio Calabi: *The variation of Kähler metrics.* I. *The structure of the space.*

Let M^n be a compact complex manifold admitting at least one Kähler metric $g_{\alpha\beta^*}$; one considers the closed, exterior form $\omega_g = (-1)^{1/2} g_{\alpha\beta^*} dz^\alpha \wedge dz^{\beta^*}$ and its (the principal) cohomology class (ω). If $R_{\alpha\beta^*}$ is the Ricci tensor, then $\Sigma_0 = (-1)^{1/2} R_{\alpha\beta^*} dz^\alpha$

$\wedge d\omega^{\bar{n}}$, the Ricci form, is closed and cohomologous to 2π times the first Chern class. *Theorem*: Given any closed exterior form Σ of type (1, 1), of class C^∞, cohomologous to 2π times the first Chern class, there exists a unique Kähler metric of class C^∞, whose principal cohomology class is (ω), and whose Ricci form equals Σ. With the aid of this theorem one considers the set of all Kähler metrics of class C^∞, whose principal class is (ω). It is an infinite-dimensional "open manifold," in which there is a natural Riemannian metric $dS^2=(\eta, \eta)$ where η is the first variation of any given ω and (ϕ, ψ) is the integrated inner product of two forms ϕ, ψ. This manifold has constant, positive curvature, unique geodesics of bounded length between any pair of points, and its completion consists of the positive semidefinite Kähler metrics defining the same principal class. (Received October 28, 1953.)

294. Eugenio Calabi: *The variation of Kähler metrics.* II. *A minimum problem.*

Let M^n be a compact, n-dimensional Kähler manifold, and (ω) its principal cohomology class (see previous abstract). For each principal Kähler form $\omega\in(\omega)$ let Σ be its Ricci form, (Σ, Σ) the norm of that form, $R=-2\Lambda\Sigma$ the scalar curvature, and H the operator projecting forms into their harmonic part. The following constants depend only on (ω), not on ω: $\int_{M^n}\omega^n=V$, $HR=2nV^{-1}\int_{M^n}\omega^{n-1}\wedge\Sigma$, $(H\Sigma, H\Sigma)=V(HR)^2-n(n-1)\int_{M^n}\omega^{n-2}\wedge\Sigma^2$. *Theorem* 1: Σ is harmonic with respect to ω if and only if R is a constant; if this is the case, and for fixed (ω), the metric is determined within any transformation due to the complex Lie group of analytic deformation of M^n, and (Σ, Σ) is minimized. The converse is implied for most complex manifolds by *Theorem* 2: If, for a certain $\omega\in(\omega)$, (Σ, Σ) is minimized, then the contravariant vector $g^{\alpha\beta^*}R_{\alpha\beta^*}$ is holomorphic; this implies that Σ is harmonic if M^n has no analytic deformations. The existence of an $\omega\in(\omega)$ minimizing (Σ, Σ) is, except in special cases, still unproved. (Received October 28, 1953.)

295*t*. John DeCicco: *Some theorems in the geometry of the Kasner turbine group in the complex domain.*

Under a representation R, every point of a complex projective three-dimensional space S_3 is represented as a lineal element of the projective extension of a complex Euclidean plane. By R, the straight lines and planes of S_3 correspond to turbines and flat fields in the opulence of lineal elements. The Kasner turbine group G_{15} is isomorphic to the complete projective group in S_3. The conjugate operation N in the opulence is represented as a special correlation in S_3. To every transformation Σ of G_{15}, there corresponds a single conjugate transformation $\bar{\Sigma}=N\Sigma N$ of G_{15}, such that Σ and $\bar{\Sigma}$ carry conjugate turbines into conjugate turbines. The self-conjugate transformations $\Sigma=\bar{\Sigma}$ of G_{15} form the Lie circular group L_{10}. If $\Sigma\neq\bar{\Sigma}$, then $\Sigma\bar{\Sigma}^{-1}$ leaves invariant the lineal elements of two conjugate turbines T_0 and $\bar{T}_0$. If $\Sigma\neq\bar{\Sigma}$, all the circles which pass through the lineal elements of T_0 and $\bar{T}_0$ are converted into circles by Σ. Certain subgroups of G_{15}, which include the Moebius and Laguerre inversive groups, are also studied. The representation R leads to the concept of conjugate differential equations and also to a new theory of algebraic differential equations invariant under the Lie group L_{10}. (Received November 16, 1953.)

296. E. F. Gillette: *A generalization of the Four Vertex Theorem for ovals on a surface.* I.

Given in the plane a closed convex curve without double points along which the curvature is a continuous function of the arc length. Then the classical Four Vertex Theorem states that the curvature assumes at least four extrema. Replacing the concept of curvature by that of geodesic curvature and giving a suitable definition of convexity, the Four Vertex Theorem can be generalized for ovals on surfaces of nonpositive Gaussian curvature. The proof is based on an extension of the following lemma of E. Schmidt: Given in the plane an arc C with continuous curvature and of length L which together with the chord joining its end points bounds a convex domain. Let $\bar{C}$ be a second arc (not necessarily plane) of the same length L. If for every value of the arc length s as parameter the curvature $k(s)$ of C is greater than or equal to the curvature $\bar{k}(s)$ of $\bar{C}$, then the length d of the chord of C is smaller than or equal to the length $\bar{d}$ of the chord joining the end points of $\bar{C}$. ($\bar{d}=d$ only if $\bar{C}$ is congruent to C.) The Four Vertex Theorem for ovals on surfaces of positive Gaussian curvature will be treated in a subsequent paper. (Received November 14, 1953.)

297. L. W. Green: *Geodesic instability.*

Let M be a complete, simply-connected, n-dimensional Riemannian manifold of class C^r ($r \geq 4$) with the property that no geodesic of M contains two mutually conjugate points. Letting $K(\beta, P)$ denote the curvature at P of the geodesic plane element tangent to the bivector β, assume further that $K(\beta, P) \geq -A^2$ for some constant A and all P, β in M. The principal result is: any two geodesic rays of M with the same initial point must diverge (i.e., the distance from a point on one to the other geodesic approaches infinity with its distance from the point of intersection). This is an extension of the author's previous result for surfaces and has several new consequences. In particular, if R is a Riemannian n-torus such that no geodesic contains a point focal to a transverse geodesic hyperplane, then R is locally Euclidean. It is also proved that the geodesic flow is topologically transitive on a wide class of n-dimensional manifolds (not necessarily compact) whose universal covering manifolds satisfy the hypotheses on M. (Received November 10, 1953.)

298. A. J. Hoffman (p), Morris Newman, E. G. Straus, and Olga Taussky: *The number of absolute points of a correlation.*

Let ρ be a correlation of a finite projective plane π with $n+1$ points on a line. Let the collineation ρ^2, regarded as a permutation of the points of π, be expressible as a product of disjunct cycles of length $d_1, d_2, \cdots, d_r$. The principal results are: (1) bounds on the number of absolute points of ρ in terms of arithmetic properties of n and $d_1, d_2, \cdots, d_r$; (2) relations between the number of absolute points of ρ and the number of absolute points of various odd powers of ρ. These results, obtained from a study of irreducible factors of the characteristic polynomial of a suitably chosen incidence matrix for π, improve a portion of the work of R. Baer and R. W. Ball concerned with the same problems. The present method applies also to correlations of symmetric group divisible designs, a class of combinatorial configurations including the finite projective planes. (Received November 14, 1953.)

Logic and Foundations

299. M. D. Davis: *Pseudocomplements of recursively enumerable sets.* Preliminary report.

Let $\{n\} = [x \mid (Ey)T(n, x, y)]$ where T is Kleene's enumeration predicate, so that

S is recursively enumerable if and only if there exists n_0 such that $S=\{n_0\}$. Recursive functions $\vee$, $\wedge$ may be defined such that $\{m\vee n\}=\{m\}\cup\{n\}$; $\{m\wedge n\}=\{m\}\cap\{n\}$. Let $\mathcal{L}$ be an ω-consistent logic adequate for recursive arithmetic. Then, as is well known, $\{n\}=[x|\vdash_{\mathcal{L}}x\in\{n\}.]$ However, if $\{n\}$ is not recursive, $\{\bar{n}\}\neq[x|\vdash_{\mathcal{L}}x\notin\{n\}]$, this being the substance of Gödel's theorem. Let a recursive function $N(n)$ be determined such that $\{N(n)\}=[x|\vdash_{\mathcal{L}}x\notin\{n\}]$. $\{N(n)\}$ is called a *pseudo-complement* of $\{n\}$. Then: (1) $\{n\wedge N(n)\}=\emptyset$ ($\emptyset$, the empty set). (2) $\{N(m\vee n)\}=\{N(m)\wedge N(n)\}$. (3) $\{N(m\wedge n)\}\supset\{N(m)\vee N(n)\}$. (4) $\{N(N(n))\}=\emptyset$. (5) $\{m\wedge N(m\wedge n)\}\subset\{N(n)\}$. (6) If R is recursive, there exists n such that $\{n\}=R$, $\{N(n)\}=\bar{R}$. A consequence of (3), (4), and (5) is that $\{N(n)\}\subset\{N(n\wedge N(n))\}\subset\{\bar{n}\}$, so that $\emptyset$ has arbitrarily small pseudocomplements. Also, it follows from (1), (2), and (4) that $\{N(n\vee N(n))\}=\emptyset$, i.e. the direct attempt to employ N itself to explore the "undecidable" region fails. (Received November 13, 1953.)

300*t*. W. J. Feeney: *Certain unsolvable problems in the theory of cancellation semi-groups.*

The nonexistence of algorithms for determining whether S is trivial, isomorphic to S', abelian, finite, a group, imbeddable in a group, or has a solvable word problem, where S and S' are arbitrary finitely presented cancellation semi-groups, follows from a theorem concerning invariant (preserved under isomorphism) properties of cancellation semi-groups. Finitely presented cancellation semi-groups in which the cancellation law is superfluous will be called *special*. *Theorem*: Let P be an invariant property of cancellation semi-groups such that (1) there exists a special cancellation semi-group S_0 which cannot be imbedded in any cancellation semi-group having the property P, and (2) there exists a special cancellation semi-group which has the property P. Then there is no algorithm for deciding in a finite number of steps whether an arbitrary finitely presented cancellation semi-group possesses the property P. In proof, let S_1 be Turing's special cancellation semi-group with unsolvable word problem. For each two words G, H of S_1 form $S_{G,H}$ by adjoining to the free product of S_0, S_1, and S_2 generators a_x, b_x, c_x, d_x and relations (1) $xa_xb_xa_xGc_xd_xc_x=b_xa_xb_xHd_xc_xd_x$, (2) $b_xa_xb_xGd_xc_xd_x=a_xb_xa_xHc_xd_xc_x$, (3) $a_xa_xb_x^2a_xGc_xd_x^2c_x=b_xa_x^2b_xHd_xc_x^2d_x$, (4) $b_xa_x^2b_xGd_xc_x^2d_x=a_xb_x^2a_xHc_xd_x^2c_x$, (5) $b_xb_xa_x^3b_xGd_xc_x^3d_x=a_xb_x^3a_xHc_xd_x^3c_x$, (6) $a_xb_x^3a_xGc_xd_x^3c_x=b_xa_xb_x^3Hd_xc_x^3d_x$, (7) $a_xb_x^4a_xGc_xd_x^4c_xc_x=b_xa_x^4b_xHd_xc_x^4d_x$, (8) $b_xa_x^4b_xGd_xc_x^4d_x=a_xb_x^4a_xHc_xd_x^4c_x$, (9) $b_xa_x^5b_xGd_xc_x^5d_xd_x=a_xb_x^5a_xHc_xd_x^5c_x$, (10) $a_xb_x^5a_xGc_xd_x^5c_x=b_xa_x^5b_xHd_xc_x^5d_x$, for each generator x of S_0 and S_1. It can then be shown that $S_{G,H}$ possesses the property P if and only if $G=H$ in S_1. (Received January 7, 1954.)

301*t*. Hao Wang: *Undecidable sentences generated by semantic paradoxes.*

If the notion of truth or the relation of designation of a system is definable in the system itself, we get some semantic paradoxes in the system. There are systems S which satisfy the following conditions: (1) the notions of truth and designation of current number theory Z can be defined in S; (2) S is omega-consistent; (3) S contains Z as a part. For example, the von Neumann-Bernays set theory is demonstrably one such system if we delete the axiom of infinity. These systems S have models in Z (or more accurately in omega-consistent extensions of Z). In particular, the definitions of truth and designation of Z in S also have models in Z. Using these facts, the author gets undecidable sentences in S which correspond to the paradoxes of Richard and Epimenides. (Received October 16, 1953.)

STATISTICS AND PROBABILITY

302. K. L. Chung: *On the Robbins-Monro procedure.*

Asymptotic properties are established for the Robbins-Monro procedure (Ann. Math. Statist. vol. 22 (1951) pp. 400–407) of stochastically solving the equation $M(x)=\alpha$. Two disjoint cases are treated in detail. The first may be called the "bounded" case, in which the assumptions are similar to the second case of Robbins and Monro. The second may be called the "quasi-linear" case which restricts $M(x)$ to lie between two straight lines with finite and nonzero slopes but postulates only the boundedness of the moments of $Y(x)-M(x)$ (see loc. cit. for notations). In both cases it is shown how to choose the sequence $\{a_n\}$ in order to establish the correct order of magnitude of the moments of $x_n-\theta$. Asymptotic normality of $a_n^{-1/2}(x_n-\theta)$ is proved in both cases under a further assumption. The case of a linear $M(x)$ is discussed to point up other possibilities. The statistical significance of our results is sketched. (Received September 17, 1953.)

303*t*. K. L. Chung and Paul Erdös: *Probability limit theorems assuming only the first moment.* II.

This is a sequel to *Probability limit theorems assuming only the first moment.* I, Memoirs of the American Mathematical Society, no. 6. The non-lattice case is taken up here. In general additional device is needed since no superfluous assumptions are made; in particular, there is no discrimination against a singular distribution. (Received, November 9, 1953.)

TOPOLOGY

304*t*. J. D. Baum: *A symbolic transformation group.*

A symbolic transformation group, an extension of the Morse minimal set [M. Morse and G. A. Hedlund, *Symbolic dynamics*, Amer. J. Math. vol. 60 (1938) pp. 815–864], is defined over the phase group of lattice points of the plane. It is shown that this transformation group is a minimal set, the orbit closure of a nonperiodic, almost periodic, point, which decomposes into a set of pairwise separated orbits and a set which is the union of pairs of points asymptotic relative to a half-space of the phase group. The asymptotic relations among eight selected points are completely investigated. (Received October 29, 1953.)

305*t*. J. D. Baum: *Asymptoticity in topological dynamics.*

If (X, T) is a transformation group [definitions as in Gottschalk and Hedlund, *The dynamics of transformation groups*, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 348–359], X a uniform space, and P a replete semigroup in T, $P\neq T$, then $x, y\in X$, $x\neq y$, are said to be P-asymptotic if for each index γ of X there exists $q\in P$ such that $xqp\in yqp\gamma$ for all $p\in P$. If A is closed invariant in X, $x\notin A$, x is said to be P-asymptotic to A if for each neighborhood U of A there exists $q\in P$ such that $xqP\subset U$. These definitions free the notion of asymptoticity from restriction to dynamical systems and iterated homeomorphisms. The P-limit set of a point $x\in X$, P_x, is a generalization of α- and ω-limit sets by the above authors and is used extensively to prove the following (where T is separable generative): (1) P_x is the smallest set to which x can be P-asymptotic, (2) x is P-asymptotic to P_x if and only if $x\notin P_x$, (3) if $T=I^n$, I the integers, and X is compact metric and dense-in-itself, then there exist P-asymptotic points, where P is a half space of T. Two points are totally asymptotic provided they are P-asymptotic for every replete semigroup P in T, $P \neq T$; x and y, $x \neq y$, are said to be compactly asymptotic if for each index γ of X there exists K compact in T, such that $t \notin K$ implies $xt \in yt\gamma$. It is shown that (1) if T is separable generative then total asymptoticity implies compact asymptoticity and (2) if T is abelian compact asymptoticity implies total asymptoticity. (Received October 29, 1953.)

306. R. H. Bing: *Decompositions of Euclidean spaces.*

Suppose G is a monotone upper semicontinuous decomposition of S^2 such that G with respect to its elements as points is topologically equivalent to S^2. It is shown that there is a deformation $f(x, t)$ ($x \in S^2$, $0 \leq t \leq 1$) of S^2 onto itself shrinking elements of G onto distinct points such that for $0 < a < 1$, $f(x, t)$ $(0 \leq t \leq a)$ is an isotopy. This theorem does not generalize to S^3 because there is a monotone upper semicontinuous decomposition of S^3 onto itself such that two elements of the decomposition are linking circles. In fact if W is a finite collection of mutually exclusive compact continua in E^3 such that no one of them separates E^3, there is a monotone upper semicontinuous decomposition of E^3 onto itself that has the continua of W as elements. Suppose G' is a monotone upper semicontinuous decomposition of a solid cube C such that each point on the interior of C is an element of G' and the other elements of G' constitute in the decomposition space a set topologically equivalent to S^2 (or a dendron). Then it is shown that the decomposition space G' is topologically equivalent to C (or S^3). (Received November 14, 1953.)

307. Raoul Bott: *On manifolds all of whose geodesics are closed.*

The following propositions are proved about simply-connected manifolds M with the above property (satisfying also certain regularity conditions concerning the way in which the geodesics are closed). I. The integral cohomology ring of M is isomorphic to a truncated polynomial ring (i.e., apart from the identity, $H(M)$ is generated by a single class $\chi(M)$ subject to only one relation which is of the type $\chi^n(M) = 0$, $n > 1$). II. If $\sigma(M)$ is the associated space of closed geodesics on M (obtained by fibering the tangent bundle of M by the closed-geodesics) the Poincaré series of the circular connectivities of M (see Marston Morse, *Calculus of variations in the large*, Amer. Math. Soc. Colloquium Publications, vol. 18), given by $P(M; s) = P\{\sigma(M); s\} s^t (1 - s^{st})^{-1}$ where $P\{\sigma(M); s\}$ is the mod 2 Poincaré polynomial of $\sigma(M)$; $1 + s = \dim \chi(M)$; and $t + 1 = \dim (M)$. This result interprets and generalizes formulae obtained by Morse for the circular connectivities of the n-sphere. (Received November 14, 1953.)

308. J. W. Brace: *The closure of weakly compact transformations.*

The set of transformations (weakly compact transformations) taking the bounded sets of a linear topological space E into the weakly compact sets of a linear, locally convex, topological space F is closed in the topology of uniform convergence on the bounded sets of E, if F is complete in the weak topology, the original topology, or any topology coarser than the original topology and finer than the weak topology. In considering a net of weakly compact transformations converging in the topology of uniform convergence on the bounded sets of E, and a fixed bounded set in E, it is observed that the relatively weakly compact images of the bounded set form what can be called a "Cauchy net of sets" in the original topology on F. When F has the desired completeness property the "Cauchy net of sets" converges in the weak topology to a relatively weakly compact set. This limit set is the image of the fixed bounded set under the limit transformation; thus establishing the above theorem. (Received November 10, 1953.)

309. C. E. Burgess: *Some theorems on n-homogeneous continua.*

A point set M is said to be ***n*-homogeneous** if for any n points $x_1, x_2, \cdots, x_n$ of M and any n points $y_1, y_2, \cdots, y_n$ of M there is homeomorphism of M onto itself that carries $x_1+x_2+\cdots+x_n$ into $y_1+y_2+\cdots+y_n$. A point set M is said to be ***nearly n*-homogeneous** if for any n points $x_1, x_2, \cdots, x_n$ of M and any n open subsets $D_1, D_2, \cdots, D_n$ of M there exist (1) n points $y_1, y_2, \cdots, y_n$ such that $y_i \subset D_i$ $(1 \leqq i \leqq n)$ and (2) a homeomorphism of M onto itself that carries $x_1+x_2+\cdots+x_n$ into $y_1+y_2+\cdots+y_n$. A nearly 1-homogeneous set is said to be ***nearly homogeneous***. The following theorems are proved. (1) If the compact metric continuum M is nearly homogeneous and is irreducible about some finite set, then M is indecomposable. (2) If the bounded plane continuum M is nearly homogeneous and has more than two but only a finite number of complementary domains, then M is indecomposable. (3) If $n>1$ and the bounded plane continuum M is nearly n-homogeneous and is not a continuous curve, then M is indecomposable. (4) If $n>1$ and the bounded plane continuum M is n-homogeneous, then M is a simple closed curve. (Received November 2, 1953.)

310. J. H. Case: *A note on the Whitehead isomorphism theorem.*

The main theorem concerns two arcwise connected topological spaces X and Y, a continuous map f from X to Y, and an arbitrary but fixed positive integer M. It is shown that if f induces onto isomorphisms of the homotopy groups of X and Y for each dimension $n \leqq M$, then f induces onto isomorphisms of the singular homology and cohomology groups of X and Y for each dimension $n \leqq M$ and any coefficient group. The concept of minimal subcomplex which is due to S. Eilenberg and J. A. Zilber and the concept of admissable subcomplex which is due to S. T. Hu are the main tools used in the proof. (Received November 13, 1953.)

311*t*. G. A. Dirac and Seymour Schuster: *A theorem of Kuratowski.*

Theorem: A finite graph is not plane if and only if it contains a subgraph homeomorphic to the complete 5-graph or a subgraph homeomorphic to the Thomson graph (the graph whose nodes are the vertices of a hexagon and whose edges are the sides of the hexagon, plus the diagonals joining the opposite vertices). This theorem was proved by Kuratowski (Fund. Math. vol. 16, pp. 271–283). In the present paper, a much simpler proof is arrived at by induction on the number of edges of the graphs embeddable in the plane. The possible extension of the theorem to countable infinite graphs is discussed. Two definitions suggest themselves: (1) An infinite graph may be called plane if it can be embedded in the plane. (2) An infinite graph may be called plane if it can be embedded in the plane in such a way that the nodes do not have a limit point. It is shown that Kuratowski's Theorem holds if definition (1) is adopted. However, a simple counterexample is given to show that the theorem does not hold when the second definition is adopted. (Received November 14, 1953.)

312. Samuel Eilenberg and Saunders MacLane (p): *Generators for the torsion product.*

For two abelian groups A and B let Tor (A, B) be the group with generators all symbols $\tau_m(a, b)$ for integers m and elements $a \in A$, $b \in B$ with $ma = mb = 0$. Subject these generators to the relations expressing linearity in each argument, and to the relations $\tau_{mn}(a, b) = \tau_m(a, nb)$ and $\tau_{mn}(a, b) = \tau_m(na, b)$, to hold whenever both sides are defined. Then if $0 \to A \to B \to C \to 0$ is an exact sequence and G any abelian group, 0

$\to \operatorname{Tor}(A, G) \to \operatorname{Tor}(B, G) \to \operatorname{Tor}(C, G) \to A \otimes G \to B \otimes G \to C \otimes G \to 0$ is exact. In particular if $C = F/R$ is any representation of C as a factor group of a free group F, $\operatorname{Tor}(C, G) = \text{kernel}\ (R \otimes G \to F \otimes G)$. This shows that the definition of Tor by generators and relations agree with the definition given in terms of F/R by Eilenberg-Cartan (*Satellites des foncteurs de modules*, to appear). Other properties of the torsion product, notably the Künneth relation for the homology group of torsion-free chain complexes may readily be derived from the description by generators and relations. (Received November 14, 1953.)

313*t*. D. O. Ellis: *Some saddle points in the cross-topolattice.*

Let S be an infinite set and let L be the topolattice of S (lattice of all T_1 Kuratowski topologies on S, ordering by inverse inclusion of classes of open sets). Let L^* be L dually ordered and let L^{**} be the cardinal product $L \otimes L^*$. One shows that if f is a permutation of S (biuniform mapping of S onto itself) and if $(a, b) \in L^{**}$ has the property that when we take 'a' as domain topology and 'b' as range topology, f is both open and continuous (homeomorphic), then (a, b) is maximal in L^{**} with respect to making f continuous and is minimal with respect to making f open. A general discussion of lattice saddle points precedes this result. The methods are elementary. (Received October 15, 1953.)

314*t*. G. W. Ford and G. E. Uhlenbeck: *On the asymptotic number of pure and mixed Husimi trees.*

A Husimi tree is a connected graph in which no line lies on more than one cycle. A pure Husimi tree has all cycles of the same length; otherwise, it is called mixed. The number of Husimi trees has been counted by Harary and Uhlenbeck, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) p. 315, and by Norman (see abstract below). Using the methods of Otter, Ann. of Math. vol. 49 (1948) p. 583, the authors have determined the asymptotic numbers of pure (rooted and free) Husimi trees consisting of n m-gons. The form of the asymptotic behavior is the same for all m, but the radius of convergence of the counting series decreases with m as $1/(m-1)$. For mixed Husimi trees of arbitrary cyclic composition, the counting series in terms of number of points has the radius of convergence $0.2221 \cdots$, and the distribution of the points over the different cycles has been determined. It is a monotonic decreasing function of the cycle length. (Received November 13, 1953.)

315. L. R. Ford, Jr.: *Coset spaces and homeomorphism groups.*

Let G be a homeomorphism group of X and C_x that subgroup having x as a fixed point. It is shown that if X is completely regular, G may be made into a topological group under a uniform convergence notion. If G/C_x is homeomorphic to X one calls the G topology "reasonable." Define strong local homogeneity (SLH) by: For every x, $U(x)$ there exists $V(x) \subset U(x)$ such that for every $z \in V$ there is a homeomorphism h of X with $h(x) = z$ and $h = I$ on $C(V)$. It is proved that a completely regular, homogeneous, SLH space has a reasonable full homeomorphism group. It is shown that locally Euclidean spaces, homogeneous 0-dimensional spaces, and normed linear spaces are SLH. In particular all manifolds are coset spaces of some topological group. (Received November 10, 1953.)

316. W. C. Fox: *Critical points of real functions defined on two-manifolds.* Preliminary report.

Let M be the interior of a two-manifold with boundary J. A real function f defined on $M+J$ is Peano-interior if it is continuous, interior on M, and every point-inverse is locally connected in M. The class of Peano-interior functions on $M+J$ properly includes the pseudo-harmonic functions. Conditions on the boundary values are given which imply that f may be extended to be Peano-interior on a larger manifold M^* and to have certain canonical properties on the new boundary. If a Peano-interior function is canonical, then there is a relation between the sum S^* of the multiplicities of the critical points of f on M^* and the first Betti number of M^*. Determination of whether or not a boundary point is a critical point is not always possible, for there is a Peano-interior function possessing two different canonical extensions for which a certain boundary point is regular under one extension and is a critical point under the other. (Received November 13, 1953.)

317. J. S. Griffin, Jr.: *A note on the cohomology theory of manifolds.*

In any topological space X the Alexander-Wallace-Spanier cohomology groups may be defined using only $(p+1)$-tuples which are small relative to any given open cover $\mathcal{U}$. But if X is a differentiable manifold with Riemannian metric g_{ij} and $\mathcal{U}$ is chosen to consist exclusively of sets convex in the sense that any two points of a member U of $\mathcal{U}$ may be joined by a unique geodesic of shortest length which lies entirely within U, then for any $(p+1)$-tuple $(x_0, \cdots, x_p)$ which is small relative to $\mathcal{U}$ there is a differentiable singular simplex $\sigma_{(x_0, \cdots x_p)}: \Delta_p \rightarrow X$ such that: (1) if v_i is the ith vertex of Δ_p, then $\sigma_{(x_0 \cdots x_p)}(v_i) = x_i$, (2) if e_i is the barycentric map of Δ_p onto the ith face of Δ_{p+1} which preserves the order of the vertices, then $\sigma_{(x_0, \cdots x_{p+1})}e_i = \sigma_{(x_0 \cdots \hat{x}_i, \cdots x_{p+1})}$. These considerations lead in a natural way to homomorphisms on the singular cochain groups and the groups of differential forms to the Alexander-Wallace-Spanier cochain groups; if the manifold is taken to be compact, then using the Leray uniqueness theory these homomorphisms may be shown to induce isomorphisms on the cohomology groups. (Received September 8, 1953.)

318. Mary-Elizabeth Hamstrom: *A characterization of plane continuous curves which are filled up by continuous collections of continuous curves.*

Let M be a compact plane continuous curve. R. D. Anderson has shown (*Continuous collections of continuous curves in the plane*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 647–657) that if M is the sum of the elements of a continuous collection of simple closed curves, then M is an annulus. The author has shown (Proc. Amer. Math. Soc. vol. 4 (1953) pp. 240–243) that if M is the sum of the elements of a continuous collection of arcs which is an arc with respect to its elements, then M is a closed disc. The principal theorem of the present paper establishes that M is the sum of the elements of a continuous collection of nondegenerate continuous curves if and only if every boundary of a complementary domain of M is a simple closed curve and the collection of these boundaries satisfies certain conditions, which are too lengthy to be presented here. (Received November 5, 1953.)

319. Melvin Henriksen: *A generalization of a theorem of Hewitt.*

Let $C(X, R)$ denote the ring of all real-valued continuous functions on a completely regular space X, and let M be any maximal ideal of $C(X, R)$; then the residue class field $C(X, R)/M$ is an *ordered* field containing the real field R (Hewitt, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 45–99, Theorem 41). $C(X, R)/M$ is called a *hyper-real* field if it contains R properly. Theorem: If X is any completely regular space, and if $C(X, R)/M$ is hyper-real, then $C(X, R)/M$ contains no countable cofinal subset. This generalizes a theorem of Hewitt, who had obtained the result under the additional restriction that X is both locally compact and σ-compact (loc. cit., Theorem 43). (Received November 14, 1953.)

320. F. E. P. Hirzebruch: *A generalisation of Todd's arithmetic genus*. Preliminary report.

For nonsingular algebraic varieties V_n the integer $\chi^p(V_n)$ is defined by $\sum_{q=0}^{n} (-1)^q h^{p,q}$. Here $h^{p,q}$ denotes the dimension of the linear space of harmonic forms of type (p, q) on V_n. It is conjectured (Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 951–956) that $\chi^0(V_n)$ equals the Todd genus of V_n which is a polynomial $T_n(c_1, c_2, \cdots, c_n)$ of weight n in the Chern classes c_i, where $c_i \in H^{2i}(V_n, Z)$. In this paper polynomials $T_{n,p}(c_1, c_2, \cdots, c_n)$ are introduced and it is conjectured that $T_{n,p}(c_1, c_2, \cdots, c_n) = \chi^p(V_n)$. Using a result of Kodaira-Spencer, this equation is proved for every complete intersection W_n of hypersurfaces in some projective space. It provides a method for calculating all the $h^{p,q}$ of W_n. (Lefschetz's theorem implies that $h^{p,q} = \delta_q^p$ for $p+q \neq n$.) The polynomials $T_{n,p}$ make sense for a compact almost-complex manifold M_n. It is proved that $2^{n-1}T_{n,p}$ is an integer for every M_n. This was proved earlier for $p=0$ (loc. cit.). For $p=1$ one gets the corollary: If in an almost-complex manifold M_n the Chern classes c_i vanish for $i \leq [n/2]$, then $2^{n-2}c_n$ is divisible by $(n-1)!$. This implies the theorem of Borel-Serre that the spheres S^{2n} are not almost-complex for $n \geq 4$. Remark: $\sum_{p=0}^{n} (-1)^p \chi^p(V_n)$ is the Euler-Poincaré characteristic of V_n and $\sum_{p=0}^{n} \chi^p(V_n)$ is the index $\tau(V_n)$ (Hodge). The polynomials $T_{n,p}$ have the corresponding properties. (Received November 12, 1953.)

321. F. B. Jones: *On a property related to separability in metric spaces.*

An example is given of a metric space in which the following proposition is *false*: If no separable subset of a metric space S is uncountable, then S is the sum of countably many discrete sets. The example possesses other properties of a related nature. (Received November 14, 1953.)

322. J. W. Keesee: *A separation theorem for manifolds.*

Let (X, A) be a compact topological pair where X is a connected $(n+1)$-dimensional Euclidean manifold. Let G be the additive group of integers if X is orientable, otherwise let G be cyclic of order two. The inclusion map $i: A \rightarrow X$ induces a homomorphism $i^*: H^n(X) \rightarrow H^n(A)$ where $H^n(X)$ denotes the n-dimensional Čech cohomology group of X with coefficients in G. Let N be the number (finite or countable) of components of $X-A$. Let $\sum_{j=1}^{N-1} G_j$ represent the weak sum of $N-1$ groups each isomorphic to G. Theorem: *The quotient group $H^n(A)/i^*(H^n(X))$ is isomorphic to* $\sum_{j=1}^{N-1} G_j$. (Received November 12, 1953.)

323t. A. B. Lehman: *On pseudo-linear quadruples in a metric lattice*. Preliminary report.

Four points in a metric space form a pseudo-linear quadruple if and only if the determinant of their distances vanishes. A necessary and sufficient condition that four elements of a metric lattice form a pseudo-linear quadruple, $Q(a, b; c, d)$, is that (1) $a \wedge b = c \wedge d$ and $a \vee b = c \vee d$, and (2) a, b, c, d distribute among themselves. The sublattice generated by a pseudo-linear quadruple is isomorphic with 2^2, 2^3, or 2^4

depending on the comparability of the elements concerned. The transitivity: $Q(a, c; b, d)$ and $Q(c, e; d, f)$ imply $Q(a, e; b, f)$ is a criterion for distributivity in metric lattices. (2) implies (1) if and only if the lattice has no four-element chain. Lattices in which (1) implies (2) are called *quasi-distributive*. Any sublattice generated by three elements is quasi-distributive; accordingly, quasi-distributivity is determined by sublattices generated by four elements. Numerous "forbidden pictures" for quasi-distributive lattices exist, the simplest being the multiple projective root. Characterization and classification of these is possible. Generation of sets metrically approximating Condition (1) is considered. (Received November 24, 1953.)

324. P. S. Mostert: *Sections in principal fibre spaces.*

Let $\mathfrak{B} = \{X, B, p, G\}$ be a principal fibre space (Serre, C. R. Acad. Sci. Paris vol. 229 (1949) pp. 1295–1297) such that X, B, and G are inverse limits of $\{X_\alpha; \pi_\alpha^\beta\}$, $\{B_\alpha; \bar{\pi}_\alpha^\beta\}$, $\{G_\alpha; \rho_\alpha^\beta\}$, respectively, $\alpha, \beta \in \Lambda$, $\alpha < \beta$, such that $\mathfrak{B}_\alpha = \{X_\alpha, B_\alpha, p_\alpha, G_\alpha\}$ is a principal fibre space, and $p_\alpha \pi_\alpha^\beta = \bar{\pi}_\alpha^\beta p_\beta$, $\pi_\alpha^\beta(g_\beta x_\beta) = \rho_\alpha^\beta(g_\beta)\pi_\alpha^\beta(x_\beta)$, where all mappings are open and continuous. We shall say that $\mathfrak{B}$ is the *inverse limit* of $\{\mathfrak{B}_\alpha; \Lambda\}$. The system $\{\mathfrak{B}_\alpha; \Lambda\}$ is said to satisfy the property (P) if (i) Λ has the finite chain property (i.e., any chain between α and β has at most a finite number of elements and Λ has a least element 1), and (ii) there exists an open set $U = P_\alpha U_\alpha \cap B$, where $U_\alpha = (\pi_1^\alpha)^{-1}(U_1)$, U_1 an open set in B_1 such that, for a fixed point $x = \{x_\alpha\} \in X$, the sets F_α of cross sections of U_α mapping a point of U_α into x_α have the property that, if $\alpha < \beta$ and $f_\alpha \in F_\alpha$, then there exists $f_\beta \in F_\beta$ such that $\pi_\alpha^\beta f_\beta = f_\alpha \bar{\pi}_\alpha^\beta$ on U_β, and conversely; moreover F_1 is nonempty. The following then holds: (1) *If* $\mathfrak{B}$ *is an inverse limit of* $\{\mathfrak{B}_\alpha; \Lambda\}$ *satisfying* (P), *then there is a cross section defined on* U. This then gives: (2) *If* X *is a locally compact group,* G *a closed subgroup such that* $B = X/G$ *is finite-dimensional, then* G *has a local cross section*; (3) *If* $\mathfrak{B}$ *is a principal fibre space with locally compact finite-dimensional structural group, and* B *is locally arcwise connected and locally simply connected, then local cross sections exist for* $\mathfrak{B}$. Other related results are obtained. (Received November 9, 1953.)

325. R. Z. Norman: *On the number of linear graphs with given blocks.*

The powerful methods of Pólya are employed to solve some problems of counting various kinds of graphs. These results are used by Ford and Uhlenbeck (see abstract above). A pure Husimi tree is a connected graph in which every line lies in exactly one cycle of length n. A generating function is developed for counting pure Husimi trees using the known function for rooted pure Husimi trees (Harary and Uhlenbeck, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) p. 315). This result is extended to count arbitrary Husimi trees. A block of a graph is defined as a maximal connected subgraph with no articulation points. A connected graph can, of course, itself be a block. Let B be a collection of blocks. Rooted connected graphs whose blocks are isomorphic to blocks of B are counted, and from this the counting of analogous unrooted graphs is accomplished, using the group of the block and a certain subgroup derived from it. Similar results are obtained for directed graphs, for graphs of strength s (see Harary, Bull. Amer. Math. Soc. Abstract 58-6-717), and for simplicial complexes. The restriction to connected graphs is easily eliminated. (Received November 13, 1953.)

326t. Joel Pitcairn: *A discrete analogue of the Weierstrass-Stone theorem.*

The following analogue of the Weierstrass-Stone theorem is proved: Let F be a field with discrete topology and let H be a set of continuous functions from a compact

Hausdorff space X to F which separates the points of X. Then every continuous function from X to F is a polynomial with respect to H, with coefficients in F. A number of applications are given, among which is the following: An algebra over $GF(p^n)$ is called a *p^n-algebra* in case every x in it satisfies $x^q = x$, where $q = p^n$. Let α be an infinite cardinal. Then there exists a unique compact boolean space X and a point s in X such that for every p, n, the free p^n-algebra with α generators is the algebra of all continuous functions from X to $GL(p^n)$ which vanish at s. If the phrase "which vanish at s" is omitted one gets the free p^n-algebra with unit with α generators. (Received October 19, 1953.)

327t. Joel Pitcairn: *The existence of Banach limits.*

A bounded sequence $x = (x_n \mid n \in N)$ is a function from the set N of positive integers into a compact subset of the reals R. Let X be the algebra over R of all bounded sequences. Let $\mathcal{A}$ be an ultrafilter on N containing the finite-complement filter. Clearly the functional $\Lambda = (\lim (\mathcal{A})x \mid x \in X)$ is a homomorphism of X onto R, and $\liminf x \leq x\Lambda \leq \limsup x$ $(x \in X)$. Define $x\Gamma = ((\sum_{i=1}^{n} x_i)/n \mid n \in N)$ $(x \in X)$. Clearly Γ is a linear transformation of X into itself, and the functional $B = \Gamma\Lambda$ is linear and has the property that $\liminf x \leq xB \leq \limsup x$ $(x \in X)$. Let $x = (x_n \mid n \in N)$ and $y = (x_{n+1} \mid n \in N)$. Then $(y-x)\Gamma = ((x_{n+1}-x_1)/n \mid n \in N)$, and so $0 = \lim (y-x)\Gamma = (y-x)\Gamma\Lambda = yB - xB$. Thus B is a Banach limit. (Received October 19, 1953.)

328t. R. L. Plunkett: *Some implications of semi-1-connectedness.*

A topological space is semi-n-connected at $x \in X$ provided there exists an open set U containing x such that every Čech n-cycle on a compact subset of U bounds on X. The following theorems are proved, where X is a separable metric continuum: (1) If X is semi-1-connected at $x \in X$, then there exists an open set V containing x such that, if Q is a region with $\overline{Q} \subset V$, then no continuum in $V-Q$ intersects two components of $\overline{Q}-Q$. (2) If X is locally connected and is semi-1-connected at a nonlocal separating point x, then X is locally peripherally connected at x. (3) If X is locally connected and is semi-1-connected at a nonlocal separating point x, then there exist arbitrarily small open sets V containing x such that, if Q is any region contained in V, then there exists a component C of $\overline{Q}-Q$ such that the component of $X-C$ containing Q is contained in V. (4) If T is a compact totally disconnected set in a locally connected X such that no point of T is a local separating point, if X is semi-1-connected at each $x \in T$, and if d is a positive number, then there exists a finite open covering $\{Y_1, Y_2, \cdots, Y_k\}$ of T such that (1) diam $(Y_i) < d$, for each i, (2) $\overline{Y}_i - Y_i$ is connected, for each i, and (3) $Y_i \cdot Y_j = 0$, if $i \neq j$. (Received November 13, 1953.)

329t. B. A. Rattray: *An antipodal point—orthogonal point theorem.*

It is shown that, if T is any continuous mapping of the metric n-sphere into itself which carries antipodal points into antipodal points, then there is some set of $(n+1)$ mutually orthogonal points whose images under T are also mutually orthogonal. The main tool used in the proof is the theory of intersection numbers. (Received November 16, 1953.)

330. J. C. Rogers: *Cross sections in fibre bundles.*

Let $\mathfrak{B}$ be a fibre bundle with base space K, a finite cellular complex, and connected fibre Y, which need not have a vanishing fundamental group nor be simple in any dimension. Then, necessary and sufficient conditions are given for the extension of a cross section (defined on a subcomplex, L) over $K^1 \cup L$. Homotopy on $K^1 \cup L$ divides cross sections into homotopy equivalence classes which are in 1-1 correspondence with the "admissible" homomorphisms, $\phi:\pi_1(K, x_0)\to\pi_1(B, b_0)$, B being the bundle space. Higher dimension deformation and extension theorems are developed, including the "one stage" homotopy classification theorem: for any $q\geq 2$, the homotopy equivalence classes of cross sections defined on $K^q\cup L$, extendable over $K^{q+1}\cup L$, and homotopic on $K^{q-1}\cup L$ are in 1-1 correspondence with a certain factor group of $H^q(K, L; \mathcal{B}^q(f))$, where $\mathcal{B}^q(f)$ is a suitably defined coefficient bundle. Applications are given, including the complete classification of the fields of real nonoriented line elements over P^n, projective n-space, $n>2$. (Received November 3, 1953.)

331t. J. P. Roth: *An investigation in Morse theory.*

An axiomatization of Morse critical point theory is given in terms of a triple (S, f, H) consisting of a Hausdorff space S, an arbitrary real-valued mapping f of S, and an arbitrary homology theory H. The principal tools are the notion of filter and the Eilenberg-Steenrod formulation of homology theory. This approach generalizes previous presentations and serves to clarify the relations between various levels of and approaches to Morse theory. The Morse relations, in the form of group isomorphisms, are derived without any finiteness assumptions (e.g. finite number of critical points). Let $S_c=\{x|f(x)\leq c\}$; if S_c is compact for each c, H is a homology theory continuous "on S and f" and S is f-reducible, then all the axioms are satisfied so that the Morse relations hold, and there exists a critical point at each critical value. Applications are made to minimal surface theory and the theory of optimum guided missile trajectories. (Received November 18, 1953.)

332. Mary E. Rudin: *Connected subsets of a plane set.*

In a paper entitled *Some remarks on connected sets* (Bull. Amer. Math. Soc. vol. 50 (1944) p. 445) Erdös raises the question of the existence of a nondegenerate connected set which is topologically equivalent to all of its nondegenerate connected subsets. In this paper it is shown that such a set exists in the plane if the hypothesis of the continuum is true. (Received November 10, 1953.)

333. Nobuo Shimada and Hiroshi Uehara (p): *Some remarks on Adem extension theorem.*

This paper is concerned with the classification of mappings of an $(n+2)$-dimensional complex into an $(n-1)$-connected space $(n>2)$ by using generalized Adem's operation, q^i-operation introduced elsewhere by the authors, and Steenrod's squaring operation. (Received November 13, 1953.)

334. G. L. Thompson: *On lattices of linear equations.*

Let k be a sfield and V_n the n-dimensional (left) vector space over k. Let N be the n-dimensional (modular) lattice of all subspaces of V_n and let L be an l-dimensional sublattice of N which has the same all and zero element as N does. It is shown that a basis can be chosen in V_n such that to every subspace A in L there corresponds an $l\times l$ matrix $\sum(A)$ where the matrix $\sum(A)$ is in row Hermite normal form and its entries are matrices of suitable dimensions with entries from k. The number of nonzero entries on the main diagonal of $\sum(A)$ is the dimension of A measured in L, and the rows of $\sum(A)$ form a basis for vectors in A. By means of the matrix $H(A, B) = \sum(A)[I-\sum(B)]$ the matrices $\sum(A\cup B)$ and $\sum(A\cap B)$ corresponding to the union and intersection of two elements A and B of L can be found. If A and B are elements of L such that $A/A\cap B$ and $B/A\cap B$ are projective in L, and $Q_s(A, B)$ is a space projectively related to L (c.f. R. M. Thrall, Canadian Journal of Mathematics vol. 4 (1952) p. 229) the problem of finding $\sum(Q_s(A, B))$ is solved. (Received November 16, 1953.)

335. R. F. Williams: *Reductions of open mappings.*

Suppose f is a continuous open mapping of a compact metric space X onto a topological space Y. The following three theorems will serve to illustrate the three types of results obtained: (1) if U is a nonempty open subset of X such that $f^{-1}(y)$ is locally connected for every $y\in f(U)$, then there exists a nonempty open subset V of X such that (a) $\overline{V}\subset U$, (b) for each $y\in f(V)$, $f^{-1}(y)\cdot V$ lies in a single component of $f^{-1}(y)\ \overline{V}$, and (c) if $y\in f(\overline{V})$ then there exists a point $x\in f^{-1}(y)$ and a neighborhood N of x with $f^{-1}f(V)\cdot N\subset V$; (2) if for each $y\in Y$, $f^{-1}(y)$ is a decomposable continuum, then there exists a proper closed subset X' of X such that $f|X'$ is monotone and onto Y; and (3) if Y is perfect and $f^{-1}(y)$ is connected and locally connected for each $y\in Y$, then there exists a closed subset X' of X such that $f|X'$ is monotone and onto Y and such that no closed proper subset of X' maps onto Y. (Received November 6, 1953.)

L. W. COHEN,
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BOOK REVIEWS

Mathematik als Wissenschaft, Kunst und Macht. By Helmut Hasse. Wiesbaden, Verlag für Angewandte Wissenschaften, 1952, 34 pp. 4.80 DM.

This little book, by a distinguished mathematician, is based on lectures delivered to the mathematical-natural sciences faculty of Hamburg University and to the "Amerikahaus" in Hamburg. It is, therefore, addressed to a general audience, with the object of setting forth the "true nature" of mathematics. The author frankly points out, however, that he speaks only for himself, limiting himself to those aspects of mathematics that first attracted him as a student and subsequently held his loyalty through the years. Written in a lucid and relaxed style, essentially as originally delivered, it is easy to read and most entertaining.

In conformity with its title, the book is divided into three main parts, preceded by a general introduction. Mathematics as "*Wissenschaft*" emphasizes the "spiritual" or intellectual character of mathematics. Here the opportunity is seized to point out that the layman, who considers mathematics as calculation and measurement, misses its true nature entirely. As an intellectual discipline, mathematics (for the author) fulfills the "instinct for knowledge of the truth" in a way not to be found in philosophy or any of the other branches of learning. Apparently it was this that first attracted the author to mathematics.

However, his sustained devotion to mathematics was due more to its nature as an *art*. To the reviewer, this part of the book (to which is devoted 14 pages, in contrast to the combined total of 10 pages for "Wissenschaft" and "Macht"), is by far the most interesting. Due to his predilection for music, the author stresses the analogies between this art and mathematics, in each of their creative, aesthetic, structural and communicative aspects.[1] Not only is this most certainly deserving of reading by the mathematician as well as the layman, but the former will find much to stimulate discussion.

For example, the striving for extreme formalism is, in the view of the author, one of the "tragedies" to which the modern mathematician is prone. The author decries the process whereby the intuitively conceived mathematical construct passes through various stages of polishing and refinement until it becomes the bloodless, dead form of "Definition, theorem, proof" in manifold repetition. Originally to be found in Euclid's *Elements*, this striving for formalism has been revived in modern times—"chiefly in North America," but, partly as a consequence of Hilbert's work and partly because of the impact of American tastes, also influencing mathematical work in Europe. Whether one agrees with this or not, one cannot but feel that the author has a point in regard to that excessive formalizing and abbreviation whose only justification is to comply with demands for brevity due to high printing costs.

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[1] The mathematician who has done creative work should enjoy the discussion of musical composition given by Paul Hindemith in *How music happens*, Saturday Review of Literature, Dec. 29, 1951, pp. 29–32.

By mathematics as "*power*," the author is careful to make clear that he means not material power or power over one's fellow men, but the feeling that comes, for instance, with the creation of a mathematical tool that enables one to solve a whole class of problems as special cases of a general theory.

Although there is plenty of room for disagreement with some of the author's principal assertions (as well as with some minor ones such as the mention of Leibniz as, by implication, the sole creator of the calculus), this little book is to be highly recommended as general reading for the professional mathematician and as "must" reading for the layman whose notions of mathematics are embodied in the query, "Won't you please add up my bridge score; you're a mathematician, aren't you?"

R. L. WILDER

An introduction to homotopy theory. By P. J. Hilton. Cambridge University Press, 1953. 8+142 pp. $3.00.

This is the first book to appear on homotopy theory. There has long been a need for collecting results from diverse sources, and this book seems to meet a large part of that need. Of course, a book of this size cannot be comprehensive and other books will be required. It is surprising, however, how much the author manages to get into the small number of pages. This seems to have been accomplished by careful planning and abetted by the author's ability to explain what is happening without becoming verbose. Another factor is the restriction to homology theory of complexes with integer coefficients, although one might argue that the introduction of singular theory would pay for itself in simplified proofs (and more general theorems) in some cases.

The book seems well suited for a textbook even though it does not contain exercises. The most important factor is that it can be read by a student. Then there is what should turn out to be a good pedagogic organization of the material. For example, some information on the homotopy groups of the rotation groups is obtained from basic properties of fibre spaces. Later the problem is resumed after the Hopf invariant and Freudenthal suspension have been introduced. Similarly, a special case of the Whitehead product is introduced and used in a proof considerably before its general definition is given. In both cases the reader is told exactly what is happening. Most of the theorems are proved. The exceptions are usually easily available.

The first chapter is a short introduction which defines the concept of a homotopy between two maps (arcwise connected Hausdorff spaces are used throughout) and the concept of homotopy type. This latter is proved to be an equivalence relation between spaces.

Chapter II defines the homotopy groups. First the absolute groups are defined using cubes, then the alternative description in terms of maps of spheres is shown to be equivalent. The operation of π_1 on π_n is given and n-simplicity of a space is defined. It is proved that the homotopy groups are invariants of homotopy type. Next the relative groups are defined both ways and equivalence is established. These are shown to be invariants of the relative homotopy type. In defining the operation of π_1 on π_n the "top hat" lemma was used, and it is now extended to the homotopy extension theorem.

The next chapter gives the basic theorem on the Brouwer degree of a map of S^n to S^n and the Hurewicz isomorphism theorem (relative). Since singular theory is not used, it is necessary to have the simplicial approximation theorem, and the Hurewicz isomorphism theorem is stated only for complexes. For proofs the reader is referred to Lefschetz's *Introduction to topology*.

In Chapter IV the homotopy sequence of a pair is defined and proved to be exact. A map between pairs is shown to induce an operator homomorphism of the sequences (with π_1 of subspaces as operators). If a pair consists of complex and subcomplex a homomorphism is obtained from its homotopy sequence to its homology sequence. The direct sum theorems obtainable from the homotopy sequence in special cases (e.g. the subspace is a retract of the space) are given. The homotopy sequence of a triple $C\subset B\subset A$ is defined and a sample of the exactness proof is given. There is a section on $\pi_2(Y, Y_0)$. This group is not necessarily abelian, but it is shown to be a crossed module under the operation of $\pi_1(Y_0)$.

Chapter V is on fibre-spaces. These are defined by local triviality (condition (ii) is the sole condition since it implies (i)), and no structure group is used. This condition suffices to give the strong "covering extension theorem." Let $\phi:X\rightarrow Y$ be such a fibre-space and

L be a subcomplex of the finite complex K and such that L is a deformation retract of K. If $f_0:L\to X$ is such that $g_0=\phi f_0:L\to Y$ admits an extension to $g:K\to Y$, then f_0 admits an extension $f:K\to X$ such that $g=\phi f$. This theorem implies the covering homotopy theorem and gives the exact homotopy sequence of the fibre-space. The exact sequence is used to study fibre-spaces over spheres. The last section considers the definition of fibre-spaces (here called pseudo-fibre spaces) by using the covering homotopy theorem as the sole axiom. For these one gets the covering extension theorem only in case K is contractible. It is shown that the space of paths between two subsets A, B of Y is such a fibre-space over $A\times B$. The technique of representing $\pi_n(Y)$ as the one-dimensional homology group of a certain space of loops is mentioned. The Cartan-Serre construction of the Eilenberg-MacLane spaces $K(\pi, n)$ is given.

The sixth chapter is entitled "The Hopf invariant and suspension theorems." The Hopf invariant γ of a map $S^{2n-1}\to S^n$ is defined and shown to depend only on the homotopy class of the map. If n is odd $\gamma=0$, and if n is even $\pi_{2n-1}(S^n)$ has an element with $\gamma=2$. The Freudenthal suspension and the J. H. C. Whitehead generalization are defined and the suspension theorems are stated. These are applied to fiberings by spheres and Stiefel manifolds. Finally the author's generalization of the Hopf invariant is given as $H^*=\chi Q\mu:\pi_r(S^n)\to\pi_{r+1}(S^{2n})$ all r, n. Here $\mu:\pi_r(S^n)\to\pi_r(S^n\cup S^n)$ is induced by shrinking the equator, $Q:\pi_r(S^n\cup S^n)\to\pi_{r+1}(S^n\times S^n, S^n\cup S^n)$ is projection onto a direct summand, and $\chi:\pi_{r+1}(S^n\times S^n, S^n\cup S^n)\to\pi_{r+1}(S^{2n})$ is induced by shrinking $S^n\cup S^n$ to a point. This is shown to be a true generalization and its relation with the G. W. Whitehead generalization is given.

Chapter VII gives a discussion of CW-complexes. It includes the Massey homology spectrum and takes from this spectrum the exact sequence which J. H. C. Whitehead used in classifying simply connected 4-polytopes according to homotopy type. For most of the results the reader is referred to *Combinatorial homotopy*. I (J. H. C. Whitehead, Bull. Amer. Math. Soc., 1948). However, a proof is given of the important theorem stating that if $f:K\to L$ realizes an isomorphism between the homotopy groups of the CW-complexes K and L, then f is a homotopy equivalence.

Chapter VIII considers the problem: If K is a CW-complex such that $\pi_i(K)=0$ for $i=0, 1, \cdots, n-1$ $(n>2)$, what is $\pi_{n+1}(K)$? This involves a detailed study of a section of the exact sequence obtained in Chapter VII. When dim $K\leqq n+2$, $\pi_{n+1}(K)$ is obtained as a definite group extension of $H_{n+1}(K)$ by a certain subgroup of $\Gamma_{n+1}(K)$

$=i_*(\pi_{n+1}(K^n))$ where $i:K^n\rightarrow K^{n+1}$ is the identity map. When, in addition, $\pi_{n+1}(K)=0$ and $n>3$, $\Gamma_{n+2}(K)$ is computed to be $H_n(K)/2H_n(K)$. This is equivalent to the statement that $\pi_{n+2}(K) \approx H_{n+2}(K)+H_n(K)/2H_n(K)$, and in this form it is proved again using the normal cell complexes of S. C. Chang.

There is a bibliography of 4 books and 42 papers and a rather complete combined index and glossary.

M. L. Curtis

Lezioni sulle funzioni ipergeometriche confluenti. By F. G. Tricomi. Torino, Gheroni, 1952. 284 pp. 2000 Lire.

Die konfluente hypergeometrische Funktion mit besonderer Berücksichtigung ihrer Anwendungen. By Herbert Buchholz. (Ergebnisse der angewandten Mathematik, vol. 2.) Berlin-Göttingen-Heidelberg, Springer, 1953. 16+234 pp. 36.00 DM.

Confluent hypergeometric series arise when two of the three singularities of the hypergeometric differential equation coalesce in such a manner as to produce an irregular singularity at infinity. These series were introduced by Kummer in 1836. In 1904, E. T. Whittaker proposed new definitions and notations which clearly exhibit the symmetry and transformation properties of confluent hypergeometric functions, and facilitate the identification of many special functions, among them Bessel functions, Laguerre and Hermite polynomials, error functions, Fresnel integrals, sine and cosine integrals, exponential integrals, and the like, as particular instances of confluent hypergeometric functions. (As a matter of fact, the chapter on Bessel functions in all newer editions of Whittaker and Watson's *Modern analysis* follows upon, and leans heavily on, the chapter on confluent hypergeometric functions.)

In the first half of the present century a sizeable literature has grown up around these functions. Many of their properties were discovered (some of them several times), and they have been found useful in pure and applied mathematics alike. Fluid dynamics, nuclear physics, probability theory, elasticity, all offer problems which can be solved in terms of confluent hypergeometric functions, and these functions proved an excellent example for illustrating the technique of the Laplace transformation. In view of the hundreds of papers and dozens of applications it is somewhat strange that no monograph on these functions seemed to be available (although several well-known books, such as *Modern analysis*, Jeffreys and Jeffreys' *Methods of mathematical physics*, and Magnus and Oberhettinger's *Special functions* devote separate chapters to these functions). Now, within a short period, we have to record the publication of two books devoted to confluent hypergeometric functions, and a third book is in preparation. From the point of view of the mathematical public (and, one supposes, also from the point of view of the authors and publishers) it is fortunate that the two books differ to such an extent as to complement each other in a very useful manner. They differ considerably in the definition and notation of the functions they investigate, in the selection of the material, methods, presentation, in the applications they envisage, and most of all in their style.

Tricomi's book is based on a course of lectures given at the University of Torino, and it is written with that gift for exposition for which its author is so justly famous. The presentation is well planned, leisurely, and detailed, with very few computations "left to the reader" and no vague appeals to "it is easy to see." The book is self-contained, and those parts of mathematical analysis which are needed, and with which the student may not be familiar, form the topics of several digressions. There is no attempt at all-inclusiveness, but the topics which have been selected are discussed rather fully. Books and memoirs are quoted when they are needed in the text: otherwise there is no bibliography, and there is no index. The book naturally favors those parts of the theory of confluent hypergeometric functions, and these are many, to which its distinguished author contributed in recent years. Shortly, this book is an eminently readable introduction to confluent hypergeometric functions which takes the reader, in several directions, to topics of interest in current research.

Buchholz's book is a handbook, and it is the fruit of many years' diligent labor. Based on a bibliography of several hundred items, the book encompasses practically everything of importance that is known about confluent hypergeometric functions. Since the number of books and papers used in writing this volume exceeds the number of its pages, it is inevitable that the writing should be concise and highly condensed. Proofs are frequently given in outline only, some proofs are omitted, and some results are mentioned only (with a reference to the literature) without details and without discussion. The book abounds in formulas, and important formulas are repeated in several different forms for handy reference. An index of notations and a subject index help the reader to locate any desired information, and contribute to the usefulness of this volume as a work of reference. Except in the choice of the applications, the author's interests seem to have had little influence upon the selection of the material: everything is included with a fine impartiality.

There follows a brief description of each of the two books.

The basic differential equation in Tricomi's book is the confluent hypergeometric equation

$$x\frac{d^2y}{dx^2} + (c - x)\frac{dy}{dx} - ay = 0 \tag{1}$$

where a and c are constant parameters. There is a unique solution of (1) which is regular at the origin, and is equal to unity there: this is Kummer's series

$$\Phi(a, c; x) = \sum \frac{\Gamma(c)\Gamma(a + n)x^n}{\Gamma(a)\Gamma(c + n)n!} \tag{2}$$

[also often denoted by ${}_1F_1(a; c; x)$].

Chapter I contains preliminary material on the Laplace transformation, on the analytic theory of ordinary linear differential equations, and on the gamma function (all of which is required later).

In Chapter II it is first shown that any ordinary linear differential equation of the second order whose coefficients are linear functions of the independent variable may be reduced to (1). If c is not an integer, the general solution is a linear combination of $\Phi(a, c; x)$ and $x^{1-c}\Phi(a-c+1, 2-c; x)$. The properties of (2), or rather of

$$\Phi^*(a, c; x) = \frac{1}{\Gamma(c)}\Phi(a, c; x), \tag{3}$$

which is an entire function of all three of its arguments, are discussed, the discussion including Kummer's transformation, recurrence and differentiation formulas, Laplace transforms. In the case $c = 2a$ there is a connection with Bessel functions, and in the case of a negative integer a with Laguerre and Hermite polynomials. This chapter concludes with the author's well known expansion of Φ in series of Bessel functions.

Chapter III introduces the "second solution"

$$\begin{aligned}\Psi(a, c; x) = {} & \frac{\Gamma(1 - c)}{\Gamma(a - c + 1)}\Phi(a, c; x) \\ & + \frac{\Gamma(c - 1)}{\Gamma(a)}x^{1-c}\Phi(a - c + 1, 2 - c; x),\end{aligned} \tag{4}$$

gives integral representations for Φ and Ψ, recurrence relations, etc., for Ψ, and the asymptotic properties of Φ and Ψ, showing that $\Psi \to 0$ as $x \to +\infty$. For real a and c, the real zeros of Φ and the positive zeros of Ψ are counted, and a description of $\Phi(a, c; x)$ for real a, c, x is given.

Chapter IV is devoted to the incomplete gamma functions and related functions. The author puts

$$(5)\qquad \int_0^x e^{-t}t^{\alpha-1}dt = \gamma(\alpha, x) = x^{\alpha}\Gamma(\alpha)\gamma^*(\alpha, x) = \Gamma(\alpha) - \Gamma(\alpha, x),$$

and points out that γ^*, being an entire function of both α and x, is the basic function. The connection with confluent hypergeometric functions with $a=1$ or $c=a+1$ is used to obtain most properties of incomplete gamma functions and of such functions as the error functions, Fresnel integrals, exponential integral, sine and cosine integrals which are related to incomplete gamma functions. This chapter contains a table of most of the important particular cases of confluent hypergeometric functions.

Chapter V contains applications. The two body problem of wave mechanics, bending of elastic plates, resultant of a large number of random vectors, and water waves are discussed and show the variety of problems in which confluent hypergeometric functions may be used with advantage.

The book is reproduced from a typescript.

Turning now to Buchholz's book, we find that Chapter I opens with the derivation of (1) and (2), but the discussion turns very soon to Whittaker's equation

$$(6)\qquad \frac{d^2y}{dx^2} + \left(-\frac{1}{4} + \frac{\kappa}{x} + \frac{1-\mu^2}{4x^2}\right)y = 0$$

and to its solutions

$$(7)\qquad M_{\kappa,\mu/2}(x) = x^{(\mu+1)/2}e^{-x/2}\Phi\left(\frac{1+\mu}{2} - \kappa,\ 1+\mu;\ x\right),$$

$$(8)\qquad W_{\kappa,\mu/2}(x) = x^{(\mu+1)/2}e^{-x/2}\Psi\left(\frac{1+\mu}{2} - \kappa,\ 1+\mu;\ x\right).$$

The function corresponding to Φ^* is

$$(9)\qquad \mathfrak{M}_{\kappa,\mu/2}(x) = M_{\kappa,\mu/2}(x)/\Gamma(1+\mu)$$

and is frequently used in Buchholz's book. Chapter I further contains an investigation of the basic properties of "parabolic functions" (thus Buchholz refers to M, W, $\mathfrak{M}$), differential equations which can be reduced to (6) (including Weber's equation of parabolic cylinder functions), an inhomogeneous Whittaker equation, and the separation of the wave equation in coordinates of the parabolic cylinder and of the paraboloid of revolution.

Chapter II brings a thorough discussion of integral representations, with application to recurrence and differentiation formulas. A few integrals representing products of parabolic functions are also given.

Chapter III is devoted to a study of the asymptotic behavior of parabolic functions, most of the results being obtained by the method of steepest descents applied to suitable integral representations.

In Chapter IV a large number of integrals involving parabolic functions are given: both indefinite and definite integrals occur, and the Laplace, Mellin, and Hankel transforms are included, as are also some infinite series involving parabolic functions.

Chapter V contains a discussion of polynomials related to parabolic functions. Laguerre and Hermite polynomials are given in detail (with many series and integrals involving these polynomials), others more briefly.

The material in Chapter VI is of great importance for the application of parabolic functions in wave propagation problems: it is the expression of several types of waves (planes waves, spherical waves, etc.) as a superposition of solutions of the wave equation in the coordinates introduced in Chapter I.

In Chapter VII the zeros of M and W are investigated both as k, m are fixed and z varies, and as m, z are fixed and k varies (in the latter case only M being considered). Vibrations of an inhomogeneous elastic string, and Green's function of the wave equation for a region bounded by two confocal paraboloids of revolution are used to illustrate the application of parabolic functions in eigenvalue problems.

Appendix I is a very useful summary of particular cases of parabolic functions, with information regarding numerical tables of these functions. Appendix II is an extensive bibliography. An index of symbols and notations is at the beginning of the book, and there is a subject index at the end.

In their very different ways both books may be expected to be of considerable assistance to those having occasion to employ these functions. Between them they provide all the information needed in the application of these functions, with the possible exception of nuclear physics where a different notation is used (under the name of Coulomb wave functions), and the selection of a second solution is different.

A. ERDÉLYI

Stochastic processes. By J. L. Doob. New York, Wiley, 1953. 8+654 pp. $10.00.

Chapter 1. Introduction and probability background (45 pages). The fundamental concepts of conditional probability and expectation are generalized over and somewhat different from those defined in Kolmogorov's *Grundbergriffe* (1933). Let y be a random variable whose expectation exists: let $\mathcal{F}$ be a Borel field of measurable ω-sets and $\mathcal{F}'$ its "completion." The conditional expectation of y relative to $\mathcal{F}$, denoted by $E\{y|\mathcal{F}\}$, is any measurable $\mathcal{F}'$, integrable ω function which satisfies $\int_\Lambda E\{y|\mathcal{F}\}dP = \int_\Lambda y dP$ for every $\Lambda \in \mathcal{F}$. If $\mathcal{F}$ is the smallest Borel field $\mathcal{B}(x_t, t \in T)$ with respect to which the x_t's are measurable, the above definition specializes to the conditional expectation of y with respect to the x_t's. The conditional probability of a measurable set M, denoted by $P(M|\mathcal{F})$, is defined to be $E\{y|\mathcal{F}\}$ where $y(\omega)$ is the characteristic function of M. An important question arises: Is it possible to define an M, ω function $P(M, \omega)$ such that (i) for every ω, $P(\cdot, \omega)$ is a probability measure of M and for every M, $P(M, \cdot)$ is measurable $\mathcal{F}'$; (ii) for every M, $P(M, \cdot) = P\{M|\mathcal{F}\}$ with probability one. If such a $P(\cdot, \cdot)$ is defined for every $M \in \mathcal{B}(y_1, \cdots, y_n)$, it is called the conditional probability distribution of the y_j's relative to $\mathcal{F}$. A related question is as follows. Let Y denote a generic n-dimensional Borel set. Is it possible to define a Y, ω function $p(Y, \omega)$ such that (i) for every ω, $p(\cdot, \omega)$ is a probability measure of Y and for every Y, $p(Y, \cdot)$ is measurable $\mathcal{F}'$; (ii) for every Y, $p(Y, \omega) = P\{[y_1(\omega), \cdots, y_n(\omega)] \in Y\}$ with probability one. If such a $p(\cdot, \cdot)$ exists, it is called the conditional probability distribution of $y_1, \cdots, y_n$ in the wide sense relative to $\mathcal{F}$. Now the main result is: while a conditional probability distribution in the wide sense always exists, a conditional probability distribution may fail to exist. The point is that an ω set $M \in \mathcal{B}(y_1, \cdots, y_n)$ does not uniquely determine the Borel set Y and if $M = \{[y_1(\omega), \cdots, y_n(\omega)] \in Y\}$ is satisfied for $Y = Y_1$ and also for $Y = Y_2$ it does not necessarily follow that $p(Y_1, \omega) = p(Y_2, \omega)$ with probability one. A sufficient condition for this, and so for the existence of a $P(M, \omega)$, is e.g. that the range of $[y_1(\omega), \cdots, y_n(\omega)]$ be a Borel set; this condition is always satisfied if the y process is of "function space type" (see below). The lack of a conditional probability distribution however vitiates a couple of the author's more famous theorems (1938) (see the Bibliography for all references). A correct form of the extension theorem, due to Ionescu Tulcea, where the existence of conditional probability distributions is assumed, is given in the Appendix. Kolmogorov's version is not given explicitly but is implied by this theorem and the sufficient condition stated above. The lack of a conditional probability distribution also necessitates a detour in proving certain results involving conditional expectations. One way is to use "representation theory." This theory enables us, in the study of the family of (real) random variables x_t, $t \in T$, to replace the original basic space Ω by the space (of function space type) $\tilde{\Omega}$ of all (real) functions of $t \in T$. The point $\tilde{\omega}$ in $\tilde{\Omega}$ ranges over all functions defined on T, and $x_t(\omega)$ is mapped into $\tilde{x}_t(\tilde{\omega})$ which for $\tilde{\omega} = \xi(\cdot)$ has the value $\xi(t)$. Thus $\tilde{x}_t(\tilde{\omega})$ is a coordinate random variable in Ω, namely "the tth coordinate of the point ω." It is shown (in the Supplement) that such a measure-preserving transformation can be established, and we can therefore translate any problem involving ω random variables into the corresponding one involving $\tilde{\omega}$ random variables. In the simplest case this amounts to e.g. considering two random variables $x(\omega)$ and $y(\omega)$ as the two coordinate random variables (of the basic point $\tilde{\omega} = (x, y)$) in the Cartesian plane, thus substituting a 2-dimensional distribution function for an abstract probability measure. Such an approach was in fact commonly used before the advent of the basic space Ω, and has been tried again in a way by Cramér in his *Mathematical methods of statistics* (1945). Here it is treated as a device rather than the logical foundation.

Chapter 2. Definition of a stochastic process—Principal classes (56 pages). A stochastic process is defined as any family of random variables x_t, $t \in T$. Let $\mathcal{F}_T = \mathcal{B}(x_t, t \in T)$ and $\mathcal{F}'_T$ be its completion. If T is nondenumerable, many significant ω sets, e.g. $\sup_{t \in T} x_t(\omega) > \lambda$, in general do not belong to $\mathcal{F}'_T$. This circumstance was noted e.g. in Khintchine's *Asymptotische Gesetze* (1933) and called for a new foundation of the theory of stochastic processes beyond that given in Kolmogorov's Grundbegriffe. The author first gave such a theory in 1937. He took Ω to be the space of all functions of $t \in T$ and his method was to enlarge Ω by "adjoining" a set of outer measure one. This method is only briefly reviewed in the present book. The new method, first presented here, does not restrict Ω nor change $\mathcal{F}_T$; instead the x_t's are modified in such a way as to leave unaltered the probability of sets in $\mathcal{F}_T$ and yet to acquire the desirable property of "separability." Let $\mathcal{A}$ be a system of linear Borel sets. The stochastic process is said to be separable relative to $\mathcal{A}$ if there is a sequence t_j of parameter values and an ω set Λ of probability zero such that if $A \in \mathcal{A}$ and if I is any open interval, the ω sets $\{x_t \in A, t \in IT\}$ and $\{x_{t_j}(\omega) \in A, t_j \in IT\}$ differ by a subset of Λ. Two important cases are where $\mathcal{A}$ is the set of all closed sets and where $\mathcal{A}$ is the set of all closed intervals; in the latter case the stochastic process is simply said to be separable. Next, $\tilde{x}_t$ is called a standard modification of x_t if for each t, the ω set $\{\tilde{x}_t(\omega) \neq x_t(\omega)\}$ belongs to $\mathcal{F}'_T$ and has probability zero. Now the fundamental theorem (Theorem 2.4) is: Any stochastic process $\tilde{x}_t$ has a standard modification x_t which is separable relative to the set of closed sets. (The x_t's may take on the values $\pm\infty$.) This new approach deprives the theory of some of its erstwhile measure-theoretic halo. Thus, the proof of the above theorem depends on the clean-cut lemma: To each linear Borel set A there corresponds a finite or denumerable sequence t_j such that $P\{x_{t_n}(\omega) \in A,\ n \geqq 1;\ x_t(\omega) \notin A\} = 0$ for each $t \in T$. Next the stochastic process x_t is called measurable if T is Lebesgue measurable, and $x_t(\omega)$ considered as a (t, ω) function is measurable in the product (t, ω) measure. A simple sufficient condition is given for the measurability of a separable stochastic process. For a separable measurable stochastic process the probabilities of significant ω sets such as the one mentioned above are defined and the integral $\int x_t(\omega)dt$ may be interpreted as ordinary Lebesgue integrals of the sample functions, etc. In essence separability reduces the considerations of sample functions as to their boundedness, continuity, etc. to those of their values on a denumerable set, and this will be repeatedly made use of throughout the book. The second half of Chapter 2 defines various stochastic processes to be discussed later together with their preliminary properties. Only one, the Gaussian process, is not discussed later for lack of further knowledge. It is used to define "strict sense" and "wide sense" concepts. If a stochastic process has a certain property P in terms of means and covariances and if the corresponding Gaussian process with the same means and covariances has the corresponding but stronger property P′, then P′ and P are the corresponding strict and wide sense properties.

Chapter 3. Processes with mutually independent random variables (46 pages). This chapter belongs to the classical theory of probability and treats such familiar topics as the 0–1 law, convergence of series, the law of large numbers, infinitely divisible distributions, and the central limit theorem. The material is presented frequently with a view to later applications. The criterion for the convergence of a series of independent random variables is given not only in the Khintchine-Kolmogorov form known as the three-series theorem but also in Lévy's form based on the idea of centering. In particular, convergence regardless of the order of terms is discussed, as will be needed later for processes with independent increments. Further criteria are given in terms of infinite products of characteristic functions, due to Wintner and others. The treatment of infinitely divisible distributions is direct and deals immediately with components which are infinitesimal but not necessarily equi-distributed. The central limit theorem due to Feller and Lévy is given in "finite terms," in the spirit of Lévy. The crisp treatment of the last two topics is made possible by a deft use of certain inequalities which are implicit in previous work but carefully sifted and dispatched in Chapter 2. The chapter ends with the author's youthful (1936) discovery to the effect that if a gambler chooses his play (in a stationary sequence of independent games) without clairvoyance, his chances are unaffected. The formal proofs of such "obvious" statements seem to be a source of the author's inspirations.

Chapter 4. Processes with mutually uncorrelated or orthogonal random variables (22 pages). There is a brief discussion of orthogonal series leading to Menshov's theorem. It is interesting to note that Menshov's condition for the strong law of large numbers of orthogonal random variables is within a factor of $\log^2 n$ of Kolmogorov's for that of independent random variables. However, it should be pointed out that Menshov's condition is not sufficient for the convergence of the corresponding infinite series, as Kolmogorov's is. This can be shown by an example based on Menshov's own counterexample.

Chapter 5. Markov processes—Discrete parameter (65 pages). The author defines a Markov chain to be a Markov process (discrete or continuous parameter) whose random variables assume values in a finite or denumerable set. Finite chains with stationary transition probabilities are first discussed. The ergodic properties are proved by studying the actual transitions, leading from various special cases to the general case. The choice of this method seems largely dictated by the generalization to the "general state space" which follows. This generalization is an exposition of Doeblin's Thesis (1937), fortified by measure theory, and strengthened and completed in important points. Here, as nowhere else in the book, the random variables assume values in an abstract space. (It is a tribute to the author's sense and conscience that he does not indulge in trivial generalizations.) Under an essential hypothesis (D), slightly generalized over Doeblin's, the decomposition of the space into a transient set and a finite number of ergodic sets each containing (possibly) cyclically moving subsets, and the convergence of the transition probabilities $p^{(n)}(\xi, E)$, uniformly in ξ and exponentially fast, are established. The main line of argument is an extension of the one used in the finite case, but it is considerably complicated by several measure-theoretic tricks and asides. While the tricks are Doeblin's, the asides, no less tricky to the average reader, are Doob's. Furthermore, the author adds a clarify-ing discussion of the hypothesis (D). The details of all this are awesome and smack of the *ad hoc*. It may be wondered if there is a better approach. Doeblin in his last great paper on Markov processes (1940; listed in the book but not used) indeed treated a more general case[1] where (D) is not postulated and used an apparently more powerful method. However, to present this paper in a shape comparable to what is given here will undoubtedly require even more space and labor, and further work will be necessary to adduce a special case like that under (D). (The reviewer has attempted to do this, but is still far short of the goal.) The condition (D) is even sufficient for a central limit theorem (within a noncyclic ergodic set) for $\sum_1^n f(x_n)$ where $\{x_n, n \geqq 1\}$ are the random variables of the Markov process, provided that $E[|f(x_1)|^{2+\delta}] < \infty$ for some positive δ (Doeblin postulated a bounded f.) The proof of this old-fashioned theorem, in which the author does all the antics of a computationist, is one of the longest (11 pages) in the book. It depends on the exponential speed of the convergence of $p^{(n)}$, and S. Bernstein's idea of grouping terms. It may be noted that the denumerable case is not treated separately but only by way of example under (D). As is well known there is a unified treatment of the finite and denumerable case, given by Kolmogorov in 1936 and now available in a somewhat different version in Feller's book. This treatment does not by itself yield the exponential speed needed here for the central limit theorem, but in the denumerable case, there is a much simpler method (see Doeblin, Bull. Soc. Math. France vol. 66 (1938) pp. 210–220).[2]

Chapter 6. Markov processes.—Continuous parameter (57 pages). Finite chains are treated first and a complete analysis of the behavior of the (stationary) transition probabilities $p_{ij}(t)$ at $t=0$ and $t=\infty$ is given. A description of the actual transitions follows, resulting in the theorem that the sample functions of a separable chain satisfying $\lim_{t\to 0} p_{ij}(t) = \delta_{ij}$ are almost all step functions. Next it is shown how to construct a chain with given $p'_{ii}(0+)$ and $p'_{ij}(0+)$. In this chapter we miss more acutely the absence of a detailed discussion of the denumerable case. The fact is that in the continuous parameter case this already offers a great challenge. The recent work of Lévy (1951) shows how much can be done, and how much remains to be done. A complete description of the sample functions becomes infinitely harder, but even a discussion of the simpler types along the lines of the author's 1942 and 1945 papers would have been valuable. In this connection it may be pointed out that a recent paper by Kolmogorov (Učenye Zapiski (Matem.), Moskov. Gov. Univ. (4) vol. 148, pp. 53–59) completes certain points of the behavior of $p_{ij}(t)$ at $t=0$ (cf. Ex. 1, pp. 265 and 271 of the book); also the behavior of $p_{ij}(t)$ at $t=\infty$ is established elegantly by Lévy (1951), both for the denumerable case. The generalization to a continuous state space (a linear Borel set) is accomplished by Doeblin's method (1939), similar to the one used in the finite case. The material in §3 is the author's version of the Ito process, and is of very recent date. It stems from the partial differential equations for the not necessarily stationary transition probabilities of Markov processes established by Kolmogorov in 1931 and studied by Feller (1936; 1945) and others. Ito was the first (1946; 1951) to show that these processes could be obtained constructively by solving the stochastic differential equation (*) $dx(t)=m[t, x(t)]dt+\sigma[t, x(t)]dy(t)$ where $y(t)$ is the Brownian motion process with variance parameter one. This equation is interpreted to mean $x(t)-x(a)=\int_a^t m[s, x(s)]ds+\int_a^t \sigma[s, x(s)]$ where the second integral (due to Ito) is defined in Chapter 9 of the book. It is shown that under reasonable assumptions on m and σ^2 a separable Markov process $x(t)$ exists whose sample functions are almost all continuous and which satisfies certain limit relations exhibiting m and σ^2 as the instantaneous mean and variance. Conversely, given m and σ, if $x(t)$ is a process whose sample functions are almost all continuous and which satisfies the stated limit relations, together with some auxiliary conditions, then $x(t)$ is a Markov process and there exists (or can be adjoined) a Brownian motion process $y(t)$ such that (*) is satisfied. This converse (Theorem 3.3) is new and made possible by the author's good work on martingales.

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[1] In a recent French book the authors stated to the effect that this extension presents no "major difficulties" and that it is only short of being "absolutely immediate." Anyone who reads the paper will see that these are gross understatements.

[2] Cf. a forthcoming paper by the reviewer, *Contributions to the theory of Markov chains*. II (to appear in Trans. Amer. Math. Soc.)

Chapter 8. Martingales (99 pages). This is perhaps the most original chapter of the book, much of the material being published here for the first time. Lévy in his 1937 book considered a class of dependent variables whose partial sums form a martingale, as a natural generalization of independent random variables in the sense that many classical limit theorems can be extended to them. (He discovered or anticipated a number of the results given here.) The name "martingale" was introduced by Ville (1939) who gave its present definition and obtained some preliminary results including the extension of Kolmogorov's inequality. It was, however, the author, a little later and under the unprepossessing name "processes with the property $\mathcal{E}$," who established martingale as a process *per se*, thus creating a new branch of stochastic processes much as Wiener did with the Brownian motion process and Lévy with the processes with independent increments. A martingale is a stochastic process $\{x_t,\ t\in T\}$ such that $E\{|x_t|\}<\infty,\ t\in T$, and $x_{t_n}=E\{x_{t_{n+1}}|x_{t_1},\cdots,x_{t_n}\}$ with probability one whenever $t_1<\cdots<t_{n+1}$. If the $=$ sign is replaced by $\leqq$ then it is called a semi-martingale. The success of martingale is rooted in its interpretation as a fair game: if x_n is the gambler's fortune after the nth play, the above definition implies that his expected fortune is always equal to his present one. This gambling interpretation brings with it the notion of a gambling system, greatly exploited here, which is fundamental to the theory and its application. The intuitive background is obvious: if the game is fair, then it remains so if the gambler examines his fortune only at certain moments chosen without clairvoyance. Mathematically speaking, let $m_1, m_2, \cdots$ be a nondecreasing sequence of integer-valued random variables such that for every μ, the condition $m_j=\mu$ is a condition on the x_t with $t\leqq\mu$. Let $\check{x}_j=x_{m_j}$. Then the $\check{x}_j$ process is said to be obtained from the x_j process by "optional sampling." An important special case is that of "optional stopping": there is a random variable m such that $m_j=\text{Min}\ (m, j)$. It is intuitive that if x_j is a martingale (or semi-martingale), then so is $\check{x}_j$; it is, however, less intuitive that this is true only when certain conditions are imposed on x and/or m. Indeed were it not for the nuisance of such conditions a "winning system" would be possible and the author would have done better going to Las Vegas than writing this book. These conditions complicate the theorems which are often crammed (a style the author loves) but the reviewer will henceforth apply the excellent methods of optional sampling and skipping freely. The fundamental inequalities are the extension of Kolmogorov's mentioned above, and the following new one. Let $x_j,\ 1\leqq j\leqq n$, be a semi-martingale and let $\beta(\omega)$ be the number of times the sample sequence $[x_1(\omega),\cdots,x_n(\omega)]$ passes from below r_1 to above r_2, then $E(\beta)\leqq(r_2-r_1)^{-1}(E\{|x_n|\}+|r_1|)$. This result, due to Doob and Snell, is here proved neatly by "optional skipping" (skipping some of the differences x_j-x_{j-1}). All convergence theorems for semi-martingales follow from this inequality. For a forward semi-martingale $\{x_n,\ n\geqq 1\}$ a condition is needed to ensure that $\lim_{n\to\infty}x_n=x_\infty$ exists (with probability one), and another to ensure that $\{x_n,\ 1\leqq n\leqq\infty\}$ is semi-martingale. For a backward semi-martingale $\{x_n,\ n\leqq -1\}$, $\lim_{n\to\infty}x_{-n}=x_{-\infty}$ exists always and $\{x_n,\ -\infty\leqq n\leqq -1\}$ is a semi-martingale. (The connection of these results with the related work of Andersen and Jessen is discussed in the Appendix.) Best illustrations of martingales are the conditional expectations

$E\{z|\mathcal{F}_n\}$ for a nondecreasing sequence of Borel fields $\mathcal{F}_n$; in particular $E\{z|y_1, \cdots, y_n\}$ and $E\{z|y_n, y_{n+1}, \cdots\}$, $n \geq 1$. Various applications are given: the 0-1 law; the theorem that for a series of independent random variables convergence with probability one is equivalent to convergence in probability; generalization of some inequalities by Marcinkiewicz and Zygmund; theory of integration and differentiation; likelihood ratios; sequential analysis (Wald's equation). But the most delightful one is undoubtedly a new proof of the strong law of large numbers. If y_n', $n \geq 1$, are independent random variables, and $y_n = y_1' + \cdots + y_n'$, this famous law follows from the fact that $E\{y_1'|y_n, y_{n+1}, \cdots\}$ is a (backward) martingale. The discussion of continuous parameter martingales is partly straightforward extensions of discrete parameter results and partly measure-theoretic complications. The main theorem (Theorem 11.5) states that almost all sample functions of a separable semi-martingale x_t, $t \in T$, are bounded in $[a, b]T$ if $a, b \in T$; that they have finite left- (right-) hand limits at every $t \in T$ which is a limit point of T from the left (right); and that their discontinuities are jumps except perhaps at the fixed points of discontinuity. This is proved by the two fundamental inequalities mentioned above and has several applications in the book. The discussion of optional sampling is peculiarly Doobian and is an example of painstaking modern rigor in probability theory. The chapter ends with applications to sample function continuity of Markov processes and processes with independent increments, and an important theorem (originated with Lévy) which states that if $\{x_t\}$ is a martingale whose sample functions are almost all continuous and such that $\{x_t^2 - t\}$ is also a martingale, then the process is a Brownian motion process. This theorem is used in Chapter 9 in connection with the Ito process.

Chapter 8. Processes with independent increments (34 pages). There is a brief discussion of Brownian motion process including the so-called "reflection principle," continuity of almost all sample functions and the expression of the variance as a stochastic limit (Lévy). That the author did not choose to enter into more details of this too, too popular process may be partly explained by the existence of a "profound study" in Lévy's 1948 book. Some readers will however miss the inclusion of material from the author's 1942 paper on the Ornstein-Uhlenbeck process rather than the desultory §3, obviously a concession to custom. There is a discussion of several useful formulations of the Poisson process and an application to the macroscopic equilibrium of molecular and stellar phenomena. (We hope physicists will appreciate the neat and yet rigorous derivation there!) The rest of the chapter is the author's version of what has been variously called "differential process," "additive process" and "integral with independent random elements," and is now renamed as given in the title. This theory, one of the crowning achievements in modern probability, is a natural generalization of the "addition of independent random variables" from the discrete to the continuous parameter case. It received its definitive form in the hands of Lévy (1934) and was further developed in his 1937 book. The re-discretization of this process, begun by Khintchine and carried on mainly by the Russian school, led to the ramified theory of infinitely divisible distributions. This latter theory, though of great importance in itself, must be regarded as a step toward retrenchment from the standpoint of stochastic processes. It is clear from Lévy's writing that he has always regarded the subject as one belonging to a (continuous-parameter) process and it was under this guidance that he was led by his extraordinary intuition to the discovery of all the main facts of the theory. That Khintchine and later authors chose the more formal analytical approach must be partly due to the fact that at the time the foundations of stochastic processes were hardly laid (cf. the last-mentioned dates with that of Khintchine's book cited in paragraph 2 above), and that mathematicians endowed with less intuition feared to tread the ground broken by Lévy. Thus we should indeed be grateful to the author for this account of Lévy's theory, taken strictly in the spirit of its creator, and embellished with the author's own reflections. Such is e.g. his construction of a process whose (almost all) sample functions have prescribed fixed points of discontinuity but are otherwise continuous. There is also a detailed discussion of the centering technique based on the results of Chapter 3. The final characterization of the sample functions of a separable centered process is obtained as a fast application of martingale theory, by noting that $z_t = e^{iu(x_t - x_0)}/E\{e^{iu(x_t - x_0)}\}$ is a martingale. Hence they have the same continuity properties as those of a martingale given in the preceding paragraph.

Chapter 9. Processes with orthogonal increments (27 pages). Integrals of the form $\int_A \Phi(t) d_t y(t)$, where $\{y(t),\ t \in T\}$ is a process with orthogonal increments and $A \in T$, are defined. It is followed by a discussion of processes which are Fourier transforms of each other, a subject which has been much blown up by "abstract" people. A more general integral of the form $\int_A \Phi(t, \omega) dy(t)$ is defined where the integrand now depends also on ω and $y(t)$ is a martingale, under appropriate conditions. These definitions specialize to Ito's if $y(t)$ is a Brownian motion process. The extension to a martingale $y(t)$ carries with it a closure property in the sense that the indefinite integral $x(t)=\int_{c}^{t}\Phi(s, \omega)dy(s)$ is then also a martingale and we can consider integrals of the form $\int\Phi(t, \omega)\, dx(t)$. Furthermore the $x(t)$ process has nice properties, in particular, its sample functions are almost all continuous if those of the $y(t)$ process are. If so and if roughly speaking the variance checks, then the $y(s)$ process in the integral representation of $x(t)$ may be taken to be the Brownian motion process. These results are used in the discussion of the Ito process in Chapter 6.

Chapter 10. Stationary processes—discrete parameter (55 pages); Chapter 11. Stationary processes—continuous parameter (53 pages). It is well known that the theory of strict sense (wide sense) stationary processes is equivalent to that of measure preserving (isometric, unitary) transformations. A detailed discussion is given to clarify the various notions of point, set, and random variable transformations in order to make this equivalence exact. G. D. Birkhoff's ergodic theorem thus becomes the strong law of large numbers for strictly stationary processes and is proved by F. Riesz's method. The usual corollaries are given. Turning to the wide sense stationary processes there is the inevitable identification of covariance function with positive definite function (Herglotz, Bochner, Khintchine) followed by specific examples of such processes to show the various possibilities. The spectral representation of the process (corresponding to the Hilbert space theory of von Neumann-Wintner and Stone) is given by Cramér's elegant method of setting up a distance preserving correspondence between the x_n's (the random variables of the process) and the numerical functions $e^{2\pi in\lambda}$. von Neumann's ergodic theorem, namely the law of large numbers in L^2, is now proved easily by this representation. A form of the strong law of large numbers (convergence with probability one), due to Loève and others, is proved under an order restriction. The author adds his own theorem on the convergence of $(n+1)^{-1}\sum_{j=0}^{n} x_{v+j}\bar{x}_j$ to the covarience $R(v)$, and related things. There are sections on "moving averages," "linear operations" (in the continuous parameter case they include differentiation and integration in L^2) and rational spectral densities in $e^{2\pi i\lambda}$. The continuous parameter case of these results is pretty much the same and both the reader and the reviewer, though not the author, may be spared the repetitious details. §11 in Chapter 11, on processes with stationary (wide sense) increments, is an exposition of Kolmogorov's theory on "curves in Hilbert space." The material in these two chapters, hallowed by tradition, somehow wears a worn look. The next generation of students of probability, who will read this probability version before its traditional counterpart, will probably find it more refreshing.

Chapter 12. Linear least squares prediction—stationary (wide sense) processes (39 pages). This chapter "is somewhat out of place in the book, since it discusses a rather specialized problem." The object of study is the wide sense conditional expectation $\hat{E}\{x_{n+\nu} | \cdots, x_{n-1}, x_n\}$, and is seen to reduce to linear approximations in L^2 with an arbitrary weighting function $F(\lambda)$, the spectral distribution function of the process. The corresponding analytic problem was discovered by Szegö in 1920; and the subject was treated by Wold and Kolmogorov before Wiener rediscovered and popularized it in this country. The present exposition claims to have made the material readily available to the American reader, in the usual language of probability.

The Supplement (24 pages) includes a "treatment of various aspects of measure theory with which the ordinary reader may not be familiar." The section headings are: fields of point sets; set functions; measure-preserving transformations. An Appendix (13 pages) collects references to the literature and historical remarks. A Bibliography (8 pages) collects the articles and books referred to in the Appendix. Relevance rather than excellence seems to be the criterion for inclusion there. There is a subject index at the end.

The following is a general appraisal of the book. The foundation of the theory of stochastic processes is largely the author's own contribution, some of which is published here for the first time. For the last 15 years the author has been almost alone in the treatment of stochastic processes from the purely probability or measure theory point of view, as distinct from the Laplace-Fourier-transform, or differential-integral-equation, or Hilbert-Banach-space standpoints. For him the study of a stochastic process is the study of its sample functions in their own right, not as the shadows behind analytical expressions nor in their weak-topological collectivizations. This book is an eloquent testimony to the success of this direct approach and will doubtless inspire and guide its further development.

Although much of the material on special processes is in the literature, the author's accounts of the results of Lévy, Doeblin, and others are far from being routine expositions. Indeed, the mathematical world is indebted to a man of the author's stature for rendering such unenviable services. He is apparently as willing to interpret the work of others as to give his own. This modesty is refreshing, although the reader may have been thereby deprived of some more thorough-going measure theory and several other interesting topics of the author's own which have been alluded to in the preceding paragraphs.

A few words must now be said of the style, inasmuch as the author enjoys, justly or not, a reputation on it. In judging the readability of this book, it must be borne in mind that it is frequently the substance rather than the form which makes for difficulty. It seems true that the author has been somewhat lax in allowing for human weaknesses on the part of the reader in his early writings, but in this book he seems to have made a sincere effort to be more accommodating. An example is his considerable pains in handling sets of measure zero, putting the horse before the cart in phrasing them and showing their dependence sometimes (cf. Theorem 3.1 of his 1937 paper). There is also a minimum amount of handwaving except that he occasionally refuses to be drawn out on measure theory. An instance where this seems unwarranted is as follows. On p. 51 a most trivial sort of example is mentioned but the reader is immediately waved off to the end of that section for further enlightenment. When he finally reaches p. 70 after a rugged 20 pages, or even if he turns at once to the promised place as this reviewer did, he finds only a number of statements of the "it-is-easy-to-see" type. Admittedly they are easy enough if the reader knows what's going on, but this is precisely what he (the average one!) does not at this stage of the game. E.g. it would take only half a page to substantiate the statement on p. 70 that "If X contains a second point, the x_t process is neither separable nor measurable," but this the author steadfastly refused to do. Indeed nowhere in the discussion did he ever indicate a proof (of the type attributed to Halmos) that a certain ω set may not be measurable. Such trivial omissions may cause some readers a disproportionate amount of labor (and complaint). However, let it be stated that this book should on the whole be quite accessible to a determined reader.

No serious error has been found by this reviewer, but there are some minor errors and a number of misprints. We mention only the following: p. 54, The last line is incorrect; part (ii) of Theorem 2.2 is proved only if "in probability" is substituted for "with probability one" in line 20 (noted by J. G. Wendel); p. 64, line 10: another term like $2\delta\int_0^{2n-\epsilon}|f(t)|\,dt$ is needed at the end; p. 285, line 2 and later reference to the formula: read h^2 for $h^{3/2}$; p. 357, the statement "If $t \in T_1(n) \cdots$" needs minor fixing. A longer list has been turned over to the author.

Finally, this is an honest mathematics book. It is not designed to sell stochastic processes cheap. It takes all sorts to make stochastic processes, but let Mr. Doob write only for the sake of mathematics. He has done it.

K. L. CHUNG

Lagerungen in der Ebene, auf der Kugel und im Raum. By L. Fejes Tóth. (Grundlehren der mathematischen Wissenschaften, vol. 65.) Berlin, Springer, 1953. 10+197 pp. 24 DM.

This is probably the only book, in any language, devoted to the subject of "arrangements" such as packings and coverings. It is full of interesting results, many of them discovered by the author himself, and the rest collected from a great variety of sources. Yet it is not a collection of isolated theorems but develops the subject systematically. The exposition is clear, and relieved by frequent historical interludes (such as one drawing attention to Minkowski's enthusiastic remark: "Mich interessiert alles, was konvex ist!"). There are 124 beautiful figures, a five-page bibliography (including many works as recent as 1951 and 1952) and a useful index.

After a brief introduction to the theory of convex regions, the author gives a neat proof that, if two ellipsoids are polar reciprocals with respect to a unit sphere, their volumes, E and E', satisfy

$$EE' \geqq (4\pi/3)^2,$$

with equality only when the ellipsoids and sphere all have the same center. This is essentially a theorem of affine geometry: If two ellipsoids are polar reciprocals with respect to a third, the geometric mean of their volumes is greater than or equal to the volume of the third. Another affine theorem, this time in two dimensions, is that, if an n-gon contains an ellipse of area e and is contained in an ellipse of area E, then

$$e/E \leqq \cos^2 \pi/n.$$

This first chapter includes also a nicely illustrated account of the regular and Archimedean solids and of the analogous tessellations; e.g., (3, 3, 4, 3, 4) is the tessellation of triangles and squares in which no two squares share a side (so that each vertex is surrounded by two triangles, a square, another triangle, and another square, as the symbol indicates). The author might well have mentioned (on p. 19) that this particular tessellation is not anomalous like (3, 3, 3, 4, 4), but can be derived from the regular tessellation (4, 4, 4, 4) in the same way that the snub cube (3, 3, 3, 3, 4) is derived from the cube (4, 4, 4) or from the octahedron (3, 3, 3, 3) (cf. Coxeter, *Regular and semi-regular polytopes*, I, Math. Zeit. vol. 46 (1940) pp. 380–407, especially p. 395).

Chapter II includes a good exposition of Blaschke's important concept of the *affine length* of a curve. It is proved that the affine circumference λ of an oval curve of area T satisfies

$$\lambda^3 \leqq 8\pi^2 T,$$

with equality only for an ellipse (whereas the ordinary circumference L satisfies

$$L^2 \geqq 4\pi T,$$

with equality only for a circle).

In Chapter III we come to one of the problems from which the book takes its name: the closest packing and thinnest covering of the plane with equal circles, or, as the author vividly explains: the most efficient distribution of trees in an orchard, and of oases in a desert. (It is a pleasant feature of the book that the exquisitely precise diagrams for these problems contain quite recognizable trees, and even an Arab riding a camel to the nearest oasis.) With any discrete set of points (such as the trees or oases), the author associates a decomposition of the plane into polygonal "cells" (elsewhere called "Dirichlet regions"). Each of the points lies in one cell, whose interior consists of all points that are nearer to this point than to any other one of the set. Given the cells, their inscribed circles form a packing and their circumscribed circles form a covering. The solution of the packing problem (p. 67) is that if a convex region of area T contains a set of at least two nonoverlapping congruent circles, the sum of their areas is less than $\pi T/2\cdot 3^{1/2}$. The solution of the covering problem is that, if a convex region of area T is completely covered by a set of at least two congruent circles, the sum of their areas is greater than $2\pi T/3\cdot 3^{1/2}$. It follows that the best positions, both for the trees and for the oases, are the centers of the cells of the regular tesselation of regular hexagons, (6, 6, 6). (A result that many of us could guess but few could prove!)

This chapter includes also some results on regions of various sizes. For instance, if L is the sum of the circumferences of n nonoverlapping circles in a convex hexagon of area S, then

$$L^2 \leqq 2\pi^2 nS/3^{1/2}.$$

(On p. 89 this expression is accidentally written as $\pi nS/3^{1/2}$.) The corresponding result for Λ, the sum of the affine circumferences of n nonoverlapping ovals, is

$$\Lambda^3 \leqq 72n^2S.$$

The historical remarks include a reference to the independent discovery by Reifenberg, Bateman, and Erdős that 18 equal circles (and no more) can have at least one point in common with a given circle of the same size, while none of the circles has its center interior to another. Those authors described the system, but here for the first time we have an actual drawing (p. 96), which shows that the centers of the 18 circles are 18 vertices of the Archimedean tessellation (3, 4, 6, 4).

Chapter IV deals with arrangements of regions derived from one convex region by a group of translations. The density of the densest packing in such an arrangement is 2/3 when the region is a triangle, and greater otherwise; that of the thinnest covering is 3/2 when the region is a triangle, and less otherwise. (These results are ascribed to Fáry.) The state of affairs is interestingly different if the region is forced to have central symmetry. Then the density of the thinnest covering is $2\pi/3\cdot 3^{1/2}$ when the region is an ellipse, and less otherwise, while that of the densest packing is $\pi/2\cdot 3^{1/2}=0.9069\cdots$ when the region is an ellipse, but not always greater otherwise! It was shown independently by Reinhardt and Mahler that it has the smaller value

$$(9-4\cdot 2^{1/2}-\log 2)/(2\cdot 2^{1/2}-1)=0.9024\cdots$$

for a "smoothed octagon": a regular octagon whose corners are cut off by certain arcs of hyperbolas.

One of the most surprising results (p. 86) is that the maximum density of such a "lattice" packing of centrally symmetrical convex regions is as great as the density of any irregular packing of congruent regions of the same shape. For instance, a packing of congruent ellipses cannot be improved by allowing their major axes to lie in various directions.

Extremal properties of the regular polyhedra are considered in Chapter V. It is proved that the density of a packing of $n>2$ small circles on a sphere is

$$\leqq \frac{n}{2}\left(1-\frac{1}{2}\csc\omega_n\right),$$

where

$$\omega_n=\frac{n\pi}{6(n-2)},$$

while that of a covering by $n>2$ small circles is

$$\geqq \frac{n}{2}n(1-3^{-1/2}\cot\omega_n).$$

In either case equality occurs only when the centers of the circles are the vertices of an equilateral triangle (inscribed in a great circle), a regular tetrahedron, on octahedron, or an icosahedron. If a polyhedron with n vertices or n faces contains a sphere of radius r and is contained in a sphere of radius R, then

$$R/r \geqq 3^{1/2} \tan \omega_n.$$

The corresponding affine theorem, about volumes of ellipsoids, is obtained by cubing both sides. For a polyhedron with k edges, we have the same inequalities with ω_n replaced by $k\pi/6(k-3)$.

A new proof is given for Michael Goldberg's form of the isoperimetric theorem: The surface F and volume V of a convex n-hedron satisfy

$$F^3/V^2 \geqq 54(n-2) \tan \omega_n(4 \sin^2 \omega_n - 1).$$

It is also proved that the sum of the edges of a convex polyhedron containing a sphere of diameter D satisfies

$$L > 10D.$$

If we are given that all the faces have the same area, "$>10D$" can be replaced by "$\geqq 12D$", with equality only for a cube of edge D.

The above results for small circles on a sphere may be interpreted as applying to circles in the elliptic plane, and suggest corresponding properties of the hyperbolic plane. In particular, the density of a packing of equal circles in the hyperbolic plane is always less than

$$3/\pi = 0.9549 \cdots .$$

(For the Euclidean plane, "$<3/\pi$" has to be replaced, as we have seen, by "$\leqq\pi/2\cdot 3^{1/2}$.")

The packing problem for n equal small circles on a sphere is taken up again in Chapter VI, where the diameter of the circles in the densest packing is denoted by a_n, so that

$$\cos a_n \geqq \frac{1}{2}(\cot^2 \omega_n - 1),$$

with equality when $n=3$, 4, 6 or 12. Values of a_n are found for $n \leqq 16$ in the manner of Schütte and van der Waerden (*Auf welcher Kugel haben* 5, 6, 7, 8 *oder* 9 *Punkte mit Mindestabstand Eins Platz*? Math. Ann. vol. 123 (1951) pp. 96–124). In particular, it is proved that $a_5 = a_6 = \pi/2$,

$$\cos a_7 = \frac{1}{2}(\cot^2 \omega_8 - 1), \quad \text{and} \quad \cos a_8 = (2\cdot 2^{1/2} - 1)/7,$$

the centers of the 8 circles being the vertices of the square antiprism (3, 3, 3, 4).

Finally, Chapter VII deals with arrangements of equal spheres in Euclidean space. If all the spheres are derived from one by a group of translations, the cell (Dirichlet region) for the densest packing is a rhombic dodecahedron, while the cell for the thinnest covering is a truncated octahedron (4, 6, 6). In other words, the centers in the densest lattice-packing form a face-centred cubic lattice, while the centers in the thinnest lattice-covering form a body-centered lattice. The latter result is hinted at on p. 189, but Bambah's proof was written just too late to be mentioned in the book.

For nonlattice arrangements, however, the problems present difficulties that have not yet been overcome. A solid sphere can be surrounded by twelve equal solid spheres, all touching it, in various ways, and it seems plausible that (as James Gregory believed) the twelve might be bunched together in such a way as to leave room for a thirteenth, still touching the original one. However, some very recent work of Schütte, Boerdijk, and van der Waerden shows this to be impossible; in fact, the distance from the center of the original sphere to the center of the thirteenth surrounding sphere seems to be at least

$$7/3 \cdot 3^{1/2} = 1.347 \cdots$$

times the diameter.

The book contains many other interesting results and discussions that cannot be mentioned here for lack of space. Some are simple enough for high-school students to appreciate and yet sufficiently unfamiliar to delight a professional mathematician. Moreover, there are enough unsolved problems to provide material for research for many years to come.

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THE NATIONAL SCIENCE FOUNDATION PROGRAM IN MATHEMATICS

ALAN T. WATERMAN

I am very happy to have this opportunity of meeting with the members of the American Mathematical Society and the Mathematical Association of America here in our neighbor-city, Baltimore.

The Foundation considers itself fortunate indeed in the advice and assistance we have had and are enjoying from members of this group. As you know, Professor Marston Morse is one of the twenty-four members of the National Science Board. Professor William Duren of Tulane University was most helpful in the initial planning of our mathematics program. We are pleased to have as program director for mathematics, your colleague, Dr. Leon Cohen, who came to us this year from Queens College, where he had been Associate Professor of Mathematics since 1947. Dr. Cohen has been with us only a few months and in that time he has made real progress toward the development of a National Science Foundation program in mathematics. Perhaps it may interest you to know something of that program. NSF activities in the field fall in several categories: conferences, institutes, grants, research seminars, the regional development of mathematics, support of publications, fellowship support.

Last May a Conference on Fiber Bundles and Differential Geometry was held at Cornell University, under the chairmanship of Dr. Norman Steenrod of Princeton. As you know, the Foundation sponsored, jointly with the American Mathematical Society, the Summer Institute on Lie Groups and Lie Algebras at Colby College, under the chairmansip of Dr. Nathan Jacobson of Yale University. I believe I am correct in saying that there was considerable enthusiasm for this institute as well as the hope that it may be the first in a series. It is my understanding that plans are now under way for a second summer institute on the functions of several complex variables, under the leadership of Dr. Salomon Bochner of Princeton.

A Foundation grant to the National Research Council for a study of training in, and the subject matter of, applied mathematics re-

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An address delivered at the dinner at the Joint Meeting of the American Mathematical Society and the Mathematical Association of America in Baltimore, December 30, 1953; received by the editors February 5, 1954.

sulted in two conferences: one which was held in conjunction with the October meeting of the American Mathematical Society in New York on training in applied mathematics, and a conference that was held in conjunction with the November meeting of the AMS in Evanston on the subject of applied mathematics. The survey was brought about through the efforts of Dr. F. J. Weyl of the Office of Naval Research.

I am sure that most of you are by now acquainted with the summer institute for teachers of collegiate mathematics that was sponsored by the National Science Foundation last summer. Under the very able direction of Dr. Burton Jones, the institute was conducted for six weeks at the University of Colorado in Boulder. The purpose of this institute was to acquaint teachers of mathematics, particularly from the smaller colleges, with some of the latest developments in the field. The institute attracted about 75 participants, representing all sections of the United States. Members of the institute lived in the residence halls and their meals were served in the college dining room. These arrangements made it possible for the participants in the conference to get to know each other very well and to hold many informal discussions of mutual problems. Judging from the enthusiastic comments we have had, the choice of Professors Wilder and Artin as regular lecturers was a most happy one, and great appreciation was also voiced for the several visiting lecturers who took part in the conference.

The apparent usefulness of this experimental institute has prompted the organization of two similar institutes for next summer. With the advice and assistance of the National Research Council Committee on the Regional Development of Mathematics, the Foundation will sponsor similar institutes at the University of Oregon and the University of North Carolina, following the pattern established at Boulder last year. An experimental venture of another type will also be tried next summer when the Foundation sponsors an institute in mathematics for high school teachers. This institute will be one approach to what appears to be an urgent problem with respect to mathematics in the nation's high schools. I should like to say more on this point a little later.

Another activity in which the Foundation is keenly interested is the regional development of mathematics. This interest is not limited to mathematics, of course. Under the provisions of its Act, the Foundation is directed "to strengthen basic research and education in the sciences, including independent research by individuals throughout the United States, including its Territories and possessions, and to avoid undue concentration of such research and education." Using mathematics as an example, I shall describe one way in which the Foundation is attempting to meet this objective. Let us assume a situation in a region of the United States where there exists a university where mathematics has not had effective support. The university has, however, shown its intentions by making good permanent appointments to the mathematics department, by scheduling teaching loads to permit research, and by providing a good library. The Foundation then feels justified in considering assistance by a grant for a research seminar. The grant would be sufficiently large to establish the group for a long enough time to afford stability. The seminar would involve several senior mathematicians with overlapping scientific interests, young Ph.D.'s as associates, and graduate students as assistants. Its purpose would be two-fold: the solution of problems drawn from some broadly conceived field, and the training of young men for research. The first such seminar is being held at the University of Washington, which has been awarded a grant of $30,000 for a period of 21 months.

In addition to providing assistance for group efforts, such as conferences, institutes, and seminars, the Foundation is also interested in the individual. Although the Foundation's assistance to individuals normally takes the form of a research grant to an institution for the purpose of aiding research being carried on by one of its staff members, we believe in maintaining flexibility in our operations, to the extent possible, and therefore will consider proposals that may lie outside the regular pattern. For example, in special cases of high merit we may give consideration to proposals that will make possible a leave of absence from teaching for a year in order that an individual may undertake or complete a research project.

A problem that cuts across all the fields of science is the whole matter of scientific information. Such matters as publication, dissemination, cataloging, indexing, abstracting, and so on, are common to the various disciplines of science. Mathematics has an added problem, however, in the complexity and costliness of reproducing mathematical copy. The Foundation has been of some small assistance to the Society in an experimental effort to change from letterpress to photo-offset in the publication of its various journals. I venture to suggest, however, that what is needed is an even broader approach to the problem, with perhaps the experimental use of some of the highly efficient new machines that are beginning to appear on the market. Perhaps the applied mathematicians might join with other scientists and engineers in applying their special skills and ingenuity to some of these vexing problems of communication.

Another form of NSF assistance in the field of publication is the grant that will make possible the publication of monographs. It has been suggested in several quarters that the more leisurely monograph, expounding a wide field, would provide a welcome departure from the pressures and limitations of writing for journals under present space restrictions. The NSF is glad to have had a part in encouraging this trend through a grant to Professor Alonzo Church, which will aid him to bring out the second volume of his *Introduction to mathematical logic.*

Turning now from various types of assistance made possible through the grants procedure, I should like to say a few words about our fellowship program, the second of the two major operating programs of the NSF. The National Science Foundation Act authorizes the Foundation "to award . . . scholarships and graduate fellowships for scientific study or scientific work in the mathematical, physical, medical, biological, engineering, and other sciences, at accredited nonprofit American or nonprofit foreign institutions of higher education, selected by the recipient of such aid, for stated periods of time." The Act further provides that "Persons shall be selected for such scholarships and fellowships from among citizens of the United States, and such selections shall be made solely on the basis of ability" but in cases of substantially equal ability, where there are not enough awards to go around, selection shall be made in such a way as to result in as wide a geographical distribution as possible.

I have followed rather closely the language of the Act because it covers a number of points on which there has been considerable discussion. One of the points most frequently mentioned, for example, is that NSF fellows have tended to concentrate in a relatively small number of institutions, and these institutions are the ones with the least need for additional support. It will be noted, however, that although the Act makes it possible for the Foundation to take the matter of geography into account in the award of fellowships, the choice of institution is entirely the fellows' and, I believe, quite properly beyond the control of the Foundation. As one of your colleagues has so aptly put it, "This is the American spirit of competition and free enterprise. . . ." It is also in the spirit of freedom for research.

Most of you are, I expect, aware that this question has been raised with respect to the concentration of mathematics fellows. We are completely sympathetic with the sincerity and zeal of those who are concerned with the strength and progress of their own institutions. The matter at issue is whether the Foundation, as a Federal agency, should control in any way a fellow's choice of institution. The majority of those who have expressed themselves to the Foundation on this point feel very strongly that the students' freedom of choice should be protected and maintained. I cannot help believing that solutions to the problems of individual institutions are to be found in measures other than attempts to control the students' choice of institution. In the best interests of progress of research and progress of the individual this must surely be the answer, especially when one considers the high degree of selectivity in the naming of these fellows who were chosen on the basis of nationwide competition.

It is a matter of some interest to note that at least 15 Nobel laureates were students at Cambridge University, 9 of them fellows of Trinity College. The University of Göttingen is also distinguished by the unusual number of Nobel laureates it has produced. Three of them, Heisenberg, 1932; Fermi, 1938; and Pauli, 1945, were associates of Max Born. One wonders how the history of science might have been altered had any of these brilliant minds been diverted from their free choice of institution and teacher.

The statistics with respect to the number of NSF fellows in mathematics are interesting from two standpoints: They indicate that a substantial number of fellowships were awarded in the field of mathematics, in competition with all other fields; and secondly, they show that from an over-all point of view the number of predoctoral fellows in mathematics supported by the Foundation in 1952–53 constituted only 1.8 per cent of the total group of mathematics graduate students throughout the country—a number too small to have very far-reaching effects on the distribution of students among the mathematics departments of the country.

For the academic year 1952–53, 297 applications in the field of mathematics were received, representing 10 per cent of the total number received (2977). The 58 fellows who accepted fellowships in mathematics constituted 10.1 per cent of the total number of fellows (575) supported by NSF in the academic year 1952–53. For the academic year 1953–54, 351 applications in the field of mathematics were received, representing 10.6 per cent of the 3298 applications received. The 55 fellows who were awarded fellowships in mathematics constituted 10.5 per cent of the total number of fellows (522) studying under NSF support during the current academic year. Thus mathematics compares very favorably with other fields in the number of fellowships awarded, being topped only by chemistry and physics last year; and by chemistry, physics, and engineering in the current year.

It would be gratifying if I were able to report that some comparable action is taking place with respect to undergraduate study and study in the high schools. Although the Foundation is authorized to award scholarships as well as fellowships, the number and complexity of the problems associated with the award of scholarships in science, especially on a national basis, are such that we have felt it desirable to study this situation thoroughly before taking action. Last fall our Division of Scientific Personnel and Education called an informal meeting of representatives of the educational associations for the purpose of seeking their advice on this knotty problem. The responses of the members of the group were interesting. In only a few instances, for example, was it felt that the stipend associated with the scholarship would prove decisive in influencing a student to attend college. The majority of the group felt that problems of identification and motivation were even greater than the question of furnishing financial aid to undergraduates. The two days of discussion furnished members of the Foundation staff with a number of valuable points to be considered in the formulation of a scholarship program.

The problems of identification and motivation take one back to the secondary school level where, I understand, a critical situation exists with respect to mathematics, and indeed all the sciences. At a recent meeting of our Divisional Committee for the Mathematical, Physical and Engineering Sciences, considerable time was devoted to a discussion of the dearth of high school science teachers. The committee felt that the high school teaching of mathematics and science is of such great importance to the scientific manpower and general welfare of the country that a national program should be developed to improve the situation. Although the problem is too large to be dealt with wholly by any single agency, the Foundation may at least be helpful in pointing the way.

The colleges and universities have long been aware of the difficulties of trying to teach science and mathematics to students who have been inadequately trained in mathematics in high school. As the Nation's scientific manpower needs become more acute, the problem is being recognized in other quarters, also. Mr. Chester H. Lang, vice president of the General Electric Company, addressing a G-E summer institute program for high school teachers of science and mathematics last summer, pointed out that the present scarcity of science high school students threatens to form a bottleneck that will choke off normal American growth. He reported that only 1.4 per cent of the country's high school students study solid geometry; only 1.6 per cent learn trigonometry; and but one-half of one per cent enroll in college algebra. Mr. Lang predicted that if this situation continues, "the technological progress we expect will slow to a walk."

The general problems associated with secondary school education, insofar as these are a concern of the Government, are primarily within the purview of the Department of Health, Education and Welfare. The Foundation's participation here is necessarily limited to the fields of mathematics, science and engineering, but we expect to cooperate fruitfully in the broader problems. In particular, I believe that the Foundation's strongest role should be played by studying these problems separately for the different scientific disciplines, e.g., in mathematics by consultation with the mathematicians, in biology with the biologists, and so on. The summer institute for high school teachers of mathematics, which the Foundation is sponsoring at the University of Washington next year, is a small experimental effort. I know that many of you here have given a great deal of thought to some of these problems and we should welcome your comments and your suggestions as to other ways in which the Foundation can strengthen the teaching of mathematics in the secondary schools.

If you are interested in how the rate of effort in mathematics compares with the total Foundation effort, I will mention a few figures. In fiscal year 1954, the NSF budget for the support of research totals approximately $4 million—half of which was allocated to the mathematical, physical, and engineering sciences. Of this, the allocation to mathematics is approximately $161,000, which includes about $12,000 for conferences and travel to foreign meetings.

If one adds the funds expended, or to be expended, for the advancement of mathematics through the Division of Scientific Personnel and Education, one finds $142,000 for fellowship awards, and $54,000 for conferences and for the mathematics section of the National Register. These figures, added to the previous ones, give a grand total of $357,000. Thus a little over five per cent of the total NSF funds are being used for the support of mathematics in various ways. Undoubtedly this is not as much as any of us could desire, and yet when we consider among how many fields the NSF dollar must be split, probably mathematics is not faring too badly.

It is especially gratifying to us in the Foundation that we should be supporting a lively program in mathematics; for mathematics, in a sense, bridges the gap, real or imaginary, which exists between the sciences and the humanities. The exigencies of modern technology have attracted many of the sciences away from their original orbits in the realm of natural philosophy. Mathematics, too, has had its

MATHEMATICS IN ECONOMICS

WASSILY LEONTIEF

1. **Gibbs and mathematical economics.** American economists have a good and special reason to honor J. Willard Gibbs. The late Professor Irving Fisher—the author of the earliest monograph on Mathematical Economics published on this side of the Atlantic and one of the truly great economists this country has produced—was a pupil of Gibbs. He was in 1929 the first to represent social sciences in this series of memorial lectures. The second was Professor Edwin B. Wilson, mathematician and economist, also one of Gibbs' immediate disciples, and author of the early treatise on Vector Analysis based on his teacher's original lectures on that subject.

Professor Fisher and Professor Wilson were leading spirits in the organization—twenty-three years ago—of the international Econometric Society which now unites 2500 economic statisticians and economists who claim the ability to speak—or at least to understand when spoken to—the "language of mathematics" which Josiah Gibbs used with such compelling and poetic power.

I did not know Gibbs and I am not a mathematician. I cannot present to you personal reminiscences about this great man nor am I able to develop before you any one particular application of mathematics to economics—which could possibly be of technical interest to a professional mathematician. I will try instead to survey the logical structure of the present day economic theory emphasizing formal aspects of some of the problems which it faces and pointing out the mathematical procedures used for their solution. The views to be presented are, of course, not necessarily shared by other economists. Even leaving out those who feel with Lord Keynes that mathematical economics is "mere concoctions," theoretical disagreements and methodological controversies keep us from sinking into the state of complacent unanimity.

2. **The general structure of economic theory.** The object of economic analysis is the observed, or at least the observable, economic process. The typical variables in terms of which an economic system is described are the amounts of various goods and services produced, consumed, added to and subtracted from existing stocks, sold and

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The twenty-seventh Josiah Willard Gibbs Lecture, delivered at Baltimore, Maryland on December 28, 1953, under the auspices of the American Mathematical Society; received by the editors January 11, 1954.

purchased; also the prices at which these purchases and sales are made.

The available quantities of natural and human resources, the state of technical knowledge and the nature of consumers' preferences (with those, in our modern much regulated economy, one must mention also the aims and preferences of the regulating governmental authorities)—all described within the setting of a specific institutional framework—constitute what might be called the operating conditions of the particular economic system. These are the "data" which in verbal analysis are used to explain the "unknown" outputs, employment, prices, investments, and so on.

Translated into mathematical language this means that the available quantities of natural and human resources, the state of technical knowledge and consumers' preferences determine the structure of equations (or inequalities) which in their turn determine the values taken on by what we choose to define as the dependent "variables" of the economic system.

The first systematically formulated mathematical theory of general Economic Equilibrium was constructed just about seventy-five years ago by Léon Walras [1]. He incorporated in it much of the so-called classical theory developed in the writings of the great English and French economists of the late eighteenth and early nineteenth century. Some essential pieces of the conceptual apparatus used by Walras—such, for example, as the concepts of supply and demand functions and the notion of diminishing marginal utility—were already cast in mathematical form by such men as Daniel Bernoulli [2], Augustin Cournot [3], and E. J. Dupuit [4].

Elaborated and extended by Vilfredo Pareto [5] and his contemporaries and successors, the general theory of economic interdependence is gradually being combined—into what promises to become a unified logical structure—with two other fields of analytical inquiry, the theory of market mechanism and the analysis of the behavior of an individual firm and of a separate household.

3. **Maximizing behavior.** It is in this latter connection, in explanation of the operation of the ultimate decision-making units, that the common notion of "economic behavior" finds its principal analytic application.

Consider the profit maximizing firm. It purchases or hires certain commodities and services and utilizes them for the production of other commodities or services. The production process itself can be described as a transformation of one set of variables—the inputs, into another—the outputs. The quantitative relationships between the inputs and the outputs are determined by the set of all available technological alternatives.

The outlays, the costs, incurred by the firm can obviously be considered as a function of the input combination used, while its gross revenue depends upon the amounts of its outputs. Among all the input-output combinations technically attainable, the firm chooses the one which maximizes the difference between its total costs and revenue.

In a simple case in which all available transformation possibilities are stated in the form of one or more well behaving "production functions" with continuous derivatives throughout the entire relevant range, a local maximum can be described by a set of simple equations involving its first partial derivatives and parameters entering the profit function such, for example, as the prices of all commodities sold and purchased.

It is not surprising that these conditions were discovered and stated by some economists verbally without any recourse to mathematics. A correct formulation and interpretation of the secondary conditions for a maximum, involving inequalities in higher derivatives, had, however, to wait for the introduction into the argument of formal calculus.

The problem becomes more intricate as soon as the well behaving continuous production functions are replaced by the more realistic description of technical input-output relationships involving linearities, discontinuities, and inequalities. Then the question concerning the optimization conditions in the small is replaced by their study in the large. Under the name of "linear programming" much advanced work has, for example, been done recently on the problem of determining maxima with the constraining transformation functions stated in the form of a set of positive vectors; the positive and negative components of each vector describe in this case the sets of outputs and, respectively, inputs corresponding to the operation on a unit level of one particular kind of productive activity. Differential calculus and elementary algebra—the two traditional tools of the mathematical economist—are being thus replaced or at least supplemented by those of topology and matrix algebra.

The explanation of consumer's behavior is developed along similar lines. A household like a firm has an income (derived from the sales of the services of persons or property rights) and an outlay; to the transformation functions of the firm there corresponds the utility function of the household. It describes the level of satisfaction corresponding to the amounts of goods and services consumed.

Within the constraints imposed by its budget, the household is supposed to select a combination of goods and services which brings it to the highest level of satisfaction. In early theories, utility was treated as a measurable quantity. On closer examination, its cardinal measurement turned out to be neither necessary for formulation and solution of the maximum problem at hand nor, essentially for that very reason, operational in terms of actual experience.

Consider two individuals facing identical budgetary restrictions. If one of them derives from any combination of commodities consumed, say, twice as much satisfaction as the other, both will obviously find their respective utilities maximized by exactly the same sets of purchases. Insofar as a consumer's observed movements through commodity space constitute the only objective source of information about the shape of his utility function, ordinal comparison of its different levels is all that can be achieved or required for explanatory purposes.

This is where the matter stood till the recently revived interest in the old eighteenth century problem of choice under conditions of uncertainty led to renewed attempts to rehabilitate the cardinal utility function. The argument hinges on the assertion [6] that from the point of view of "rational" behavior, if,

(a) $U(X_1)$ and $U(X_2)$ are the utility levels associated in the mind of a decision-making consumer with certain but alternative possession of the two specific commodity combinations, X_1 and X_2, and

(b) p is a true positive fraction such that

(c) this consumer, when offered the choice between the "chance with the probability p of possessing X_1" and "the chance with the probability $(1-p)$ of possessing X_2" will find both these offers to be equally desirable, then,

$$\frac{U(X_1)}{U(X_2)} = \frac{(1-p)}{p}.$$

Once this is admitted, a cardinal comparison of utilities must obviously be accepted as operationally feasible. Whether a particular individual actually behaves in accordance with this assertion or not can be empirically tested—through introduction of a third commodity combination, X_3, with an accompanying probability, q. Two choices, one between the chances involving $U(X_3)$ and $U(X_1)$, and another involving $U(X_3)$ and $U(X_2)$, should lead to measures consistent with the comparison of $U(X_1)$ and $U(X_2)$ as shown above. If they do not, the subject of the experiment is declared to be "irrational." The reference to "rational" or, should I say, "economic" behavior as used in this connection is intended to justify the acceptance of a crucial proposition "ex definitione." Substantively, it denies the phenomenon of the pleasure (the utility) of gambling by disallowing the possibility of using a utility function of the more general form, such, for example, as $U(X_1, p)$.

In this, as in many other similar instances, the economist must be prepared to make up his mind whether he is aiming at a positive explanation of observed facts or at setting up normative rules for, in some sense, "reasonable" behavior and tracing out their logical implications.

In the discussion of public economic policies—in contrast to the analysis of individual choice—the normative character of the problem has been clearly and generally recognized. Here the mathematical approach has crystallized the analysis around the axiomatic formulation of the (desirable or conventional) properties of the "social welfare function." Social utility is usually postulated as a function of the ordinally described personal utility levels attained by each of the individual members of the society in question.

The only other property on which something like a general consensus of opinion seems to exist is that "the social welfare is increased whenever at least one of the individual utilities on which it depends is raised while none is reduced." Without any furthermore stringent limitation on its possible shape, such a social welfare function allows only a partial ordering of all possible combinations of individual utility levels. A much more specific description of its properties would have to be required if the social welfare function were to reflect—in axiomatic formulation—concrete normative judgments pertaining, for example, to the problem of income distribution. The struggle to increase the utility levels of some groups of individuals at the cost of reducing the welfare of others constitutes, no doubt, the core of much of the present day politico-economic controversy.

The important contribution of the mathematical approach to our thinking on such controversial issues consists in showing how difficult it actually is to formulate in concise operational terms any specific normative attitude toward questions of public welfare in general and the problem of equitable distribution of income in particular.

4. Consistency criteria in the theory of interdependent choices. The analysis of the behavior of individual firms and households is and—if it has to have explanatory rather than normative significance—should be not more than a direct translation into concise mathematical language of problems of maximizing choice as seen from the actual decision makers' point of view. The restraining relations and parameters which the economist assumes as "given" must, of course, be precisely those which the household or firm actually considers as being independent of its action, and the set of variables—the optimal combination of which the theorist explains—must indeed include all those, and only those, on which the real economic units actually operate in putting into effect their profit or, respectively, utility maximizing decisions.

So long as one does not radically widen the conventional universe of economic discourse, the invariance of technological transformation functions in respect to changes in specific input combinations can be taken for granted. The same, however, cannot be said about the functions and parameters which—although they are treated as fixed constraints in the explanation of individual maximizing behavior—within the larger framework of the general theory of economic interdependence turn up in the role of dependent variables rather than of "given" data.

Farmer Jones, when he decides on the most profitable number of hogs to grow, takes into account the market price at which they can be sold. In doing so, he most likely considers that price as "given," i.e., to be practically independent of the specific outcome of that decision. In explaining farmer Jones's output, the economist accordingly treats the price as one of the parameters entering the solution of the corresponding profit maximization problem.

In his very next step, in presenting the general equilibrium theory (which I will presently take up), the economist lists all prices—including the price of hogs—among the unknowns to be determined through the solution of an appropriate system of equations. In particular, he then proceeds to explain, in terms of that system, why the price of hogs would fall if all farmers, say, for experimental purposes, had produced and thrown on the market 10 per cent more hogs than before. Another argument based on the same general equilibrium equations shows that, within the range of output variations accessible to him, farmer Jones's belief in his own inability to affect the market price of hogs to any appreciable extent is indeed entirely correct. If, however, it had turned out—again within the framework of the general equilibrium theory—that farmer Jones's individual action could have influenced the price of hogs—as indeed would have been the case had he owned half of all the hogs in the country—the entire analysis in both its parts would have been false. The explanation of farmer Jones's maximizing behavior, because it was derived from an assumption that now proved to be inconsistent with the implication of further general equilibrium analysis based on that very explanation, the general equilibrium analysis, obviously would be false for the same reason.

All problems dealt with in the analysis of market behavior lead to such questions of theoretical consistency. Their logical structure is frequently quite subtle and the circular test outlined above is difficult to apply without recourse to mathematical formulation.

The analysis of duopology and oligopoly, i.e., of the relationships between two or few mutually interdependent sellers, also the explanation of bilateral monopoly, a situation in which a single seller faces a sole buyer, each clearly and appreciably affecting by his actions the others' profit, all lead to the same theoretical problem—the explanation of maximizing behavior of two or more mutually interdependent units.

Beginning with Augustin Cournot [3], that is, for over a century, mathematical economists have wrestled with that question without apparent success. The modern Theory of Games [7] has contributed greatly toward a more concise formulation of the issues involved, but an acceptable theory of interdependent maximizing behavior has yet to be offered. As in the discussion of the cardinal measure of utility, an elaboration of the logical consequences of arbitrary normative assumptions here, too, has occasionally been mistaken for a solution of the positive problem. Possibly, such a solution can be even shown not to exist.

5. The theory of general interdependence. The Theory of General Equilibrium—the analysis of the mutual interdependence of all the producing and consuming units making up a national economy or—if one wants to take into consideration international trade—the world economy as a whole, makes up the core of modern economic theory.

The simplest standard model of the general equilibrium system—stripped of all optional equipment and adornments—is designed to explain the determination of the (time-) rates of production (sales) and consumption (purchases) of all commodities and services by each of the individual decision-making units as well as the prices at which all these inputs and outputs are traded.

The explanation is presented in the form of a system of simultaneous equations. Their number just suffices to determine the values—unique or multiple—of the unknowns. All sales and purchases of each particular commodity are supposed to be transacted at the same price and the prices of all commodities and services are to be such as to make the combined output (supply) of each commodity by all the units equal to its aggregate input (demand) by all the units.

The quantity of each commodity produced or consumed (it could be both) by any unit has already been shown to depend—through the budgetary restriction—on prices; its own as well as those of the other goods. The "supply" and the "demand" functions, so frequently referred to by the economist, are meant to describe this dependence; their shape is obviously implicitly determined by the equations (or inequalities) which in the description of its maximizing behavior served to determine the optimal position of the individual decision-making unit in the commodity space.

Although some of its constituent equations are thus based on the satisfaction of certain maximizing conditions, the general equilibrium system itself cannot legitimately be thought of in any other but quasi-mechanical terms. This does not mean that an eighteenth century believer in the Invisible Hand or his present day counterpart, the modern welfare theorist, could not have legitimate interest in finding out whether the actual economy—as described by the set of the general equilibrium equations—does or does not satisfy the normative social welfare criteria of his particular choice.

Let me add that under certain ideal conditions, the outcome of the automatic operation of the competitive price mechanism, as reflected in the general equilibrium system described above, can be shown—so far as the organization of *production* is concerned—to be identical with that which would be achieved by an omniscient and all-powerful planning committee of efficiency experts. In a state satisfying the Walrasian equilibrium equations, the total output of no commodity can be increased and the input of no scarce primary resource diminished without reduction in the output of at least one other commodity or an increase in the input of at least one other commodity or an increase in the input of at least one other scarce primary resource.

In other words, if outputs are measured as positive and inputs as negative quantities in the many dimensional commodity space, the actual equilibrium position of a competitive economy is represented by a point located on the hull of the compact space comprising all input-output combinations attainable to it on the basis of the given transformation functions; each vector connecting any two points on that hull necessarily contains components of opposite signs.

This obviously applies to any optimal position which an individual profit-maximizing enterprise would choose among all the input-output combinations attainable to it.

The truth of that theorem in the case of the competitively operating economy as a whole follows from the fact that it can be shown to apply to the sum of the optimal input-output vectors of any group of profit-maximizing enterprises simultaneously operating within the same price system.

This makes it possible for the economist, when he studies the quantitative aspects of the input-output relationships within the theoretical framework of a competitive general equilibrium system, to disregard its subdivision among the many individual enterprises and to speak of an "industry," groups of industries, and even of the economy as a whole as if it were a large single enterprise.

6. **Dynamics.** The quasi-mechanical nature of the economic system as a whole becomes particularly clear when, as has occurred over the last twenty-five years, the mathematical economists engaged increasingly in exploring its dynamic properties.

The static, essentially timeless system of general equilibrium equations described above is an idealization of limited empirical validity. The technical transformation functions, for example, in order to reflect more closely the conditions of actual production, should contain the values of at least some of the variables as related to different points in time: This year's harvest depends on last year's sowing.

Consider, for instance, the process of economic growth. Insofar as it involves the accumulation of capital, its explanation leads back to the fundamental observation that the output of a finished product—expressed as a rate of flow, per unit of time—cannot be described as depending only on the flow rates of requisite inputs. It requires also the presence of certain specific stocks: stocks of buildings, stocks of machinery, inventories of raw materials and of intermediary semi-finished products. But stocks can mostly be described as flow rates (or differences of flow rates) integrated over time.

The dynamic process of capital accumulation in its simplest form can be described and explained in ordinary language. With the introduction of other kinds of dynamic relationships, the theoretical system becomes unmanageable without the use of mathematics. The theory of the so-called "business-cycle," that is, of the fairly regular succession of ups and downs in output, employment, trade and prices experienced by all advanced western economies is a case in point. From the time the first major nineteenth century depression hit England in 1819, economists have searched for a systematic explanation of that phenomenon. But not before the nineteen thirty's when the mathematical economists became interested in the subject was there introduced into its discussion the notion of self-generated periodic fluctuations corresponding to the pairs of complex roots which frequently appear in the solution of difference or differential equations [8].

No wonder that from that time and up to two or three years ago, when the theory of games and problems of linear programming came in vogue, dynamic general equilibrium theory has been the favored hunting ground of mathematically inclined economists. Integral and differential, difference and mixed difference and differential equations, phase graphs of linear and nonlinear oscillating systems—all of these and many other tools of applied analysis have found their place in the recent discussion of economic dynamics. Having likened the austere outlines of Walras's original general equilibrium system to a standard, stripped down "model T," I cannot help but compare some of the latest dynamic models with the super-deluxe editions of hard top convertibles equipped with everything from white-wall tires to a concealed bar.

7. Paucity of operationally significant conclusions. One has, unfortunately, to admit that neither the simpler type of economic theory nor its most modern dynamic versions have brought us very far along the road toward detailed explanation, not to say prediction, of the specific states of the actually observed economic system.

Seldom, in modern positive science, has so elaborate a theoretical structure been erected on so narrow and shallow a factual foundation. Traditionally—and that tradition still prevails among mathematical and nonmathematical economists alike—"pure" theory has not been implemented with empirical determination of any of the numerical parameters involved. As can be seen even from the sketchy outlines presented above, all empirical assumptions on which such theories are based are qualitative in character and, at that, they are quite vague and general. So are the few operational propositions at which pure economic theory arrives.

Paul Samuelson, who more than anybody else contributed to the systematic codification of modern economic theory and a clarification of its logical structure [9] pointedly brings out the parallelism between the method used by economists to derive certain meaningful implications of maximizing behavior and the elegantly general modes of reasoning found in J. Willard Gibbs' celebrated treatise *On the equilibrium of heterogeneous substances.* The following simple, but typical argument from the theory of consumers' behavior will show what I have in mind.

Let the elements of a non-negative row matrix, X, represent the quantities of commodities which a household, with a dollar income, r, can purchase at prices described by the element of the column matrix, P. Under the budgetary constraint,

$$XP_i = r_i, \tag{1}$$

where the subscript, i, is used to identify some specific price income situation, the household will choose to purchase those particular amounts, X_i, which will maximize his utility $U(X)$. About $U(X)$, we only know (a) that it is a nondecreasing function of X and (b) that—since utilities can be compared only in the ordinal sense—it admits transformation by any increasing function, $F(U(X))$.

Let X_1 and X_2 represent the optimal consumption patterns corresponding to two different price-income situations P_1, r_1 and P_2, r_2. If

$$X_1P_1 = r_1 \geqq X_2P_1, \quad \text{then necessarily} \quad U(X_1) > U(X_2), \tag{2}$$

since otherwise when placed in the price-income situation P_1, r_1, the consumer would purchase X_2 rather than X_1.

For analogous reasons,

$$U(X_1) > U(X_2) \quad \text{implies} \quad X_2P_2 = r_2 < X_1P_2. \tag{3}$$

It follows that

$$(X_1 - X_2)P_1 \geqq 0 \quad \text{implies} \quad (X_1 - X_2)P_2 > 0. \tag{4}$$

Propositions (2) and (3) make it possible in some, but unfortunately not in all, cases to infer from the change in the observed price-purchase pattern of the consumer to the direction of the corresponding change in his level of welfare. Proposition (4) imposes certain limitations on the shape of individual demand functions.

Analogous arguments make it possible to impose similar empirical limitations on the shape of the behavior equations of profit maximizing firms.

Insofar as the individual demand and supply functions enter into the analysis of the economy as a whole, these limitations carry over into the general equilibrium system as well. This applies, in particular, to the input-output relationships characterizing the operations of the productive sectors of the economy. As mentioned before (see p. 223), whenever the economy operates within the framework of competitive pricing, these relationships are identical with those which would have prevailed within one single large profit maximizing enterprise operating on the basis of the same technological horizon. This is why the "pure" general equilibrium theory seems to yield richer empirical results in respect to production than when it deals with household consumption. It also explains why in studying the quantitative aspects of an economic system one sometimes legitimately disregards the details, or should I say accidents, of its particular institutional organization and conducts the entire analysis in terms of only such basic data as the supply of primary resources and the "state of the arts," i.e., the technologically given transformation functions. Reduced to these simplest terms, the same general theoretical propositions apply to the highly advanced American private enterprise economy, the centrally planned Soviet system and, say, the economy of an isolated primitive tribe.

With all that, or rather because of that, the legitimately harvested empirical yield of the general equilibrium theory is very limited. An interesting attempt was made by Abraham Wald [10] to impose further limitation on the admissible shape of the general equilibrium equations by introducing the requirement that all prices and quantities as determined by it have to be positive. On closer examination the operational implications of this argument turn out to be disappointing: Any number of alternative sets of sufficient conditions for such a result can be stated, but they obviously would be of little significance from the economic point of view. The necessary conditions for such a positive solution applying to any particular observed price output situation would, on the other hand, be so special that even if stated explicitly they also would be devoid of empirical interest.

Furthermore, the entire question is misplaced from the point of view of the purpose it is intended to serve. One of the interesting empirical questions which an economist occasionally has to answer is whether with a given combination of (virtual) operating conditions an economic system would be capable of yielding positive outputs, i.e., whether it could exist at all. In posing the question, one obviously must allow for the possibility of a negative answer.

In dynamic analysis, a similar search for additional limitations on the empirically admissible shapes of the basic quantitative relationships has produced the suggestion that only convergent systems leading to stable solutions should be considered. If that proposal were taken seriously, how would we go about explaining the rapid and apparently limitless growth of the modern western economies?

8. **Indirect inference.** Please note that while questioning the validity of such quasi-empirical generalization, I do not impugn the error of circular reasoning to those who make them. If the invisible, but indirectly inferred structural properties of a theoretical system were used to "explain" only those of its directly observed characteristics from which these properties were derived in the first place, the argument could indeed be rejected as heuristically useless. When, however, the indirectly inferred structural properties of the theoretical system serve to derive new factual propositions, not obviously and immediately related to the first set of empirical statements on which the original inference has been based, the explanatory power of the theory has been clearly increased.

If indirect statistical inference is to be used to obtain the numerical parameters for our system, methods of modern mathematical statistics should obviously supply the tools for such an undertaking. In his 1929 Gibbs Lecture, Irving Fisher refers to "smoothing of statistical time series, correlation and probabilities" as, at that time, newly introduced applications of mathematics to economics. The original systematic attempt to derive a "statistical demand curve" for an individual commodity was made by Henry Moore as early as 1917 [11]. Soon it was followed by a whole series of similar studies.

The first, let me call it the "unsophisticated" phase, of econometric work, aimed at an indirect determination of the structural parameters of the economic system through statistical analysis of the behavior of its variables, reached its high point in the works of Henry Schultz [12], Paul Douglas [13], and Jan Tinbergen [14].

It was characterized by rather careful collection and organization of primary statistical material and straight forward—some might say indiscriminate—application of the "least square" curve fitting techniques in calculation of the actual parameters. There was little in it of mathematical interest (except for Tinbergen's use of difference equations for empirical determination of the complex components of fluctuating time series) and the empirical results obtained appeared to be of rather doubtful significance. After fitting a first, second, or third degree parabola to a price-quantity scatter, one did not know whether it represented the supply or the demand curve for the commodity in question or possibly some weighted average of the two.

It was only natural to make the crudity of the statistical procedure used responsible for such disappointing substantive results. The late thirties and the forties witnessed an unprecedented concentration of intellectual effort on problems of methodology [15]. Far from being satisfied with the simple adaptation of recent advances in mathematical statistics, the new econometric school made a number of original contributions to the theory of stochastic systems and the methodology of indirect statistical inference. Such significant advances as the theory of identification (analyzing the relationship between the statistical parameters of an observed stochastic system, on the one hand, and the constants of the underlying theoretical model, on the other) can rightfully be said to have issued from that series of methodological investigations.

The theory of probability and many other modes of mathematical reasoning found in all these studies a most varied and fertile application. I will not invite you now, however, to consider them; they belong in the field of general statistics rather than that of economics. Moreover, the positive contribution which these advanced methods were able to make toward actual empirical determination of the specific quantitative properties of the observed economic system proved to be quite limited, hardly greater than that made by the primitive methods used in studies belonging to the first, unsophisticated phase of that econometric movement.

The explanation of this disappointing result lies, I think, in the fact that for a study of a set of quantitative interrelationships as complex as those underlying a modern economy indirect statistical inference, however refined methodologically, simply will not do [16]. Statistical reliability measures, if properly applied to even the most favorable situation—so far as primary information is concerned—yield confidence limits so far apart as to negate the empirical usefulness of the numerical parameters obtained.

It is as if we were asked to reproduce the blueprint of a complicated motor on the basis of our knowledge of the general principles of operation of internal combustion engines and no other specific information but that conveyed by the few dials located on the dashboard and possibly the noise coming from under the closed hood. And as if that were not difficult enough, the structural characteristics of the engine the economist is studying are known to change under the impact of its continual operation.

The task as presented can hardly be accomplished. It certainly becomes much easier if we are allowed to look under the hood. It would, of course, be even more convenient if it were possible to stop the motor, take it apart and subject each of its components to any desired tests and measurements. That is what experimental scientists can do and economists cannot. But look under the hood he can, although in economics as in the garage it is an inconvenient and often a dirty operation. Admittedly, had we been able to reproduce the blueprint of the engine by indirect inference from the behavior of the gauges, such intellectual accomplishment would earn a much higher rating. Nevertheless, some economists rolled up their sleeves and looked under the hood.

9. **Direct structural analysis.** Direct observation and detailed description of the various aspects of economic reality are as old as our discipline itself. As statistical information became more and more available, descriptive quantitative economics came to be recognized as one of its important branches. But far from seeking to establish a close—not to say intimate—cooperation with the theorist, many of the empirically minded investigators came to consider direct observation as a separate self-sufficient approach to the explanation of the functioning of the economic system rather than the necessary descriptive complement of its theoretical analysis.

The empiricist school developed even a kind of a quantitative descriptive technique all its own. Its principal or rather only tools are averaging and aggregation (an aggregate of a set of magnitudes is defined as a weighted sum of its elements). While the theorist discovers or, should I say, introduces order into the complex multiplicity of measurable economic phenomena by reducing it to a system of equations, the radical empiricist simplifies the quantitative picture by describing it in terms of a few broad aggregates and averages. A detailed tabulation of the amounts of all the many kinds of goods and services consumed in the course of a given year by households or invested in all the various industries to expand their productive capacity is replaced, for example, by one single figure identified as *the* annual Net National Income. This figure represents the sum total of the dollar values of all the individual commodities and services mentioned above. Similarly, the multidimensional set of the corresponding prices is replaced by a single number—a weighted average of its individual components—called "the General Price Level."

As conveniently simplified—but albeit necessarily blurred—descriptions of quantitative phenomena such aggregates and averages proved to be useful, nay, indispensable to economists. Even pure theorists use it—more often probably than they should—as a pedagogical device which lends the appearance of realism to their schematic general equilibrium models. Some of these models purport to depict the operation of the entire economic system in terms of five, four, or even only three aggregative variables. As a substitute for theoretical analysis and generalization, such simplifying devices are obviously valueless. To the extent to which broad aggregates are not directly observable (and few of them are) but have to be compiled from separate measurements of the component variables, in utilization of primary observational data, no economy can be achieved through their use either.

Direct factual study and quantitative descriptions of the structural properties of the economic system, detailed in content, comprehensive in coverage, and systematically designed to fill the specific requirement of an appropriate theoretical scheme, seem to offer the only promising approach to empirically significant understanding of the operational characteristics of the modern economy.

The task thus imposed on the collector of primary factual information exceeds by far anything demanded hitherto from quantitative empirical research, in economics or any other social science. It is only reasonable to suggest that the theorist should meet him half way by redesigning his analytical scheme so as to take advantage of the strengths and mitigate the weaknesses of the observational data to which it will have to be applied.

As an example of such mutual accommodation between theoretical formulation and observational capability, let me say a few words about the so-called input-output analysis in which I myself happen to have an interest of long standing, [17] and [18].

The difficulty, the challenge which an economist faces in trying to analyze and to describe in concrete numerical terms the specific operational characteristics of a modern national economy is caused by the complexity of the interindustrial or, more generally, intersectoral relationships which it comprises. A reduction in the consumers' purchases of automobiles leads, for example, to a fall in the demand for the electric energy required for production of aluminum which goes into the manufacture of cylinder heads.

Economic theory tells us that in order to trace through such a chain of relationships, one must determine the actual shape of the transformation (production) functions of all the individual sectors of the economy in question, insert them into an appropriate system of general equilibrium equations and finally compute the effect which the assumed increase or decrease in final demand would have on the output in question.

Since a fully detailed description of the actual shapes of all the transformation functions comprised, say, in the structure of the American economy, is obviously impossible, the theory had to be reformulated in terms of linear transformation functions—considered to represent the first approximation to the actual functions. The solution of the general equilibrium problem was accordingly reduced to inversion of the coefficient matrix of a system of linear equations: If a_{ik} represents the input coefficient showing the number of units of the product of industry i absorbed by industry k per unit of its respective output, the relationships between the total gross outputs, $x_1, x_2, \cdots, x_n$, of the n industries constituting the national economy and the so-called final demand comprising consumption and new investment (accumulation), must satisfy the following matrix equation:

$$(I - A)X = Y \quad \text{or} \quad X = (I - A)^{-1}Y$$

in which A is the square non-negative matrix of all input coefficients (with a_{ik} its general element), X a column matrix of the n gross outputs, and Y the corresponding matrix of final demand.

Matrix A represents a concise numerical description of the structural properties of the specific economy; it summarizes the results of painstaking and systematic empirical inquiries. Even a highly aggregated picture of the U. S. economy described in terms of 100×100 matrix required over a year's factual research by 20 trained economists; a more detailed 200×200 matrix—two years' work by 75 persons. The most detailed input-output matrix of the American economy yet compiled is of the order 450×450.

The inversion of such matrices presents a real challenge even to modern large-scale computing machines. A system of so many simultaneous linear equations would—if its coefficients were randomly chosen—be highly unstable; its numerical solution would show hardly more than an accumulation of round-off errors. As an economist, I was not astonished, however, to find the inverses of the empirical input-output matrices to be very stable. The economy of the United States actually operates as a kind of a large-scale calculating machine, continuously working out the solution of problems which it poses itself. Applying the conventional standards of computational accuracy, one must say that these solutions certainly do not prove to be excessively unstable.

This last analogy leads quite naturally to the consideration of a workable empirical approach to dynamic problems. Again, the requirements of factual implementation demand the use of a greatly simplified theoretical framework. The dynamic input-output theory which is now undergoing its first empirical tests is based on the introduction into the original static scheme of so-called stock-flow relationships. It leads to the following system of linear differential equations with constant coefficients,

$$(I - A)X(t) - B\dot{X}(t) = Y(t).$$

The general term b_{ik} of the square matrix B represents the *stock* of the products of industry i required by industry k per unit of its respective output: $B\dot{X}$ describes the time rate of change of all stocks, i.e., the rate of accumulation or decumulation of all kinds of "capital" in their dependence on changes in the rate of output, $\dot{X}$, of all the individual industries.

The determination of the magnitude of the elements of the capital coefficient matrix B involved a series of empirical studies even more exacting than those aimed at the derivation of the flow coefficients a_{ik}.

A 100×100 B (stock) matrix of the American economy is now available. A numerical general solution of a homogeneous system of 20 linear differential equations based on consolidated 20×20 A and B matrices of the American economy was completed a few weeks ago by Kenneth Iverson on the new Harvard Mark IV calculating machine.

Even if I had time—which I do not have—to outline the study of the formal properties of the linear system described above, and had shown how more and better empirical data will permit the use of a more refined analytical scheme, I admittedly could not dispel the sense of intellectual retreat which by now you must have felt. From the heights of general theorems describing the formal properties of broadly defined systems, we have step by step descended into the realm of elaborate factual observation followed by equally laborious computations: From Gibbs to crude numerical analysis. But such a rebound has probably been unavoidable. Economics, mathematical economics, in particular, acquired very early in its development the attitudes and manners of the exact empirical sciences without really having gone through the hard school of direct, detailed factual inquiry. Possibly it will do us good to be sent down in order that we may catch up with the experience we have missed. And when one has put a hand to it, one cannot help but derive a peculiar satisfaction from seeing masses of seemingly amorphous facts do the bidding of the orderly and ordering mathematical thought.

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GENERATION OF ERRORS IN DIGITAL COMPUTATION

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Consider the problem of computing the numerical value $f(x)$ of some function f corresponding to a particular value of x. If x is a physical quantity whose value is known only from measurement, then one has available not x itself but some approximation x^*. If x is a mathematical constant, one may again be forced to replace x by some approximate value x^*. This is to be expected at least when x is irrational, and is often true when x is rational. For these or other reasons, though $f(x)$ is desired, one may have to accept some $f(x^*)$ instead.

Unless the function f is rational, one will generally be forced to represent f approximately by a truncated Taylor series, or an orthogonal series, or in some other fashion. In general, therefore, one does not strictly compute even $f(x^*)$, but rather some $f_a(x^*)$, where f_a represents a function which approximates f over some range containing x^*.

Suppose that f_a is defined by a finite sequence of elementary arithmetic operations, and that the computations are digital. In general the result of a division is not representable exactly by a terminating decimal. The same statement holds, of course, if one were using any fixed base. The result of multiplying two numbers, each expressible by n digits, will in general require $2n$ digits for its representation and further multiplications would increase the number of digits in proportion. Hence for either a product, or a quotient, one will in general replace the true result by an approximation obtained by truncating the sequence of digits and perhaps adjusting by some rule the last one retained. Hence one does not, in general, end up even with $f_a(x^*)$, but rather with some $f^*(x^*)$, in which true products and quotients are replaced by pseudo products and pseudo quotients, obtained according to rules that are determined in part by the nature of the facilities used for the computing.

Thus one starts to compute $f(x)$ and ends by computing some $f^*(x^*)$, thereby committing an error whose amount can be expressed as a sum of three independent components:

$$\begin{aligned} f(x) - f^*(x^*) = {} & [f(x) - f(x^*)] + [f(x^*) - f_a(x^*)] \\ & + [f_a(x^*) - f^*(x^*)]. \end{aligned} \tag{1}$$

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Of these three components, the first will be called the propagated error, the second the residual error, and the third the generated error.

If it is possible to assign limits to the error $x-x^*$, then by standard methods one can estimate the propagated error. The propagated error is completely independent of computing techniques and devices and can be reduced only by improving the approximation x^* to x.

The nature of the facilities available to the computor will guide him in his selection of the approximation f_*. So, too, will the requirements of accuracy: if a linear approximation will suffice, there is no point in computing the quadratic term. But once f_* has been determined, the residual error is also determined. It can be estimated by applying standard remainder formulas.

There remains, however, the generated error. It depends upon the facilities being used, and even upon quite subtle details of their application. This is the subject on which I wish to speak.

Unfortunately, there is no well developed general theory of generated error. For one thing the need did not become acute until the advent of automatic, high speed computing machinery. For another thing, the differences among machines now operating are such that each machine demands a theory all its own. Nevertheless, a few general principles can be stated, or at least illustrated and used as guides. This I shall attempt to do, first by describing a specific computing system, and then analyzing certain particular computational sequences.

In 1947, von Neumann and Goldstine gave an elaborate analysis of the errors generated in the course of inverting a matrix by Gaussian elimination. This is the first systematic treatment of generated error to appear in the literature. The computing system I shall presuppose is called the Oracle and is quite similar to theirs, but differs in a few respects. I shall assume that a register in the machine exhibits $\sigma+1$ binary digits, $\alpha_0, \alpha_1, \cdots, \alpha_\sigma$, and that the machine's arithmetic is designed with the assumption that these digits represent a number

$$a = -\alpha_0 + \alpha_1 \cdot 2^{-1} + \cdots + \alpha_\sigma \cdot 2^{-\sigma}.$$

Any number so represented will be called a digital number, so that digital numbers satisfy

$$-1 \leq a < 1,$$

and are integral multiples of $2^{-\sigma}$. The binary digit α_0 is called the sign digit, the others the precision digits.

When multiplication is called for, the sign digit and the first σ

precision digits of the product will appear in one of the registers, and the remaining σ precision digits will appear in the other. Normally one retains only the most significant part, and this will be called the simple pseudo product. If the factors are a and b, the simple pseudo product will be designated $(ab)^*$. A special order will yield $ab+2^{-\sigma-1}$, whose most significant part is the rounded pseudo product $(ab)_r^*$. These pseudo products satisfy the relations

$$0 \leqq ab - (ab)^* < 2\epsilon, \qquad -\epsilon \leqq ab - (ab)_r^* < \epsilon, \tag{2}$$

where

$$\epsilon = 2^{-\sigma-1} \tag{3}$$

is the unit rounding error.

When multiplication is called for, and the pseudo product is to be sent to storage, then if $a=b=-1$, storage will receive -1. Hence in planning a computation one must be sure that this case cannot arise, or else provide remedial steps to be taken in case it does. For all other combinations of a and b, multiplication is legitimate.

For division a/b, it is required that $|a|<|b|$. Division can be performed with a digital, or with a of the form

$$a = a_1 + a_2 \cdot 2^{-\sigma},$$

where a_1 and a_2 are digital and $a_2 \geqq 0$. In either case, following the division one register will exhibit a truncated pseudo quotient $(a/b)^*$ which satisfies

$$\begin{aligned} 0 &\leqq a/b - (a/b)^* < 2\epsilon, & b > 0,\\ 0 &< a/b - (a/b)^* \leqq 2\epsilon, & b < 0. \end{aligned} \tag{4}$$

The other register will exhibit the remainder.

If n is a small positive integer the direct division of n into a digital number a can be formed by forming $2^{-\sigma}a/(2^{-\sigma}n)$. The somewhat more precise limits for the pseudo quotient are often useful:

$$\begin{aligned} 0 \leqq a/n - (a/n)^* &\leqq 2(n-1)\epsilon/n,\\ 2\epsilon/n \leqq a/(-n) - [a/(-n)]^* &\leqq 2\epsilon. \end{aligned} \tag{5}$$

Addition and subtraction propagate, but do not generate errors, unless the sum or difference falls outside the digital range. This would be considered a blunder and I shall assume it does not happen. Likewise I shall assume that any product $ab<1$, and that for any quotient $|a/b|<1$.

In general a mathematical formulation does not by any means define uniquely a computational procedure, and it is the computational procedure that determines the generated error. A particular ordered sequence of operations is called a routine, and the unambiguous prescription for such a sequence is a program. In planning the computation of a particular function, one wishes to program a routine that will minimize the maximum possible generated error, or that will at least hold the maximum possible generated error within tolerable bounds. This is not the only possible criterion, but it is the one that will be applied here.

Consider to begin with the very simple problem of computing $a/(bc)$. To avoid tedious multiplicity of cases, assume all factors positive, and assume, as usual, that any operation called for is legitimate.

The computation, then, seems eminently reasonable. We have a choice, however, of doing two divisions in sequence, or of doing first a multiplication and then a division. Machine-wise, as many steps are required one way as the other. Suppose the multiplication is done first. We have

$$\frac{a}{bc} - \left[\frac{a}{(bc)^*}\right]^* = \frac{a}{bc} - \frac{a}{(bc)_r^*} + \frac{a}{(bc)_r^*} - \left[\frac{a}{(bc)_r^*}\right]^*$$

$$= \frac{a}{bc(bc)_r^*}\left[(bc)_r^* - bc\right] + \frac{a}{(bc)_r^*} - \left[\frac{a}{(bc)_r^*}\right]^*.$$

The second difference on the right is the difference between a true quotient and a pseudo quotient, and lies between 0 and 2ϵ. Consider, therefore, the first expression on the right.

Disregarding the trivial case $a=0$, which leads to a result that is strictly correct, the division could be legitimate with $(bc)_r^*$ as small as $2^{-r+1}=4\epsilon$ provided $a=2\epsilon$. Since

$$-\epsilon \leqq bc - (bc)_r^* < \epsilon,$$

therefore ab can be as small as $4\epsilon-\epsilon=3\epsilon$. This is assuming that the limits have been attained and we have

$$a = 2\epsilon, \qquad (bc)_r^* = 4\epsilon, \qquad bc = 3\epsilon.$$

In this event, the first expression on the right becomes

$$\frac{2\epsilon}{3\epsilon\cdot 4\epsilon}\left[4\epsilon - 3\epsilon\right] = \frac{1}{6}\cdot$$

The computed quotient can be too small by 1/6, or 25% of the true quotient 2/3.

If bc and $(bc)_r^*$ differ in the other direction, the computed result can be too large by nearly $1/10$, or again 25% of the true quotient which would, in this case, be $2/5$.

This is, of course, an extreme, but perfectly possible case. Since the error is certainly intolerable, let us consider the other routine of doing successive divisions. We have now

$$\frac{a}{bc} - \left[\left(\frac{a}{b}\right)^* \Big/ c\right]^* = c^{-1}\left[\frac{a}{b} - \left(\frac{a}{b}\right)^*\right] + \left(\frac{a}{b}\right)^* \Big/ c - \left[\left(\frac{a}{b}\right)^* \Big/ c\right]^*.$$

Again it is the first expression that requires consideration. We now observe immediately that if b and c are different, then it will make a difference in our result whether we first divide by b, as indicated, or first divide by c instead. Suppose we have somehow arranged it so that $c \geqq b$, which gives the most favorable result. Then, since $cb > a$, necessarily

$$c > a^{1/2}.$$

Again neglecting the trivial case $a = 0$, we can assume $a \geqq 2\epsilon$ and therefore

$$c > (2\epsilon)^{1/2}.$$

Hence we come out with the result that

$$0 \leqq \frac{a}{bc} - \left[\left(\frac{a}{b}\right)^* \Big/ c\right]^* < (2\epsilon)^{1/2} + 2\epsilon. \tag{6}$$

This is considerably better, but even so only about half the figures are significant and we have assumed that c was known to be not less than b.

The extremely unfavorable result of the first routine occurs when bc is small. If bc is not so small the result might be more satisfactory. Assuming again that $c > b$, we shall say that the first routine is to be preferred over the second unless

$$\frac{a}{bc(bc)_r^*} \geqq \frac{1}{c},$$

or, since all factors are supposed positive, unless

$$b \leqq a/(bc)_r^*.$$

This is equivalent to saying that

$$b \leqq [a/(bc)_r^*]^*. \tag{7}$$

The second inequality implies the first since the pseudo quotient cannot exceed the true quotient. Conversely, the pseudo quotient and b are both digital. Hence if b exceeds the pseudo quotient, it must do so by at least as much as 2ϵ, whereas the true and pseudo quotients differ by less than 2ϵ.

It follows that if our pseudo quotient, obtained by following the first routine, is exceeded by both of the factors in the denominator, then we have made the best selection. If not, we should recompute, following the second routine. To show that the condition is not trivial, take $a=2^{-10}$, $bc=2^{-6}$. Then

$$a/(bc)_r^* = 2^{-4}$$

and condition (7) fails if $b=c=2^{-3}$, but is satisfied if $b=2^{-5}$, $c=2^{-1}$.

If one does not know a priori enough about the magnitudes of the quantities involved to select the preferred routine, or at least to know that one routine or the other, even if not the preferred one, will yield a satisfactory result, then it is quite possible to program both routines and the selection of the better of the two results. The machine will then follow the first method, test the criterion, and recompute by the second method if the criterion fails.

On the Oracle, the special feature permitting the direct use of a double precision dividend makes it possible to get a much better result and with less trouble. Suppose $b \leqq c$. The discrimination is easily programmed. The machine division of a by b yields the partial quotient and the remainder, so that one can store the partial quotient and continue the division to obtain the next σ digits of the quotient. Altogether, then, we have 2σ precision digits of the quotient a/b, which form a number $(a/b)'$ satisfying

$$0 \leqq a/b - (a/b)' < 4\epsilon^2.$$

If one divides $(a/b)'$ by c and denotes the result by $(a/bc)^*$, the error can be written

$$\frac{a}{bc} - \left(\frac{a}{bc}\right)^* = c^{-1}\left[\frac{a}{b} - \left(\frac{a}{b}\right)'\right] + c^{-1}\left(\frac{a}{b}\right)' - \left(\frac{a}{bc}\right)^*.$$

Since

$$bc > a, \qquad c \geqq b,$$

it follows that

$$c > a^{1/2}$$

and except for the trivial case $a=0$,

$$a \geqq 2\epsilon.$$

Hence

$$c > (2\epsilon)^{1/2}.$$

Hence

$$0 \leqq \frac{a}{bc} - \left(\frac{a}{bc}\right)^* < 2\epsilon[1 + (2\epsilon)^{1/2}]. \tag{8}$$

The example I have just given is by no means artificial. Thus, iterated division is required in the formation of divided differences for purposes of interpolation, and the number of factors in the divisor is equal to the order of the divided difference. Let the divided differences be formed according to the iterative scheme

$$y_{01\cdots i} = (y_{12\cdots i} - y_{01\cdots i-1})/(x_i - x_0).$$

Mathematically, $y_{0\cdots i}$ is a symmetric function of its arguments. However, if the digital number $y^*_{01\cdots i}$ is to be formed and utilized in the computation of the divided difference of next higher order, then it is clear from the foregoing discussion that the optimal arrangement will be that for which

$$x_0 < x_1 < x_2 < \cdots$$

or that for which the inequalities are reversed. Unfortunately, the formation of a double precision dividend is more costly, since the dividend is the difference of two quantities, and the difference of double precision numbers requires a fair amount of programming.

It is usual to indicate the limits of error by estimating the absolute value of the difference between the computed and the desired results. However, this may lead to estimates that are unduly pessimistic. Thus, expressed in this form we would have to say that in division

$$|a/b - (a/b)^*| \leqq 2\epsilon,$$

which fails to exhibit the fact that the pseudo quotient cannot exceed the true quotient. More generally, suppose one carries out a sequence of operations, coming out at the end with some quantity f^*. We expect our error analysis to yield for us two numbers, which we may call δ_l and δ_u, which are such that the true value of f must satisfy

$$f^* - \delta_l \leqq f \leqq f^* + \delta_u. \tag{9}$$

If this can be done, then it is not at all important that δ_l and δ_u should be equal, or even that they should be both positive. If these are known then we can always, if we choose, replace f^* by another number that is as close as possible to the midpoint of the interval on which f is now known to lie. The important thing, therefore, in designing the program, is to make $\delta_u+\delta_l$ as small as possible.

This quantity $\delta_u+\delta_l$ I shall call the uncertainty (cf. Kuntzmann's "incertitude"), the interval from $f^*-\delta_l$ to $f^*+\delta_u$ will be the interval of uncertainty, and the two end points the limits of uncertainty. In general one will select a routine for which the uncertainty is as small as possible. Thereafter one can determine one or the other of the limits of uncertainty, and finally, perhaps, adjust the computed f^* to place it as close as possible to the midpoint of the interval of uncertainty.

Consider the evaluation of the first $n+1$ terms of a Maclaurin series. Presumably, up to some n at least, each additional term will increase the uncertainty but reduce the residual error. There will be some point beyond which further computation is unprofitable, either because the truncation error is already below tolerance, or because the decrease in the truncation error is overshadowed by the increase in the uncertainty. This point, moreover, will almost certainly vary with x. Hence one might be inclined to calculate and add successive terms until a term is reached whose value is less than some assigned quantity.

Each term in the series will have the form $a_i x^i$. If a_i is computed and multiplied by x^i, then one forms

$$(a_i x^i)^* = [a_i^*(x^i)^*]^*.$$

If we use roundoff multiplication, the error limits are symmetric. Suppose, for simplicity, that

$$| a_i - a_i^* | \leqq \alpha_i \epsilon.$$

If

$$| x^i - (x^i)^* | \leqq \xi_i \epsilon,$$

then

$$x^{i+1} - (x^{i+1})^* = x[x^i - (x^i)^*] + x(x^i)^* - (x^{i+1})^*.$$

Hence

$$\xi_{i+1} \leqq | x | \xi_i + 1.$$

But $\xi_1 = 0$, so that

$$\xi_i \leqq (1 - | x |^{i-1})/(1 - | x |) < i - 1.$$

Also

$$a_ix^i - (a_ix^i)^* = (a_i - a_i^*)x^i + a_i^*[x^i - (x^i)^*] + a_i(x^i)^* - (a_ix^i)^*.$$

Hence, for $i \geqq 1$,

$$| a_ix^i - (a_ix^i)^* | \leqq \{\alpha_i | x |^i + | a_i^* | \xi_i + 1\}\epsilon.$$

The uncertainty in the term in x^i is twice the right member of this inequality, and the total uncertainty for $n+1$ terms is obtained by summing to $i=n$. If the a_i are of the order of unity, then for x of the order of unity the sum is of the order of $n^2\epsilon$.

If we are willing to fix n independently of x, or if we can select n in advance for any given x, then another procedure is possible. Let

$$y_n = a_n, \qquad y_i = xy_{i+1} + a_i.$$

Then by a backward induction, y_0 is the required true sum. Also

$$| y_n - y_n^* | \leqq \alpha_n\epsilon,$$

$$\begin{aligned} y_i - y_i^* &= xy_{i+1} - (xy_{i+1}^*)^* + (a_i - a_i^*) \\ &= x(y_{i+1} - y_{i+1}^*) + xy_{i+1}^* - (xy_{i+1}^*)^* + (a_i - a_i^*). \end{aligned}$$

Hence if

$$| y_i - y_i^* | \leqq \eta_i\epsilon,$$

then

$$\eta_i \leqq | x | \eta_{i+1} + 1 + \alpha_i,$$

$$\eta_0 \leqq \sum_0^n \alpha_i | x |^i + (1 - | x |^n)/(1 - | x |).$$

If $\alpha > \alpha_i$ for all i, then

$$\eta_0 < (n + 1)\alpha + n,$$

and if α is at worst a small integer the uncertainty is of the order of a small multiple of $n\epsilon$, instead of being of the order of $n^2\epsilon$.

Quite often the computation of the a_i can be incorporated into the induction so as to reduce the uncertainty still further. Consider the first $n+1$ terms of e^{-x}. Let

$$y_n = 1 + x/(-n), \qquad y_i = 1 + xy_{i+1}/(-i).$$

Then y_1 is the required sum. Then

$$y_i - y_i^* = \frac{x}{-i}(y_{i+1} - y_{i+1}^*) - \frac{1}{i}[xy_{i+1}^* - (xy_{i+1}^*)^*]$$

$$+\frac{(xy^*_{i+1})^*}{-i} - \left[\frac{(xy^*_{i+1})^*}{-i}\right]^*.$$

Let

$$-\eta_i'\epsilon \leqq y_i - y_i^* \leqq \eta_i''\epsilon.$$

Then, using simple multiplication,

$$\eta_i' = \frac{x}{i}\eta''_{i+1}, \qquad \eta_i'' = \frac{x}{i}\eta_i' + 2.$$

Hence if

$$\eta_i = \eta_i' + \eta_i''$$

so that $\eta_i\epsilon$ represents the uncertainty, then

$$\eta_i = 2 + x\eta_{i+1}/i.$$

Now $\eta_n \leqq 2$, and by a reverse induction

$$\frac{1}{2}\eta_1 \leqq 1 + \frac{x}{2} + \cdots + \frac{x^n}{n!} < e^x.$$

Hence, independently of n,

(10) $$\eta_n < 2e^x.$$

Thus $2\epsilon e^x$ is an absolute limit to the uncertainty. In general the inclusion of an additional term increases the uncertainty by $2\epsilon x^n/n!$, and decreases the residual error by an amount of the order of $x^n/n!$.

Continued fraction expansions sometimes have desirable properties, not the least being that they may converge where the power series does not. If the fraction is written in the form

$$F = b_0 + \frac{a_1}{b_1 +}\,\frac{a_2}{b_2 +}\,\cdots,$$

and if c_n represents either the numerator or the denominator of the nth convergent, then by the well known recursion

$$c_r = b_r c_{r-1} + a_r c_{r-2}.$$

The analysis is greatly simplified when the a's and b's are all positive. One may determine in advance that n for which

$$F_n = A_n/B_n$$

is sufficiently close to F. Then A_n^* and B_n^* are computed by applying the recursion to the A's and B's, and finally

$$F_n^* = (A_n^*/B_n^*)^*$$

is the computed value of F. Then

$$\begin{aligned} F_n - F_n^* = {} & B_n^{-1}\{(A_n - A_n^*) - (B_n - B_n^*)A_n^*/B_n^*\} \\ & + A_n^*/B_n^* - (A_n^*/B_n^*)^*. \end{aligned}$$

If $\alpha_n\epsilon$ and $\beta_n\epsilon$ represent the uncertainty in A_n and in B_n, respectively, then approximately the uncertainty in F_n is equal to

$$[B_n^{-1}(\alpha_n + F\beta_n) + 2]\epsilon.$$

From the recursion we have

$$\begin{aligned} c_r - c_r^* = {} & b_r(c_{r-1} - c_{r-1}^*) + a_r(c_{r-2} - c_{r-2}^*) \\ & + (b_r - b_r^*)c_{r-1}^* + (a_r - a_r^*)c_{r-2}^* \\ & + b_r^*c_{r-1}^* - (b_r^*c_{r-1}^*)^* + a_r^*c_{r-2}^* - (a_r^*c_{r-2}^*)^*. \end{aligned}$$

If $\gamma_r\epsilon$ represents the uncertainty (hence either $\alpha_r\epsilon$ or $\beta_r\epsilon$ as the case may be), then γ_r satisfies a recursion of the form

$$\gamma_r = b_r\gamma_{r-1} + a_r\gamma_{r-2} + \delta_r,$$

where δ_r represents a linear combination of the uncertainties in a_r and b_r, together with added terms arising from the products being formed. Hence the γ's satisfy a nonhomogeneous system whose matrix is identical with that of the system which determines the c's.

The properties of continuants are well known and I shall not discuss them here. However, it is clear that if the a's and b's are large, then the uncertainty will build up rapidly. In fact, it builds up much more rapidly in proportion than does c_n, because of the presence of the nonhomogeneous term in each equation. Moreover, we may be faced with a difficult scaling problem. On the other hand, if the a's and b's are too small, B_n may be quite small, and the factor B_n^{-1} in the final uncertainty will cause trouble. The best plan seems to be to make every $a_r = 1$ if the b_r are then small, or else every $b_r = 1$ if the a_r are then small. This does not necessarily eliminate the scaling problem, but if both $a_r \leqq 1$ and $b_r \leqq 1$, then scaling can be effected by introducing factors 2^{-1} with sufficient frequency.

In illustration of a computation of a different sort, consider the extraction of a square root by Newton's method. For solving $x^2 - a = 0$, it is convenient to write the iteration in the form

$$x_{i+1} = x_i - \Delta^*(x_i), \qquad \Delta(x) = (x - a/x)/2. \tag{11}$$

I have yet to define Δ^*. If $x_0 = 1$, it is easily shown that the mathematical sequence, defined with Δ instead of Δ^*, converges monotonically. One may expect, therefore, that the digital sequence will be monotonic at least up to some x_t, after which either Δ^* vanishes, or else the x_i become periodic in some manner.

There are several possible routines for computing a Δ^*, all differing slightly from one another, and all giving slightly different results. It turns out that if one programs the computation in the form

$$-\Delta^* = \{[-x - (-a/x)^*]/2\}^*,$$

then one can show that the sequence $-\Delta^*(x_i)$ increases monotonically to zero, and that when $\Delta^*(x) = 0$, then

$$-\epsilon + (a + \epsilon^2)^{1/2} \leqq x < \epsilon + (a + \epsilon^2)^{1/2}; \tag{12}$$

furthermore, for the same x, if $x \leqq 2^{-1}$, then

$$(x^2)_r^* = a,$$

and if $x > 2^{-1}$, then

$$|(x^2)_r^* - a| \leqq 2\epsilon.$$

One could not, of course, expect that the pseudo square of the pseudo square root would always equal the number itself, since numbers below, say, $\epsilon^{1/2}$ could not be pseudo square roots by any reasonable approximation. In other words, there are more digital numbers than pseudo square roots. The uncertainty in the square root obtained in this manner is exactly 2ϵ, which is the smallest possible value. Since

$$0 < (a + \epsilon^2)^{1/2} - a^{1/2} \leqq \epsilon,$$

with equality holding only when $a = 0$, therefore

$$-\epsilon < x - a^{1/2} < 2\epsilon.$$

Hence if $a^{1/2}$ is digital the routine always yields the strictly correct result.

In the case of cube roots, the best method devised so far leaves an uncertainty greater than 2ϵ by roughly $\epsilon^{3/2}$.

More generally, consider the problem of solving an equation

$$f(x) = 0$$

for a particular root α that is simple and has been isolated. Suppose, by some means, one has found some approximation x' such that

$$\delta_l(x') \geqq f^*(x') \geqq -\delta_u(x'). \tag{13}$$

For definiteness suppose f is increasing, and suppose, moreover, that an $m>0$ is known such that $f' \geqq m$ throughout an interval containing both α and x'. Then

$$x' - (f^* + \delta_u)/m \leqq \alpha \leqq x' - (f^* - \delta_l)/m. \tag{14}$$

Since (14) holds for any x' satisfying (13), one might replace x' by $x' \pm 2\nu\epsilon$ and improve the limits slightly.

There are, of course, many ways of writing an equation $f(x)=0$ satisfied by α, and for each f, many possible possible routines for evaluating it. The limits (14) are optimal for a given f and a given routine for evaluating it, but will, in general, be different for different routines.

If one is to employ Newton's method, forming the sequence

$$x_{i+1} = x_i - \Delta^*(x_i),$$

where

$$\Delta(x) = f(x)/f'(x),$$

then the equation $\Delta=0$ is equivalent to $f=0$, and m is close to unity. If Δ is computed as a quotient, then one can write

$$\Delta - \Delta^* = [f - f^* - \Delta(f' - f'^*)]/f'^* + [(f^*/f'^*) - (f^*/f'^*)^*].$$

Near $x=\alpha$, Δ should be a small multiple of ϵ, and therefore the second term in the first bracket should be negligible. Hence approximately, in the vicinity of the root,

$$\Delta - \Delta^* = (f - f^*)/f'^* + [(f^*/f'^*) - (f^*/f'^*)^*]. \tag{15}$$

The first term on the right represents again the uncertainty in f divided by the derivative, but this uncertainty is increased by the fixed amount 2ϵ resulting from the division.

Of the many iterations of higher order that have been proposed, it seems unlikely that they can prove useful for this type of computation except possibly in very special cases. Even the simple-minded scheme of successive bisection requires at most σ evaluations of f, σ being, for most machines, approximately 40. Newton's method requires the evaluation of f' along with f, but should converge in fewer steps. An iteration of higher order will generally require a more complicated program for each step, with a corresponding increase in the uncertainty. What little can be gained by fewer steps is apt to be more than lost in the program.

My remarks here have not been directed to the experienced computors, by whom most of my conclusions are very likely well understood already. But however well known these matters may be, they do not seem to have been written down. The systematic study of generated error may be said to have begun with the paper by von Neumann and Goldstine [9] to which reference was made above. More recently Lotkin and Remage [7; 8] have analyzed the generation of error in matrix inversion by a different method. Goldstine (unpublished) has discussed the generation of error in the determination of the proper values of Hermitian matrices using an iterative scheme, and Givens [2], in a recent report, has analyzed the application of a certain direct method to the same problem. Other studies are made in memoranda and reports from various research organizations where computing machines are in operation.

My purpose here has been to indicate how the techniques can be applied to a variety of situations, and to point up the need for a systematic attack so as to transform the art of computing into a science of computing.

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OAK RIDGE NATIONAL LABORATORY

THE FEBRUARY MEETING IN NEW YORK

The five hundredth meeting of the American Mathematical Society was held at Hunter College in New York City on Saturday, February 27, 1954. The meeting was attended by about 180 persons, including the following 161 members of the Society:

C. R. Adams, Linda Allegri, R. L. Anderson, R. G. Archibald, Richard Arens, P. N. Armstrong, W. G. Bade, F. E. Baker, J. H. Barrett, L. K. Barrett, Anatole Beck, E. G. Begle, Lipman Bers, Armand Borel, Samuel Borofsky, S. G. Bourne, E. H. Boyle, J. W. Brace, A. D. Bradley, F. E. Browder, A. B. Brown, F. H. Brownell, C. W. Burrill, J. H. Bushey, Jewell H. Bushey, Sarvadaman Chowla, Alonzo Church, H. J. Cohen, L. W. Cohen, M. L. Constable, T. F. Cope, A. H. Copeland, Sr., A. H. Copeland, Jr., J. B. Crabtree, J. H. Curtiss, J. M. Danskin, M. D. Darkow, M. D. Davis, R. B. Davis, A. H. Diamond, V. J. Doberly, Jesse Douglas, Nelson Dunford, Aryeh Dvoretzky, H. A. Dye, B. E. Dyer, Jacob Feldman, William Forman, R. M. Foster, Joel Franklin, M. P. Gaffney, G. N. Garrison, H. A. Giddings, Wallace Givens, Sidney Glusman, S. I. Goldberg, Lawrence Goldman, A. J. Goldstein, Laura Guggenbuhl, Felix Haas, G. A. Hedlund, M. H. Heins, Sigurdur Helgason, Alex Heller, Robert Hermann, Abraham Hillman, S. P. Hoffman, Jr., Banesh Hoffmann, C. C. Hsiung, Witold Hurewicz, R. V. Kadison, Shizuo Kakutani, Aida Kalish, M. E. Kellar, L. S. Kennison. J. F. Kiefer, H. S. Kieval, M. S. Klamkin, E. R. Kolchin, Horace Komm, A. G. Kostenbauder, Saul Kravetz, J. B. Kruskal, M. D. Kruskal, M. K. Landers, R. K. Lashof, Marie Lesnick, M. E. Levenson, Eugene Lukacs, Brockway McMillan, L. A. MacColl, H. M. MacNeille, A. J. Maria, M. H. Maria, L. F. Meyers, K. S. Miller, August Newlander, Jr., Albert Nijenhuis, M. A. Oliver, A. F. O'Neill, Steven Orey, E. E. Osborne, L. J. Paige, T. K. Pan, L. E. Payne, E. J. Pellicciaro, Anna Pell-Wheeler. F. P. Peterson, R. P. Peterson, Jr., Hans Rademacher, G. N. Raney, M. S. Rees, B. L. Reinhart, Helene Reschovsky, H. G. Rice, Moses Richardson, Herbert Robbins, M. S. Robertson, Louis Robinson, Robin Robinson, Selby Robinson, P. C. Rosenbloom, A. S. Rosenthal, J. E. Rosenthal, H. D. Ruderman, J. P. Russell, C. W. Saalfrank, H. E. Salzer, Hans Samelson, Arthur Sard, R. D. Schafer, Samuel Schecter, E. V. Schenkman, Seymour Schuster, Abraham Schwartz, Sol Schwartzman, I. E. Segal, G. B. Seligman, I. M. Sheffer, J. L. Snell, J. J. Sopka, J. J. Stoker, R. L. Taylor, P. M. Treuenfels, A. W. Tucker, Annita Tuller, D. H. Wagner, H. F. Weinberger, Alexander Weinstein, Morris Weisfeld, Louis Weisner, Bernard Weitzer, M. E. White, A. L. Whiteman, Albert Wilansky, Jacob Wolfowitz, Arthur Wouk, Hidehiko Yamabe, Karl Zeller, J. A. Zilber, Leo Zippin.

Dr. Armand Borel of the Institute for Advanced Study delivered an address entitled *Topology of Lie groups and characteristic classes* at a general session presided over by Professor Witold Hurewicz, by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings. Sessions for contributed papers were held in the morning, presided over by Professor A. B. Brown, and in the afternoon, presided over by Professor Sarvadaman Chowla.

Abstracts of the papers presented follow. Those having the letter "*t*" after their numbers were read by title. Where a paper has more than one author, that author whose name is followed by "(p)" presented it. Mr. Norman Shapiro was introduced by Professor Alonzo Church and Dr. J. H. Williamson by Dr. I.T.A.C. Adamson.

ALGEBRA AND THEORY OF NUMBERS

336t. H. W. Becker: *Eight-parameter conjugates of rational cuboids.*

Generalize O'Riordan's forms, Dickson's *History* II, p. 504, to $ac\pm bd=E_\pm$, $ad\mp bc=F_\pm$, $eg\pm fh=G_\pm$, $eh\mp fg=H_\pm$. Let $x^2+y^2=u^2$, $y^2+z^2=v^2$, $x^2+z^2=w^2$. Then $v, z=(FH\pm EG)_-$; $u, x=(FG\pm EH)_-$; $w=(FG-EH)_+$, $y=(FH+EG)_+$. The resulting quadratics are $g/h=[(e^2-f^2)(2F_-E_- -F_+E_+)+ef(F^2-E^2)_+\pm 2\Delta_g^{1/2}]/[(fF-eE)_+^2 -4ef(EF)_-]$, $\Delta_g=2(EF)_-(ae+bf)(af-be)\{(cf+de)^2-(ce-df)^2\}$, and the same, under: $a, b, c, d\leftrightarrow g, h, e, f$ or else e, f, g, h; and $a, b, g, h\leftrightarrow c, d, e, f$. Thus a cuboid of the kind has 4 conjugates, under change of sign of the 4 discriminants, each in turn having 3 new conjs., etc. ad inf. Expressions satisfying $\Delta=\square$ are deduced from the Euler[3], Lenhart[12], and Cunliffe[9] cuboids (the former, Euler[3] transforms of the Saunderson[2] and Rignaux[24a], ibid.): a, b; c, d; e, f; $g, h=2s(r+s)$, $-r^2+2rs+s^2$; $2s(r-s)$, $r^2+2rs-s^2$; d, $2r(r+s)$; $2r(r-s)$, b; and $r+s$, $2s$; $s(2r-s)(r+s)$, $r^3+3r^2s-4rs^2-2s^3$; $r-s$, $2r$; $r(r+2s)(r-s)$, $2r^3-4r^2s-3rs^2+s^3$; and r, $r-s$; $(r-s)(4r^2-2rs+s^2)$, $rs(4r-3s)$; $(2r-s)(r^2-s^2)$, $2r^3-5r^2s+rs^2+s^3$; e, $s(3r^2-3rs+s^2)$, respectively. A different 4-p.pr. cuboid is typified by no. 7 of Kraitchik's table, Scripta Math. vol. 11 (1945) p. 326: v, $y=\pm F_-H_\mp+E_-G_\mp$; w, $x=F_-G_\pm\mp E_-H_\pm$; $u=-F_+G_-+E_+H_-$, $z=F_+H_-+E_+G_-$. Put $ce+df$, $cf-de=C, D$, then $\Delta_a=2gh(C^2-D^2)(gC-hD)(gD+hC)$ isomorphic with the 3-p.pr. Δ of the type, and Δ_g is the same under $g, h\leftrightarrow b, a$. $\Delta_a=2abgh(c^2+d^2)^2\{(ah+bg)^2-(ag-bh)^2\}$, and Δ_a is the same under $a, b, c, d\leftrightarrow g, h, e, f$. Such cuboids have 5 conjs., one via a 3-p.pr. Δ_c. (Received January 14, 1954.)

337t. H. W. Becker: *Eight-parameter O'Riordan vectors.*

Dickson's *History* II, p. 502, Euler[22], first solution of $4\square$, the sum of any $3\square=\square$, is covered by O'Riordan's forms, p. 504, only if they are extended to 8 parameters Generalize Euler's forms to 2 parameters: $c=a$, $d=b$, $e=(a^2+b^2)^2$, $f=4ab(a^2-b^2)$, $g=e(e^2+f^2)$, $h=f(e^2-f^2)$, and interchange B and C. Then, in the notation of the companion abstract: $A=E_+H_- -F_+G_-=2e^2f(a^2+b^2)$, $\alpha=E_+G_-+F_+H_-=(e^4+f^4)(a^2+b^2)$; B, $\gamma=(EG\mp FH)_-=(e^4+f^4)(a^2-b^2)\mp 4abe^2f$; C, $\beta=(EH\mp FG)_-=2e^2f(a^2-b^2)\mp 2ab(e^4+f^4)$; $D=(EH-FG)_+=-2ef^2(a^2+b^2)$, $\delta=(EG+FH)_+=(e^4+2e^2f^2-f^4)\cdot(a^2+b^2)$. Each O'Riordan form has 4 conj. solutions. Thus $h/g=[2cd(a^2-b^2)(e^2-f^2)\pm\Delta_g^{1/2}]/(e^2E^2+f^2F^2-2efE_-F_-)$, $\Delta_g=4c^2d^2(a^2-b^2)^2(e^2-f^2)^2-s$, $s=(e^4+f^4)E_+^2F_+^2-2ef(e^2-f^2)E_-F_-(E_+^2-F_+^2)+e^2f^2(E^4+F^4-4E_-^2F_-^2)$. In Euler's solution $s=0$, and the conj. $h'=0$. If $g=0$, $h=1$ the O'Riordan forms above reduce to his 3-p.pr. forms. But if $e=1, f=0$ then $\Delta_g=E_-F_-(2E_+F_++E_-F_-)=\square$ (and analogously if $a=1$, $b=0$) satisfied by $c=a$, $d=b$, $g=(a^2+b^2)^2$, $h=4ab(a^2-b^2)$ as in Euler's second method, where $A=D$, $\alpha=\delta$. But if $c=1$, $d=0$ the vector is imaginary: $A=\beta$, $B=\alpha$, $C^2+D^2=0$. Thus for any e, f, g, h, $a/b=[ef(g^2-h^2)\pm ghi(e^2+f^2)]/(e^2h^2+f^2g^2)$; and the same under $a, b\leftrightarrow e, f$; $i=(-1)^{1/2}$. And for any a, b, e, f, $h/g=\pm(af-be)i/(ae+bf)$, so $A=\beta=(af+be)(ae+bf)+(ae-bf)(af-be)i$, $B=\alpha=a^2e^2-b^2f^2-(a^2f^2-b^2e^2)i$, $C=(ae+bf)\cdot(af-be)(1+i)=D/i$; γ, $\delta=(ae+bf)^2\mp(af-be)^2i$. (Received January 14, 1954.)

338t. H. W. Becker: *Equiareal rational triangles and their Euler transforms.*

Basic solutions of $abcd(a^2-b^2)(c^2-d^2)=\square$ are $a, b, c, d=$: (I, II) $2r^2\mp s^2$, $r^2\pm s^2$, $r^2\mp 2s^2$, b (Fermat I & II, 175); (III) $r(3r-s)$, $s(r+s)$, $r(3r+s)$, $s(r-s)$ (Euler-Hillyer triad, 174); (IV) r^4-s^4, $3s^4$, r^4+2s^4, $3r^2s^2$ (Rignaux, 502); (V) $(r^2+s^2)^2$, $4rs(r^2-s^2)$, r, s (Rolle, 447); (VI) $8rs^3$ or $(r^2-2rs+3s^2)(r+s)^2$, $(r^2+s^2)(r^2-3s^2)$ or $2rs(r^2+2rs-s^2)$, $(r^2+2rs+3s^2)(r-s)^2$, $2rs(r^2-2rs-s^2)$ (dual Crussol-Rignaux triad, 502); (VII) $\Pi(r^2-rs\pm s^2)$, $(s$ & $r)$ $(r^3-3r^2s+4rs^2-s^3)$, $s(r^3-2rs^2+2s^3)$; (VIII) $(2rs^2+^1)^2$, $4rs(r+s)(2r-s)$, $4r^4+8r^3s+8r^2s^2-4rs^3+s^4$, b (Guibert, 440); (IX) $\Pi(r^3\pm r^2s\pm s^3)$, $3rs^4$, $\Pi(r^3+rs^2\pm s^3)$, $3r^4s$ (dual Gerardin, 647); (X) $2r\Pi(r^3\pm 2r^2s-3rs^2\pm 2s^3)$, $(s$ & $r)$ $\cdot(r^2-s^2)(r^4+18r^2s^2+s^4)$, $2s\Pi(2r^3\pm 3r^2s+2rs^2\mp s^3)$ (dual Euler, 646). Pagination is to Dickson's *History*, vol. II, where the above are given explicitly or deducibly. Further solutions are given by the Petrus (446), Euler (474) and other transforms, and their iterates. The transform $a, b, c, b, a\pm b, c\pm b\rightarrow A, B, C, B, A\pm B, C\pm B=(a+b)$ $\cdot\Pi(a-b\pm c)$, $b(a-b)^2+c^2(3a-b)$, $c\{(a-b)(a+2b)+2c^2\}$, B, $(a-b)C$, $a\Pi(a-b\pm 2c)$, $(c\pm b)(a-b\pm c)(a-b\pm 2c)$ is a formula version of Euler's recurrence (a great contribution, but no "general solution," possibility of which is questionable). Solutions (III; IV; VII) are applications of Euler's transform to the degenerate a, b, c, d $=r, s, r, s$; $r^2, 0, s^2, 0$; $s(r\pm s)$, $(r$ & $s)(r-s)$. Empirical solutions are found by tabulating $rs(r^2-s^2)/\square$. Maverick solutions fitting none of the above forms or their transforms are: 19, 8, 22, 3; 45, 32, 22, 13; 56, 25, 32, 31; 64, 61, 60, 61. (Received January 14, 1954.)

339*t*. H. W. Becker: *Six-parameter O'Riordan vectors.*

O'Riordan (Dickson's *History* II, p. 503) gave the general 3-p.pr. criterion for $4\boxplus$, the sum of any $3\boxplus=\square$. His $(a, b; c, d)$ satisfy $abcd(a^2-b^2)(c^2-d^2)=\square$, and his $e, f=\iota, \theta$ or κ, λ where $(\iota, \theta; \kappa, \lambda)$ is the Petrus transform of $(a, b; c, d)$. Substitution of the Euler-Hillyer forms for $(a, b; c, d)$ gives the Tebay and final Euler forms for O.V.; substitution of the Rolle forms, the second Euler O.V., ibid. line 28. Substitution of the Fermat I forms for $(a, b; c, d)$ gives the O.V.: $A, D; B, C=6(r^2+s^2)(r-s)(r^2-3rs+s^2)\{s(2r^2-s^2),\ r(r^2-2s^2)\}$; $(2r^2-s^2)(r^2-2s^2)(2r^2-3rs+2s^2)\{3rs,\ 2(r^2+s^2)\}$. The dual Fermat II forms give perhaps the next to smallest nonrepetitive O.V.: 792, 385, 840, 1980. Euler's semifinal algorithm is equivalent to O'Riordan's more elegant and convenient procedure: $v, x, y, z=D, B, A, C$; and exhibits their filigree of properties $D/A=[cd(a^2-b^2)/ab(c^2-d^2)]^{1/2}$, $B/C=[(a^2-b^2)(c^2-d^2)/abcd]^{1/2}/2$, DB/AC $=(a^2-b^2)/2ab$, $B=fE_--eF_-$ and its conjugate $B'=f'E_--e'F_-=$ (in the Euler-Hillyer case) $eE_+^2-fF_+^2F_-/E_-$. Each 3-p.pr. O.V. has 3 conjugates, under change of sign of the discriminant in the quadratics for a/b, c/d, e/f, wherein is the proof that the index $=$area$/\square$ is unaltered by Petrus transformation of Pythagorean $\triangle$. Conjecture: no general formula for O.V. exists, in a finite number of parameter pairs. (Received January 14, 1954.)

340*t*. Leonard Carlitz: *A note on generalized Dedekind sums.*

Rademacher has proved (in a paper to appear in vol. 21 of the Duke Math. J.) a three-term relation for Dedekind sums. In the present paper a certain n-fold sum is defined which reduces to the ordinary Dedekind sum for $n=1$. It is proved that this sum satisfies both an $(n+1)$- and an $(n+2)$-term relation, those generalizing both the familiar reciprocity formula and Rademacher's new formula. (Received January 13, 1954.)

341*t*. Leonard Carlitz: *A note on Euler numbers and polynomials.*

This paper contains various extensions of the known congruence $E_{4m}\equiv 1$

$-(-1)^{(p-1)/2}$ (mod p), where p is an odd prime such that $p-1|2m$. (Received January 13, 1954.)

342*t*. Leonard Carlitz: *Dedekind sums and Lambert series.*

Apostol (Duke Math. J. vol. 17 (1950) pp. 147–157) has proved a transformation formula for the function $G_p(x)=\sum_{m,n=1}^{\infty} n^{-p}x^{mn}$, where p is an odd integer >1. The writer (Pacific Journal of Mathematics vol. 3 (1953) pp. 513–522) showed that this transformation formula implies a reciprocity relation for the function $f(h, k; \tau) = \sum_{s=0}^{p+1} \binom{p+1}{s}(k\tau-h)^{p-s}c_s(h, k)$, where $c_s(h, k)=\sum_{\mu} \overline{B}_{p+1-s}(\mu/k)\overline{B}_s(h\mu/k)$, the summation extending over a complete residue system (mod k). In the present paper the latter reciprocity relation is proved in an elementary manner using only familiar properties of the Bernoulli functions $\overline{B}_s(x)$ and the representation of $c_s(h, k)$ in terms of "Eulerian" numbers. (Received January 13, 1954).

343. S. I. Goldberg: *Lie algebra extensions and enlargements of modules.*

Let G be a Lie algebra over an arbitrary field F, K a subalgebra of G, and M a G-module. The notions of K-extension and K-enlargement are introduced as follows: A K-extension (E, K, ϕ, ν) is an extension (E, ϕ) with a representative function ν such that $[\nu(\gamma), \nu(\sigma)]=\nu([\gamma, \sigma])$ for all $\gamma\in G$ and $\sigma\in K$. A K-enlargement (ϵ, K, ψ, μ) is an enlargement (ϵ, ψ) with a representative function μ such that $\sigma\cdot\mu_m=\mu_{\sigma\cdot m}$ for all $\sigma\in K$ and $m\in M$. The following is then proved: (i) the elements of $H^1(G, K, \mathrm{Hom}_G(M, P))$ are in (1-1) correspondence with the equivalence classes of K-enlargements of P by M and (ii) for any given K-extension (E, K, ϕ, ν) of G by M with factor set f and K-enlargement (ϵ, K, ψ, μ) of P by M, a necessary and sufficient condition for the existence of a K-extension ϵ^* with $\epsilon^*/P\approx\epsilon$ is that $\Delta(f)$ be a relative coboundary mod K where Δ: $H^2(G, M)\rightarrow H^3(G, P)$ is the invariant coboundary. (Received October 13, 1953.)

344*t*. Arno Jaeger: *On derivations and differentiations in algebraic function fields of prime number characteristic.*

Let F be a separably generated algebraic function field in one indeterminate over a constant field f of characteristic $p\neq0$, and let D be a fixed regular differentiation of F over f in the sense of F. K. Schmidt (cf. Monatshefte für Mathematik vol. 56 (1952) pp. 181–219). For each derivation δ in F over f in the classical sense there exists an element $a_\delta\in F$ such that $\delta=a_\delta D$ holds. A derivation δ belongs to a differentiation if and only if $D^{p-1}a_\delta^{-1}=0$ is satisfied. If δ^n denotes the operator obtained by the n-fold iteration of δ and δ^0 the identity operator, the ring of derivation operators $\sum b_n\delta^n$ $(b_n\in F)$ is isomorphic to the ring of all $p\times p$ matrices with elements in the subfield of F containing all D-constants of F. For the derivation δ^p one finds $\delta^p=c\delta$ where c is a D-constant. The theory of the solutions of the equations $\sum b_n\delta^n y=a$ $(b_n\in F, a\in F)$ can be derived from the theory of differential equations in the sense of F. K. Schmidt. Generalizations of these results are obtained in the case of a separably generated algebraic function field in several (but finitely many) indeterminates where multi-differentiations $\mathfrak{D}$ (cf. J. Reine Angew. Math. vol. 190 (1952) pp. 1–21) take over the role of the differentiations in the one-dimensional case. (Received January 14, 1954.)

345*t*. K. G. Wolfson: *A class of primitive rings.*

Motivated by the examples of the ring of all linear transformations of an arbitrary vector space over a division ring, and the ring of all bounded operators on a Hilbert space, it is determined when the ring $T(A, B)$ of all continuous linear transformations on the pair of dual vector spaces (A, B) over a division ring D has the property that every right (left) annihilating ideal is a principal right (left) ideal generated by an idempotent element. A necessary and sufficient condition is that every closed subspace S of A (in the sense of Mackey) [Trans. Amer. Math. Soc. vol. 57 (1945) pp. 155–207] possesses a closed complementary subspace Q, such that S and Q form a modular pair in the lattice of closed subspaces of A. In particular, the ring of all bounded operators on a Banach space has the required property, precisely when the lattice of closed subspaces (in the usual norm topology) is complemented. The method is based on the theory of dual spaces of Dieudonné, Jacobson, and Mackey, and previous results of the author [Amer. J. Math. vol. 75 (1953) pp. 358–386]. Applications to the theory of dual rings are made. (Received January 6, 1954.)

ANALYSIS

346t. W. W. Bledsoe, M. J. Norris, and G. F. Rose: *On a differential inequality.*

Using vector-matrix notation and letting the subscripts 1 and 2 indicate the positive and negative part, respectively, of a vector or matrix, the results of this paper are slight generalizations of the one below. If, on the interval $[a, b]$, F and M are finite-valued, F_2 is continuous, D^+F_2 is never $-\infty$ for any component, $F(a)$ has no negative component, M is non-negative almost everywhere off the main diagonal, M is essentially bounded above, and $D^+F \geqq MF$ almost everywhere; then on $[a, b]$, F never has a negative component. Examples are given that show that in a sense the hypotheses are as weak as possible. (Received January 4, 1954.)

347. V. J. Doberly (Dobroliuboff): *Solution of differentiable equations in infinite series.*

The author's own method of solving insoluble by elementary means algebraical equations extends practically to all differentiable equations which can be presented as continuous functions (y) of an unknown and sought for independent variable quantity (x). Although in its final form the root of such an equation always appears as an infinite series, the question of its convergibility usually does not present a problem, since this method permits to effect the desired change in the conditions of convergency through an appropriate choice of the parameter which appears in all terms of the resulting series as a numerical quantity independent of x and y. (Received January 5, 1954.)

348t. Herbert Federer: *An addition theorem for Lebesgue area.*

If X is the space of a 2-dimensional finite Euclidean cell-complex K, f is a continuous function mapping X into Euclidean n-space with $n \geqq 2$, and the restrictions of f to the 1-cells of K are curves of finite length, then the 2-dimensional Lebesgue area of f is the sum of the 2-dimensional Lebesgue areas of the restrictions of f to the 2-cells of K. (Received December 8, 1953.)

349t. R. V. Kadison: *On the general linear group of infinite factors.*

In the paper *Infinite general linear groups* [Trans. Amer. Math. Soc. vol. 76 (1954)], the author determined all the uniformly closed, normal subgroups of the group of all invertible operators in a factor, with the exception of the factors of type

II_∞. The present note is devoted to filling the gap in the information. Theorem 4 of the above mentioned paper is proved again by means of a proof which is valid for both the I_∞ and II_∞ cases. The key lemma of the present note is one which states that an operator in a C^*-algebra lies in the center if and only if each of its inner transforms (by invertible operators in the algebra) is normal. (Received January 14, 1954.)

350*t*. A. J. Lohwater: *On the radial limits of analytic functions*. I.

Nonconstant analytic functions $f(z)$ are studied whose radial limits are constant almost everywhere on an arc A: $[\alpha<\theta<\beta]$ of $|z|=1$, $z=re^{i\theta}$. It is shown, for example, that every complex number ζ (including ∞) is an asymptotic value on any arc of $|z|=1$ containing any point $e^{i\theta}$ of A, if the points z_k for which $f(z_k)=\zeta$ satisfy the condition $\sum(1-|z_k|)<\infty$. (Received January 11, 1954.)

351*t*. E. R. Lorch: *On certain extensions of the notion of volume.*

The classical notion of volume assigns to a body $\mathfrak{K}$ in n-space a unique number $V(\mathfrak{K})$ which is the volume of $\mathfrak{K}$. It is shown here that the concept of volume depends on a real parameter r, $1\leqq r\leqq\infty$. Thus for each smooth convex body $\mathfrak{K}$ there is defined a function $V_r(\mathfrak{K})$, the volume of order r. The case $r=1$ gives the classical volume. The definition of $V_r(\mathfrak{K})$ follows: Let $\mathfrak{K}$ be defined by $\phi(x)=\phi(x_1, \cdots, x_n)\leqq 1$ where ϕ is positively homogeneous of order 1. Let $r>1$ and let $G(x)=r^{-1}\phi^r(x)$. Let $|G_{ij}(x)|$ be the Hessian of $G(x)$, let Ω be the unit sphere in n-space, let $\mathfrak{K}^*$ be the body adjoint to $\mathfrak{K}$. Then (definition) $V_r(\mathfrak{K}^*)=[n^{-1}(r-1)^{-1}\int_\Omega |G_{ij}(x)|\,d\Omega]^{1/r}$. The cases $r=\infty$ and $r=2$ receive special attention. It is shown that $V_\infty(\mathfrak{K})=\max_{x\in\Omega}\phi^{-n}(x)$. If $\mathfrak{K}$ is an ellipsoid, its volume of order $r=2$ is shown to be proportional to the product of its principal semi-axes. Thus for the computation of relative volumes of solids which are approximately ellipsoids, volumes of order 2 give approximately the same results as those of order 1. The development is based on previous work of the author (*Differentiable inequalities and the theory of convex bodies*, Trans. Amer. Math. Soc. vol. 71 (1951) pp. 243–266). (Received January 11, 1954.)

352. L. E. Payne and H. F. Weinberger (p): *Bounds for harmonic and biharmonic functions.*

The following identity is proved for the Green's function $G(0, Q)=-(\omega_N r^{N-2})^{-1}+g(0, Q)$ of a closed region in N dimensions, $N\geqq 3$. If h is the function $r\partial r/\partial n$ on the boundary B, then (1) $(N-2)g(0, 0)=\iint_B h(\partial G/\partial n)^2$. From this and the representation of a harmonic function in terms of its boundary values, it follows that if B is star-shaped with respect to the point 0 so that $h>0$, then (2) $u(0)\leqq(N-2)g(0, 0)\iint_B h^{-1}u^2$. Putting $u(Q)=g(0, Q)$ results in an upper bound for $g(0, 0)$ and the inequality (3) $u(0)\leqq\iint_B h^{-1}u^2\iint_B h^{-1}r^{-2(N-2)}$. In two dimensions the analogue of (1) is (4) $\oint_B h(\partial G/\partial n)^2=(2\pi)^{-1}$ and the second integral in (3) is replaced by $(2\pi)^{-1}$. This inequality gives an error estimate if a harmonic function is chosen to approximate given boundary values on B in L_2 norm. An analogous result can be established for biharmonic functions. In two dimensions, if the clamped plate Green's function $\Gamma=(8\pi)^{-1}r^2\ln r-\gamma$, then $\gamma(0, 0)=(1/2)\oint_B h(\Delta\Gamma)^2$. (Received January 14, 1954.)

353*t*. R. S. Phillips: *A note on the abstract Cauchy problem.*

The notion of an abstract Cauchy problem has recently been introduced by E. Hille. The abstract Cauchy problem (ACP) may be formulated as follows: Given a linear operator U with domain and range in $\mathfrak{X}$ and given an element $y_0\in\mathfrak{X}$, find a function $y(t)=y(t; y_0)$ such that (i) $y(t)\in\mathfrak{D}(U)$ for $t>0$; (ii) $y(t)$ is strongly absolutely continuous and continuously differentiable in each finite subinterval of $[0, \infty)$; (iii) $U[y(t)] = y'(t)$ for $t>0$; and (iv) $\lim_{t\to 0} y(t; y_0) = y_0$. Theorem: Let U be a closed linear operator with dense domain and nonvacuous resolvent set. Suppose that for each $y_0 \in \mathfrak{D}(U)$ there is a unique solution to ACP. Then U generates a semi-group $S(t)$ strongly continuous for $t \geqq 0$ with $S(0) = I$ and such that $S(t)y_0 = y(t; y_0)$ for all $y_0 \in \mathfrak{D}(U)$. If condition (ii) is relaxed so that it is valid only for subintervals of $(0, \infty)$, then a similar result can be proved for a more general class of semi-groups. (Received January 13, 1954.)

354*t*. R. S. Phillips: *Groups of positivity preserving operators*. Preliminary report.

Let $\mathfrak{S}$ be a locally compact Hausdorff space and let $C(\mathfrak{S})$ denote the Banach space of all complex-valued continuous functions on $\mathfrak{S}$ which vanish at infinity, with $\|f\| = \sup |f(s)|$. The purpose of this paper is to characterize all strongly continuous one-parameter groups $[S(t); -\infty < t < \infty]$ of positivity preserving linear bounded operators on $C(\mathfrak{S})$. In fact, $[S(t)]$ is such a group if and only if it can be represented in the form $[S(t)f](s) = \theta(t, s)f[\phi(t, s)]$ where (1) $\phi(t, s)$ is a group of homeomorphisms of $\mathfrak{S}$ continuous on $E_1 \times \mathfrak{S}$ to $\mathfrak{S}$ such that $\phi(t_1+t_2, s) = \phi(t_1, \phi(t_2, s))$ where $\phi(0, s) = s$, and (2) $\theta(t, s)$ is a positive real-valued function continuous on $E_1 \times \mathfrak{S}$ to E_1 such that (a) $\theta(t_1+t_2, s) = \theta(t_1, s) \cdot \theta(t_2, \phi(t_1, s))$ where $\theta(0, s) = 1$ and (b) there exist constants $M>0$ and $\omega \geqq 0$ such that $[M \exp (\omega|t|)]^{-1} \leqq \theta(t, s) \leqq M \exp (\omega|t|)$. Suppose next that $\mathfrak{S}$ is an n-dimensional manifold of class C^∞ and let $\mathfrak{D}$ be the set of all infinitely differentiable functions with compact carriers. If $\mathfrak{D}$ is contained in the domain of the infinitesimal generator A, then both $\theta(t, s)$ and $\phi(t, s)$ are continuously differentiable with respect to t; $\partial\theta(t, s)/\partial t|_{t=0} = \beta(s)$ is real-valued and $\partial\phi(t, s)/\partial t|_{t=0} = \alpha(s)$ is a vector lying in the tangent space at s. Finally if $f \in \mathfrak{D}$, then $Af = \beta f + \alpha \cdot \nabla f$ where ∇ is the gradient operator. In particular, this shows that the diffusion equation can never lead to a reversible Markoff process. (Received January 13, 1954.)

355*t*. V. L. Shapiro: *Logarithmic capacity of sets and double trigonometric series*.

Given a double trigonometric series $T = \sum a_M e^{iMX}$, T is said to be a series of class (U') if $\sum_{M\neq 0} a_M |M|^{-2} e^{iMX}$ is the Fourier series of a continuous periodic function. If this latter series is the Fourier series of a bounded function, then T is said to be a series of class (B'). Let Z be a closed set contained in the interior of the fundamental square Ω. It is shown in this paper that a necessary and sufficient condition that Z be a set of uniqueness for series of class (U') under circular $(C, 1)$ summability is that Z be a set of logarithmic capacity zero. It is further shown that a necessary and sufficient condition that Z be a set of uniqueness for series of class (B') under local uniform circular $(C, 1)$ summability is that Z be a set of logarithmic capacity zero. Using Fourier-Stieltjes series, necessary and sufficient conditions that Z be a set of positive logarithmic capacity are obtained in terms of double trigonometric series of classes (U') and (B'). (Received January 13, 1954.)

356. Alexander Weinstein: *The general solution for a class of composite equations*.

Let $v = v(x_1, x_2, \cdots, x_m, y)$ and let $L_\alpha v$ denote either the operator $v_{yy} + \alpha y^{-1} v_y + \Delta_x v$ (occurring in generalized axially symmetric potential theory) or the hyperbolic Darboux operator $v_{yy} + \alpha y^{-1} v_y - \Delta_x v$. Let $u^{(\alpha)}$ denote in both cases any solution of $L_\alpha u = 0$.

It is proved that the general solution of $L_\beta v = u^{(\alpha)}$, $\alpha \neq \beta+2$, is $v = u^{(\alpha-2)} + u^{(\beta)}$. Applications: (A). The general solution of the *polyharmonic equation* of order n, $L_0^n v = 0$, is given by the new formula $v = u^{(0)} + u^{(-2)} + \cdots + u^{(-2n-2)}$. (B). The general solution of the equation $L_{2n}^{n+1} v = 0$ is $v = u^{(0)} + u^{(2)} + \cdots + u^{(2n)}$. In the case of a Darboux operator L_{2n}, the fact that the equation $L_{2n}^{n+1} v = 0$ admits the particular solution $v = u^{(0)}$ was formulated in a special case as a theorem of Friedrichs in the book of Courant-Hilbert, vol. 2, pp. 416 ff. See also A. Weinstein, Bull. Amer. Math. Soc. Abstract 59-4-395. (Received January 14, 1954.)

357*t*. J. G. Wendel: *Haar measure and the semigroup of measures on a compact group.*

Let S be a compact semigroup. Results of Numakura, Wallace, and Peck show that (i) S contains idempotents, and (ii) if S is not a group but contains an identity then it contains additional idempotents. We show that these facts imply the existence of Haar measure on any compact group G. The S which intervenes consists of the regular normalized nonnegative Borel measures on G, with convolution as multiplication, topologized by pointwise convergence as functionals on $C(G)$. G is embedded in S via the unit masses at the points of G, and $e \in G$ is the identity of S; no measure in $S-G$ has an inverse relative to e, so that S is not a group if $G \neq \{e\}$. If it exists, the Haar measure on a compact subgroup H of G defines an idempotent in S. Conversely, any idempotent in S arises in this way. Hence by (ii) some nontrivial subgroups have Haar measure. Using (i) it is shown that the set of idempotents is directed in the partial ordering $\mu \leq \nu$ in case $\mu\nu = \nu\mu = \nu$. Then S contains a greatest idempotent, which annihilates all the elements of S, and which therefore is the Haar measure on G. (Received January 6, 1954.)

358*t*. J. H. Williamson: *On topologising the field $C(t)$.*

It is possible to topologise the field $C(t)$ of rational functions of the indeterminate t, with complex coefficients, so that (i) $C(t)$ is a linear topological space over the complex field, and (ii) multiplication is continuous. If local convexity is not required, there is a known metrisable topology with the required properties; regard $C(t)$ as a subspace of the set of almost everywhere finite Lebesgue measurable functions on (0, 1), with the topology of convergence in measure. It is also possible to produce a locally convex metrisable topology satisfying (i) and (ii). It can be shown that any such topology must be neither too fine nor too coarse. In particular, multiplication is not continuous in the finest locally convex topology compatible with the linear space structure of $C(t)$. (Received December 3, 1953.)

359. Karl Zeller: *Proofs of completeness.*

A general method for proofs of completeness is presented, and exhibited in a special case. The space of the functions $x(t)$, measurable in (0, 1), is complete in the known Fréchet-metric. The functional $p(x) = \int_0^1 |x(t)| dt \leq \infty$ is lower semicontinuous in this space. It is concluded that the space (L^1) is complete with the norm $p(x)$. (Received January 18, 1954.)

360. Karl Zeller and Albert Wilansky (p): *Inverses of matrices and matrix-transformations.*

Given a matrix A with range R (a set of sequences), one may investigate the existence of right (A'), left $('A)$, and two-sided (A^{-1}) inverses for A, and for T, the transformation with matrix A, and connections between them. Typical results: A' exists iff $R \supset E = (x_n = 0$ for almost all $n)$, is unique iff T is 1-1. Either one of $'A$, $'T$ may exist without the other. If a row-finite $'A$ exists, $'T$ exists. If A is row-finite: if A' exists, there exists a row-finite A'; there exists row-finite $'A$ iff T is 1-1, if A is reversible, it has row-finite A^{-1}. If A is column finite: A' may exist and no column finite A'. MacPhail has shown that $\{C_n\}$, Banach, p. 50, may be unbounded. If A is regular, or even co-regular, it is shown that $C_n = 0$. (Received January 18, 1954.)

Applied Mathematics

361. F. H. Brownell: *A lemma on rearrangements and its application to certain delay differential equations.*

Consider the nonlinear delay differential equation (1) $x'(t) = -(c/\theta)\int_0^\theta(\theta - h) \cdot \{f(x(t-h)) - 1\}dh$, $c > 0$, $\theta > 0$, over $t \geq 0$, where $f(x) > 0$ is assumed to possess a continuous derivative $f'(x) > 0$ for all real x with $f(0) = 1$. By using a simple modification of a theorem of Hardy, Littlewood, and Pólya on rearrangements, it is possible to prove that any solution of (1) approaches in a certain sense as $t \to +\infty$ the collection of periodic solutions of (1), and that every nonzero periodic solution of (1) must have its period $T = \theta/k$ for some integer $k \geq 1$ and also satisfy the ordinary nonlinear differential equation (2) $\gamma = F(x(t)) - x(t) + (1/2c)[x'(t)]^2$ for some constant $\gamma > 1$, where $F(x) = 1 + \int_0^x f(y)dy$. Using the resulting formula for T as a function of γ for these periodic solutions of (2), it becomes clear that if $F(x)$ is analytic, then (1) can have only a finite number of essentially distinct periodic solutions, and in this case any solution of (1) must approach just one of these as $t \to +\infty$. (Received January 14, 1954.)

362*t*. Aaron Fialkow and Irving Gerst: *Bounded analytic functions in network synthesis.*

The transformations $f = (Z-1)/(Z+1)$, $s = (p-1)/(p+1)$ which convert the domain of positive real functions $Z(p)$ into the domain of so-called unit functions $f(s)$ (i.e. $f(s)$ analytic and $|f| < 1$ in $|s| < 1$) have been used in connection with the network synthesis of $Z(p)$. It is the purpose of this paper to exploit this relationship by considering f as the transform of Z and to investigate those elementary closed operations in the f-domain whose maps in the Z-domain lead to network realizations of Z. Among the results obtained the following are particularly simple and have wide applicability: (here f, f_1, f_2 are unit functions and Z, Z_1, Z_2 are their corresponding inverse transforms) (i) $f = (f_1 - a)/(1 - af_1)$, $-1 < a < 1$, a real, corresponds to an impedance level change in Z. (ii) A factorization $f = f_1 f_2$ corresponds to the realization of Z as a series-parallel circuit involving Z_1 and Z_2. (iii) A partition $f = (f_1 + f_2)/2$ corresponds to the realization of Z as a symmetric bridge network whose arms are Z_1 and Z_2. The results are applied to effect simplifications in the synthesis of positive real functions without mutual inductance. (Received January 18, 1954.)

363*t*. Aaron Fialkow and Irving Gerst: *Impedance synthesis without mutual inductance.*

The present paper provides a new synthesis procedure for rational positive real functions (p.r.f.) $Z(p)$ which completely eliminates the Brune minimization process. In this synthesis the zeros of the function (*) $Z(p) + Z(-p)$ determine both the network structure and the computational sequence. There is no necessity for the Bott-

Duffin (Journal of Applied Physics vol. 20 (1949) p. 816) algorithm unless $(*)$ has some pure imaginary zeros, in which case it need only be applied to a p.r.f. $Z_1(p)$ of reduced degree, for which all the zeros of Re $[Z_1(i\omega)]$ are real. Corresponding to the zeros of $(*)$ which are not pure imaginary the method utilizes the theorem that $Z_4(p)=[1-Z_1(p)Z_2(p)]/[Z_1(p)-Z_2(p)]$ is a p.r.f. of lower degree than $Z(p)$ if Z_1 and Z_2 are a certain p.r.f. and a reactance function respectively whose construction depends upon Z and the zeros of $(*)$. In this way the synthesis of $Z(p)$ is made to depend upon that of the simpler function $Z_4(p)$. If Re $[Z(i\omega)]\neq 0$ for all real ω, the same will be true of the p.r.f. $Z_4(p)$ of reduced degree at each stage of the synthesis. (Received January 18, 1954.)

364*t*. K. S. Miller: *A remark on stability.*

Let L be a linear differential operator, $H(t, y)$ its impulsive response, and $\{\phi_k(t)\}$ a fundamental set of solutions of $Lu=0$. Then it is shown that (i) $\int_a^\infty |H(t, y)|\,dt<\infty$ for all $y\in I$, and (ii) $\int_a^\infty |\phi_k(t)|\,dt\leq M<\infty$, $k=1, 2, \cdots, n$, are coextensive. The connection between this result and the stability of linear systems is briefly discussed. (Received December 21, 1953.)

365. L. E. Payne: *On three-dimensional flows which do not possess rotational symmetry.*

The exact solution to the classical incompressible flow problem is known for only a limited number of three-dimensional bodies. A review of the literature reveals that the only such nonaxially symmetric problem, for which the exact solution has been obtained, is the flow about an ellipsoid (See Lamb, *Hydrodynamics*). F. C. Karal (Journal of Applied Physics vol. 24 (1953)) has indicated a method for obtaining the general flow about two equal separated spheres, but has not carried it through to completion. In this paper, solutions are obtained for the flow at any angle of attack about a lens, a spindle, a torus and two separated spheres. These solutions are given in terms of generalized Legendre functions. The components of the virtual mass tensor associated with the hemisphere are calculated explicitly. (Received January 14, 1954.)

GEOMETRY

366. T. K. Pan: *Complementary surfaces for a vector field.*

Let v be a vector field in a surface in an ordinary space. The author defined the curve of v and the asymptotic line of v and proved that, at a point P on a nondevelopable surface S, they are curves along each of which the straight lines on v form generators of a developable surface [Amer. J. Math. vol. 74 (1952) pp. 955–966]. Define the orthocenter of associate curvature of v at P as a point on the straight line on v at a distance $-{}_vr_a \cos\theta$ from P, where ${}_vr_a$ is the radius of associate curvature of v along the asymptotic line of v and θ the angle at P between the asymptotic line of v and the associate curvature vector of v along the asymptotic line of v. It is found that the locus of the orthocenter as P varies is a surface which is the locus of the edges of regression of the developable surfaces consisting of straight lines on v along the asymptotic lines of v. This surface is called the surface complementary to S for v, a generalization of the surface complementary to S for a geodesic family according to Bianchi. Properties of these surfaces and related curves are studied. (Received January 5, 1954.)

367. J. J. Stoker: *On the embedding of surfaces of negative curvature in three-dimensional Euclidean space.*

Hilbert showed that an abstract two-dimensional Riemannian surface provided with a complete metric having Gauss curvature $K = -1$ can not be embedded without a singularity in three-dimensional Euclidean space. The same statement holds for nonconstant Gauss curvature if $K \leq -c^2 < 0$, $c =$ constant, and if the mean curvature is bounded. (Received January 14, 1954.)

LOGIC AND FOUNDATIONS

368t. J. W. Addison: *Some unsolvable problems about cancellation semigroups.*

The problems of classifying the (finite) defining systems for cancellation semigroups (c.s.) according to whether or not the c.s. they define are (1) trivial (problem of triviality), (2) finite, (3) abelian, (4) groups, (5) embeddable in a group, (6) isomorphic (isomorphism problem), or (7) embeddable in one another are unsolvable. More generally so is the problem of classifying the defining systems of c.s. according to whether or not the c.s. they define have property P for any P normal in the following sense: A property P of c.s. is *normal* if P is invariant under isomorphism, if the trivial c.s. has P, and if a defining system S can be given in which no proposition of the form $A \sim AB$ holds for words A, B of S and which defines a c.s. not embeddable in any c.s. with P. The proof consists in showing that if this problem were solvable so would be the word problem (problem of identity) of a particular c.s. shown by Turing (Ann. of Math. vol. 52 (1950) pp. 491–505) to have an unsolvable word problem. The methods also prove that the metaproblem of identity for c.s. (the problem of classifying their defining systems according to whether or not their word problems are solvable) is unsolvable. (Received January 13, 1954.)

369t. S. C. Kleene: *On the forms of the predicates in the theory of constructive ordinals.* II.

Correcting the first paper of this title (Bull. Amer. Math. Soc. Abstract 48-5-215), it is now shown that the predicates $a \in O$ and $a <_O b$ of the system S_3 of notation for ordinal numbers are expressible in the respective forms $(\alpha)(Ey)R(a, \alpha, y)$ and $(\alpha)(Ey)S(a, b, \alpha, y)$ where α is a variable for a 1-place number-theoretic function and R and S are primitive recursive (i.e. as predicates of their number variables they are primitive recursive uniformly in α). Conversely, every predicate of the form $(\alpha)(Ey)R(a, \alpha, y)$ where R is general recursive is expressible in the form $\phi(a) \in O$ where ϕ is primitive recursive. There is a primitive recursive linear ordering of the natural numbers in which there exists an infinite descending sequence but no infinite descending arithmetical sequence. (Received December 4, 1953.)

370. Norman Shapiro: *Recursive sets of real numbers.*

By a dyadic sequence is meant a recursive sequence of zeros and ones which does not ultimately consist of ones. Given a set S of real numbers let $P_S(x)$ mean that the sequence whose Gödel number is x represents in the binary notation an element of S, P_S being a predicate whose domain of definition is the set of Gödel numbers of dyadic sequences. If S does not contain all recursive real numbers and if S contains the rational numbers, then every predicate of the form $(\exists y)R(x, y)$, R recursive, is many-one-reducible to P_S. If S is a set of algebraic real numbers containing the rationals, then every predicate of the form $(\exists y)(z)R(x, y, z)$, R recursive, is many-one-reducible to P_S. Either of the above theorems shows that if we regard real numbers as being presented by algorithms for the generation of dyadic sequences defining them, the decision problem for such S's is recursively unsolvable. By methods similar to these, most standard subsets of the real line have, in this sense, recursively unsolvable decision problems. An example of a set with solvable decision problem is $S = \{x \mid 0 \leq x < 1/2\}$. In fact P_S is potentially partial recursive. (Received January 11, 1954.)

STATISTICS AND PROBABILITY

371. R. P. Peterson: *Density unbiased point estimates.*

Let F be an $s+1$-parameter family of probability density functions $p(x, \theta, \sigma)$ where θ and $\sigma = (\sigma_1, \cdots, \sigma_s)$ are unknown parameters. Let $X_n = (x_1, \cdots, x_n)$ be a point in n-dimensional sample space M_n, where $x_1, \cdots, x_n$ are n (not necessarily independent) observed values of x, and let $P(X_n, \theta, \sigma)$ denote the joint probability density function at $X_n \in M_n$. Then the *mean probability density function of F generated by the point estimate $f(X_n)$ relative to θ* is given by $\phi_f(x, \theta, \sigma) = \int_{M_n} p(x, f(X_n), \sigma)P_n(X_n, \theta, \sigma)dX_n$, where $\phi_f(x, \theta, \sigma)$ is a probability density function itself provided $p(x, f(X_n), \sigma)$ is measurable in X_n over M_n. A point estimate $f(X_n)$ of θ is *density unbiased* if $\phi_f(x, \theta, \sigma) = p(x, \theta, \sigma')$, where σ' is any admissible value of σ. If $p(x, \theta, \sigma) = (1/\sigma)p((x-\theta)/\sigma)$, $\sigma > 0$, then the characteristic function of $p(x, \theta, \sigma)$ is of the form $e^{it\theta}k(+\sigma)$ and if $k(+\sigma)$ is closed under multiplication ($k(+\sigma_1) \cdot k(+\sigma_2) = k(+\sigma_3)$ where σ_1, σ_2, and σ_3 are admissible values of σ), then any linear homogeneous point estimate $\sum_1^n a_i x_i$, $\sum_1^n a_i = 1$ is a density unbiased estimate of θ. It is further shown that the ordinary sample mean $\bar{x} = \sum_1^n x_i$ is, in a certain sense, the "best" density unbiased point estimate for the mean of any normal population. (Received January 15, 1954.)

TOPOLOGY

372t. Seymour Ginsburg and J. R. Isbell: *Uniformly convergent functions on partially ordered sets.*

A function f from a partially ordered set P to a uniform space uX [J. Tukey, *Convergence and uniformity in topology*] is called *uniformly convergent* (u.c.) if, for each covering $\{U_\alpha\}$ in u, there is a family of residual subsets Q_α of P, whose union is cofinal in P, satisfying $f(Q_\alpha) \subseteq U_\alpha$; if $\{Q_\alpha\}$ can always be chosen finite, f is *essentially bounded* (e.b.). If uX is a locally compact group then each u.c. function satisfies the defining condition for all open coverings of X. Consider the ring $U(P, R)$, and its subring $U(fP, R)$, consisting of all u.c. resp. e.b. real-valued functions on P, modulo functions converging to zero. $U(fP, R)$ is naturally isomorphic with the ring of all continuous real-valued functions on the character space βP [M. Stone, *Applications of Boolean rings to general topology*, Trans. Amer. Math. Soc. vol. 41 (1937) pp. 375–481] of the Boolean algebra of maximal residual subsets of P [S. Ginsburg, *A class of everywhere branching sets*, Duke Math. J. vol. 20 (1953) pp. 521–526]. Some residue class ring (empty in a nontrivial case) of $U(P, R)$ is isomorphic with the ring of continuous real-valued functions on a certain pseudo-discrete space uP [L. Gillman and M. Henriksen, *Concerning rings of continuous functions*, to appear in Trans. Amer. Math. Soc.]. (Received February 12, 1954.)

373t. J. R. Isbell: *Some details concerning function rings.*

Terminology as in the preceding abstract. The ring $U(P, R)$ shares with the ring of continuous functions on a pseudo-discrete space the property that every ideal is closed in the m-topology [L. Gillman and M. Henriksen, op. cit.]. Its residue class fields are all real closed; the proof for continuous functions [J. Isbell, *More on the continuity of the real roots of an algebraic equation*, Proc. Amer. Math. Soc. vol. 5] applies, by use of the result of the preceding abstract, on the locally compact group E^{2n+1}. When such a field is not the real numbers, every countable subset has an upper bound in the field ordering; a proof is given which applies as well for continuous functions, simplifying the proof of Hewitt and Henriksen [E. Hewitt, *Rings of real-valued continuous functions*, I, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 45–99; M. Henriksen, Bull. Amer. Math. Soc. Abstract 60-2-319]. (Received January 15, 1954.)

374. R. K. Lashof: *Lie algebras of locally compact groups.*

An LP-group is the projective limit of Lie groups. Yamabe has shown that every connected locally compact group is an LP-group. If $G = \text{Lim } G_\alpha$, G_α Lie groups, then the (possibly infinite-dimensional) Lie algebra $\mathfrak{g}$ of G is the projective limit of $\mathfrak{g}_\alpha$, $\mathfrak{g}_\alpha$ the Lie algebra of G_α. The existence and uniqueness of the Lie algebra of an LP-group is shown and also its connection with the group by means of an exponential map. The notion of a universal covering group for connected groups with the same Lie algebra is extended to LP-groups (with a modified definition of covering). Classical theorems connecting homomorphisms of group and algebra are generalized. A 1-1 correspondence is established between "canonical LP-subgroups" of a group and subalgebras of its Lie algebra. The Lie algebra of a locally compact group is shown to have the form $L \times A \times S$, where L is a finite-dimensional Lie algebra, A an (infinite-dimensional) abelian Lie algebra, and S the (infinite) direct product of simple finite-dimensional compact Lie algebras. (Received December 21, 1953.)

375*t*. P. M. Swingle: *m-divisible n-point connexes.*

Among other sets we define: let M be a connexe; then M is said to be an n-point connexe if M is not disconnected by the omission of a subset of power n; an n-point connexe M is an m-divisible n-point connexe if M is the sum of n, but not of a greater number, of mutually exclusive n-point subconnexes. Among other things is proven: if n is an integer ≥ 2 and M is a 1-divisible n-point connexe and also a $(4n+1)$-point connexe, then M does not have a finite cut-set; if n is an integer ≥ 2, M is a $(2n+1)$-point connexe in the plane with complete minimal cut-set N, and if $M-N=H+K$ strongly separate where H is disconnected and N' is a subset of power $n+1$, then $H+N'$ is an n-point connexe. (Received January 13, 1954.)

L. W. Cohen,
Associate Secretary

OTTO SZÁSZ

On September 19, 1952, Otto Szász, Professor at the University of Cincinnati, died in his 68th year. Szász was a native of Hungary, studied in Budapest, Göttingen, Munich and Paris, and received his Ph.D. degree at the University of Budapest in 1911. He was Privatdozent (later with the title of a professor) at the University of Frankfurt, Germany, from 1914 until 1933. He was also Privatdozent at the University of Budapest. In 1933 he came to the United States, first to the Massachusetts Institute of Technology and Brown University and in 1936 to the University of Cincinnati where he resided until his death. The only interruption of his stay at this university was a year spent with the Institute of Numerical Analysis at the University of California in Los Angeles. In 1930 Szász received the Julius König prize of the Hungarian Mathematical and Physical Society. He was a member of the American Mathematical Society and of the Mathematical Association of America.

The life of Szász was spent in deep devotion to mathematical study, research and teaching. His steady preoccupation with mathematics, his erudition and broad knowledge of classical and contemporary literature and his perseverance in dealing with open problems of a number of varied fields have secured a firm place for him in the mathematical life of Hungary, Germany and of the United States. He inspired many young persons and collaborated with numerous mature mathematicians. He maintained warm and mathematically fruitful relations with such great personalities as Fejér, Landau, Perron, Pringsheim and I. Schur. The short résumé of his most important accomplishments to be given below reflects these relationships.

We can distinguish various periods in the work of Szász. Economy of space compels us to quote (by numbers in brackets) only a few of his papers enumerated at the end of this note.

1. **Continued fractions.** The work of Szász in this direction deals with convergence questions [5; 8] generalizing certain results of Perron. Another paper deals with the special case of continued fractions of the Stieltjes-A. Markov type [2] answering a conjecture of Perron. Finally [1] is concerned with the irrational character of certain continued fractions.

2. **Approximations** [3; 6]. These contributions are related to the famous problem of S. Bernstein solved by Ch. H. Müntz about the completeness of a set of powers x^{p_n} in a finite positive interval.

3. **Bounded power series.** This work was inspired by the problem of Landau to determine the maximum modulus of the nth partial sum of those power series $f(z)$ which are regular in the unit circle and for which $|f(z)| < 1$ in $|z| < 1$ holds. Szász gives various far-reaching generalizations of the result of Landau [7].[1] His further contributions in this direction are connected, among others, with a related problem of F. Riesz.

4. **Trigonometric polynomials.** These papers [10; 12] deal with inequalities for the coefficients of trigonometric polynomials subjected to various conditions of boundedness. Another particularly interesting result of Szász [4] is the following inequality for a non-negative trigonometric polynomial $\sum c_\nu e^{i\nu\theta}$, $-n \leqq \nu \leqq n$:

$$\sum |c_\nu| \leqq (n+1)c_0,$$

with equality for the Fejér means $c_\nu = 1 - |\nu|/(n+1)$. We mention also [7] a very simple proof for the theorem of M. Riesz (in fact of S. Bernstein) on the derivative of a rational polynomial which remains under a given bound in the unit circle.

5. **Fourier series.** Probably the most interesting contribution of Szász [11] to this field is the refinement of certain results of S. Bernstein and Hardy-Littlewood concerning the sum $\sum |c_n|^k$ where c_n denotes the Fourier coefficients of a given periodic function $f(x)$. Assuming that the mean value

$$\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(u+h) - f(u)|^p du\right\}^{1/p}$$

is of the order $|h|^\alpha$, $1 < p \leqq 2$, $0 < \alpha < 1$, Szász proves that the sum in question is convergent provided $k > p(p\alpha + p - 1)^{-1}$ and this bound is in a certain sense the best possible. Other theorems deal with the order of approximation by Fejér means connected again with certain results of S. Bernstein [9].

Further remarkable contributions of Szász are concerned with Tauberian theorems, various methods of summability, Gibbs phenomenon, etc. Many of the papers from the last period of his life (after 1933) are devoted to these subjects. He contributed, jointly with E. Hilb, to the Encyclopedia of Mathematics by an article dealing with expansions in terms of functions of a complex variable.

His students and friends will always warmly remember him as a man of gentle, unassuming and quiet personality. His life and energy were dedicated to the promotion of simple and beautiful problems of Mathematics, in particular of the classical Analysis. His nearly one hundred and thirty mathematical papers remain a living document of his efforts.

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[1] These generalizations turned out to be useful in the investigations of W. Rogosinski-G. Szegö, Math. Zeit. vol. 28 (1928).

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GABOR SZEGÖ

NELS JOHANN LENNES

Nels Johann Lennes was born June 15, 1874, in Henne, Norway. He came to the United States in 1890, unable to speak the English language. He entered the University of Chicago in 1896 and received from that institution a bachelor's degree in 1898, a master's degree in 1903, and a Ph.D. degree in 1907, having been influenced by E. H. Moore and others at Chicago. He was married and the father of three children.

From 1898 to 1907 he taught in Chicago high schools in addition to doing graduate study at the University. He was an instructor at Massachusetts Institute of Technology from 1907 to 1910 and at Columbia University from 1910 to 1913. He became Professor of Mathematics and Chairman of the Department at Montana State University in 1913 and held that position until his retirement in 1944. His death occurred at Missoula, Montana, November 21, 1951.

He was a mathematician of excellent talent, one of the precursors of modern abstract mathematics, who made a number of interesting contributions, and who in different circumstances would very likely have made many more. In the latter part of his life he wrote nearly one hundred textbooks which were widely used. The bibliography below contains only his scientific work.

His interests were mainly in functions of a real variable, foundations of geometry, and the topology of euclidean spaces. He proved [9] that a polyhedron separates space. In the same paper he also proved that a polygon can be divided into triangles and that the corresponding theorem is true for polyhedra. Apparently none of these facts had been proved earlier. Here he was motivated by definitions of area and volume. He also gave self-dual axioms for projective geometry. In one note [3] he analyzed conditions under which the Heine-Borel theorem is valid and pointed out that if points at infinity are added the theorem is not restricted to bounded sets. Systematic use of the "Heine-Borel property" (i.e. compactness) was a feature of the book [4] on real function theory.

In the paper [10] which is one of his best the term connected set for subsets of the plane is defined as follows: "A set of points is a connected set if at least one of any two complementary subsets contains a limit point of points in the other set." In [8] the same definition was given for subsets of any euclidean space. For closed sets a similar definition had been given by Jordan, *Cours d'analyse*, vol. I, 1893, p. 25. Using this definition Lennes characterized an arc as a compact connected set containing two distinct points a and b and such that no proper connected subset contains a and b. This characterization which was given for the plane and later euclidean spaces extends unchanged to more general spaces, and remains in wide use.

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DEANE MONTGOMERY AND OSWALD VEBLEN

BOOK REVIEWS

Logic for mathematicians. By J. B. Rosser, New York, McGraw-Hill, 1953. 14+540 pp. $10.00.

This book is undoubtedly a major addition to the literature of mathematical logic. Yet it is hardly the kind of addition which one would expect from its title, its preface, and the reputation of its author. The preface states that the book is intended as a textbook for mature mathematicians; that, as such, it aims to be relatively complete; and that matters of considerable logical importance which are not interesting for the mathematician are purposely omitted. The book turns out to be a detailed exposition, modeled on the *Principia Mathematica*, of a particular logistic system.

For the purposes of criticism the book will be divided into two parts. The first part, comprising the first eight chapters, develops what may be called the basic logic, i.e. logic through the restricted predicate calculus, including descriptions and equality. The second part deals with what may be called the higher logistic, which in this case means abstract set theory, including the more elementary portions of cardinal and ordinal arithmetic. It will be convenient to discuss what seems to be the main aspect of the second part first, and then to take up the first part along with the certain auxiliary aspects of the second.

The system developed in the second part is Quine's *New foundations*, which will be called NF. This is the system proposed by Quine at the winter meeting of 1936 and published in the American Mathematical Monthly in 1937. It followed a series of papers in which Quine, who was at first an advocate of the theory of types, made a study of axiomatic set theory. He showed in these that the axiom of subsets (Aussonderung), which guarantees the existence of the subset of those elements of a given set which have a certain property, is the essential axiom scheme of the theory. In NF he proposed that the restriction to subsets of a given set can be abandoned, provided that the property is required to be stratified, i.e. such that its variables (as they appear in that particular formula) can be classified so that the restrictions of the theory of types are satisfied. In the resulting system there is a single universal class, rather than a hierarchy of universal classes in the various types; also each class has a unique complement; and cardinal numbers exist as sets without typical ambiguity.

The consistency of the system was at first in doubt (cf. Zentralblatt für Mathematik 16: 193). In 1939 Rosser made a serious attempt to prove it inconsistent, and later Lyndon and Rosser did prove the inconsistency of the stronger system of Quine's book of 1940; but NF withstood all these onslaughts. On the other hand certain anomalies soon became apparent in it. As Quine himself showed in 1937, Cantor's paradox is only avoided because a class α is not always cardinally similar to the set of its unit subsets (which Rosser calls USC (α)). There are various anomalies connected with mathematical induction; thus in the present book one needs a special axiom to the effect that the first n natural numbers form a set with the cardinal number n. The restriction to stratified properties is peculiarly annoying; not only is the criterion tedious to verify (much more so than the restriction to subsets of a given set) in case defined terms are present, but it is apt to fail in the most unexpected places. The most striking anomaly, however, is that shown recently by Specker (Proceedings of the National Academy for September, 1953; the presentation is based on the present book). Specker showed that the axiom of choice is incompatible with NF, in fact the cardinals are not well ordered. Informally, Specker's conclusion can be made plausible as follows. For any cardinal n, let $T(n)$ be the cardinal of USC(α) when that of α is n, and let ∞ be the cardinal of the universal class; then the cardinals

$$\infty,\ T(\infty),\ T(T(\infty)),\ \cdots$$

form a decreasing sequence with no first member. In addition to the ordinary sets and cardinals which ascend from below, there are thus extraordinary sets which descend from ∞. Only the ordinary sets are of any interest in mathematics. It seems clear that, in order to develop a mathematics which remotely resembles what we are used to, a restriction to ordinary sets will have to be introduced at some point. Such a restriction will be similar to those in more orthodox set theory; and in that case the restriction to stratified properties is an unnecessary and bothersome complication.

Notwithstanding all this—indeed because of it—the system NF is of great logical interest. Its promulgation was a major help to deepening our understanding. In presenting the most elaborate treatment of this system which has yet appeared, Rosser has rendered a real service to logic. But NF is a strange choice for a logic for mathematicians. Even before Specker's discovery enough was known of the system's anomalous and dangerous character to have warned Rosser of the risks he was taking. These risks, moreover, could have been largely avoided by using a more modern, postulational point of view. Much of the theory of the book is independent of the particular logistic system chosen, and one would expect a cautious author to point out this independence. A large part of the theory of relations, for example, could be deduced (using basic logic) from certain postulates which would be valid in NF for homogeneous relations and in orthodox set theory for relations in a restricted universe. Such an approach would be more suitable for mathematicians than an unqualified commitment to a system of such questionable character. In choosing the latter course, the reviewer thinks that Rosser has done the exact opposite of what he claims to have done in his preface.

Let us now turn to the criticism of the first part of the book. No attempt will be made here to outline the contents of the various chapters. Rather, certain general features will be discussed. These include: first, features which seem to be advantages of the book as a textbook; second, contributions of the book to technical mathematical logic; and third, criticism of certain faults.

Any one acquainted with Rosser's previous work will know that at his best he is a master of lucid exposition. This book contains many instances of his skill. A notable example is the theorem on the completeness of propositional algebra. Here a clear presentation of the proof by Kalmar's method is made even more perspicuous by a preliminary consideration of a special case. Another instance is the explanation of the dot notation on p. 19 ff. In other cases, where the clarity is not so noticeable, the discussion is stimulating. Nearly every important logical idea is discussed from the standpoint of its significance first, and these discussions are usually noteworthy on one or both of these counts. Some further examples are the discussion of variables and quantifiers at the beginning of Chapter VI; of what Rosser calls Rule C in Chapter VI, §7; of classes and the Russell paradox in Chapter IX, §1; and of functions in Chapter X, §5.

The book also contains a large number of exercises. Furthermore, considerable space is devoted to the analysis of concepts and proofs from ordinary mathematics. These illustrations are drawn from such fields as elementary geometry, advanced calculus, basic topology (Hausdorff spaces), etc. Although the proofs analyzed are among the simpler ones from the point of view of the mathematician, yet they serve well as illustrations. There is also some attention to developing techniques which would be useful for such analysis. The reviewer regards this contact with ordinary mathematics as one of the strong points of the book.

We turn now to the technical contributions in the book. The reviewer has selected three for special comment.

The first of these is the formulation of propositional algebra. The author presents an axiomatic formulation in which all the connectives are defined in terms of conjunction and negation. There are three axiom schemes. These differ from the four scheme formulation of the reviewer (*Leçons de logique algébrique*, Paris, 1952, p. 114), in that the commutative law is eliminated by introducing a twist in one of the other schemes—a similar trick was used by Nicod (1917) and also works for the *Principia Mathematica*. The only other formulation, so far as the reviewer knows, which uses these primitives is one of Sobocinski (1939). (However, these primitives were used, in combination with others, by Lewis and other writers on strict implication, and they were used by Peano.)

The second point is the elaboration of certain rules of the restricted predicate calculus having to do with quantification over restricted ranges on the one hand, and with rules for introducing quantifiers on the other. The first of these is related to work of A. Schmidt (1938) and Wang (1952); the second is equivalent to the corresponding rules of Gentzen (see below); but the author's treatment has features of interest.

The third point is the axiomatic theory of descriptions. The notion $(\iota x)F(x)$ is taken here as primitive and is taken so that it is always an object. The idea is extended to variables over a restricted range in a peculiar way. Let A be an object specified in advance but belonging to the range; then when $(\iota x)F(x)$ (over the restricted range) does not exist in the ordinary sense it is defined to be A. This does not always agree with our intuitions. Thus, if the range consists of human beings living in 1953, let A be Winston Churchill; then

The King of France is bald

turns out to be true. On the whole, however, the theory of descriptions is an improvement over that of Russell.

Let us now turn to some entries on the other side of the ledger.

In the first place there is serious confusion about the basic nature of formal reasoning. In several places Rosser states explicitly that he adheres to the view that in such reasoning one is talking about meaningless marks. That requires, among other things, the use of quotation marks. He explains the usual use of these on page 50 and claims (on p. 51) to have used them according to that rule up to that point. Yet on page 12 we read, "Consider two statements 'P' and 'Q' of symbolic logic which are translations of English sentences 'A' and 'B'." But "P," "Q," "A," and "B" are capital letters; by no stretch of the imagination can they be either statements or sentences.

On the other hand on p. 83 we read that x (rather than "x") is the twenty-fourth letter of the alphabet. These are not isolated instances; there is a general confusion between use and mention throughout. There is no use of Quine's quasi-quotation marks or other devices which are indispensable for correct thinking along syntactical lines. In fact the evidence is that Rosser is not actually thinking in that way; he is thinking not of symbols but of their meanings. If he were to abandon the fiction that he is talking about symbols and say instead that he is using symbols according to prescribed rules without regard to their meaning, then most of what he says after p. 51 (where he allows himself to be careless) would be correct, whereas the earlier part where he has "scrupulously tried to be correct" is a muddle in any event. This confusion also obscures considerably the discussion of variables and functions. In these cases it is literally not clear just what the author means.

Even apart from this criticism the discussion of the nature of logic leaves much to be desired. In his amusing introductory chapter he fails (so the reviewer thinks, of course) to bring out with sufficient clarity that the aim of logic is objective analysis; that it strives to replace subjective feelings of correctness by objective criteria; that without it there is no objective meaning to be attached to mathematical rigor or truth. The lack of this emphasis gives this introduction an air of pedantry. Later on he does not characterize with sufficient definiteness what the reviewer would call epimethods, including the nature of constructiveness, and the relation of the constructive epitheorems to the fundamental requirements of objectivity. The reviewer finds the discussion of three logics on p. 79 a bit confusing. To him these three "logics" seem so little alike that to call them all by the same name is hardly more than a three-way pun.

Another point is that certain standard terms are used in senses quite different from those generally accepted. For example the terms "dual" (Chapter IV) and "indeterminate" (Chapter VI) are used in nonstandard senses. In regard to "substitution theorem," there is confusion between that sort of substitution (German "Ersetzung") and that indicated (p. 209) by "$\{\text{Sub in } S{:}A \text{ for } x_1\}$" (German "Einsetzung"). The distinction between these two can be maintained by using "replacement" for the former (see Carnap, *Logical syntax of language*, 1937, p. 36). The usage of Carnap deserves to be more generally followed than it is. Finally the reviewer finds it confusing to talk about certain statements in proofs by Rule C as "steps" in the proof; these statements are really suppositions which are eliminated later. This circumstance is made much clearer by the Gentzen formulation of the rule.

The author's discussion of functions has already been commented on. It would be well for every mathematician to read this section. There is much truth in it. Because of the lack of a suitable notation for functions at least one eminent mathematician has published an erroneous result. Nevertheless, the passage should be read with a critical eye. Before making sweeping changes the more basic confusion between use and mention must first be cleared up. As to whether the notation "$f(x)$" denotes a function or a function value, that is a question of whether the variable is bound or free; in such a statement as

$$\frac{d}{dx}f(x) \doteq g(x)$$

all instances of "x" are bound, either explicitly or by the context, and therefore the reviewer would say that the statement has to do with functions. A notation for explicitly indicating the binding of variables would of course be useful in advanced work. (It would be useful too in Chapter VI; in fact the reviewer wonders why the λ-notation was not introduced before considering quantifiers.) Since the λ-notation conflicts with other uses of "λ" one of the other notations such as "$[x]A$" (reviewer), "$x \rightarrow A$" (H. T. Davis), "$_xA$" (suggested verbally by Quine), "$[A]_x$" (reviewer) might be used.

Again, the author's opinion as to what sort of logic is suitable for mathematicians is certainly very different from the reviewer's. The particular sort of axiomatic propositional algebra used seems to the reviewer a technicality; the system of Hilbert and Bernays is more natural. Likewise, the long list of technical developments of set theory, developed in sickening detail, has an interest which seems rather special. On the other hand, the reviewer thinks the following topics deserve greater emphasis: the Gentzen rules, which are far closer to intuitive reasoning than Rosser's Rule C, for example; the intuitionistic and minimal logics; the relations to other algebraic systems, which can be developed to a certain extent as independent systems without basing them on logic; the theorems of Gödel; independent recursive or combinatory number theory, etc.

In conclusion, the reviewer can see both merits and faults in this work of Rosser. It is the most complete treatment of Quine's *New foundations* which now exists; and it makes several minor contributions to technical logic. It has several virtues from the expository viewpoint. As a logic for mathematicians, however, it is something of a disappointment. It makes an unfortunate choice of logistic system and goes overboard in commitment to it. Furthermore, it seems to the reviewer a bit old-fashioned; contrary to the statement in the preface it considers several topics whose interest seems technical and special; while other topics of modern logic, whose mathematical interest seems far greater, are ignored.

H. B. CURRY

Theory and applications of distance geometry. By L. M. Blumenthal. Oxford University Press, 1953. 12+348 pp. $10.00.

The first systematic development of geometric aspects of metric spaces is due to Karl Menger, who published a number of results in his *Untersuchungen über allgemeine Metrik* in 1928. Since then several authors have made contributions to this new domain. In 1938 Blumenthal gave a survey of the material available at that time in his book *Distance geometries*, one of the University of Missouri Studies. Considerable progress has been made since 1938 and with this new book the author aims to give a detailed introduction to the subject. About two-thirds of the text is devoted to imbedding and characterization problems. This shows clearly that the writer's own taste has played a large part in selecting the topics.

In the first chapter the author introduces all the principal notions which play a part in the theory of abstract metric spaces. Some have a topological character. It should be remarked that the definition given for compact spaces (p. 29) is different from the usual one, which may lead to some confusion. Chapter II is devoted to the notions of betweenness, convexity and metric segments. It contains Aronszajn's proof of one of Menger's theorems concerning segments: Each two points of a complete and convex space are joined by a metric segment. Chapter III gives in 30 pages a somewhat scanty treatment of metric curve theory. It contains several possible definitions of length and curvature and a survey of the results obtained so far, but for the proofs of many results the reader is referred to the original papers. A fuller treatment is given of the characterization problem. A subclass of metric spaces is characterized metrically whenever necessary and sufficient conditions (in terms of the metric) are obtained which have to be satisfied in order that a metric space be congruent with a member of this subclass. If the subclass consists of the set of subspaces of a metric space the problem is called the imbedding problem. An extensive treatment of this problem for the Euclidean spaces is given in Chapter IV. Most of these results are due to Menger but the author has succeeded in simplifying several of the proofs. If the subclass consists of one space only the problem is to characterize this space metrically among the metric spaces. Several characterizations are given for Euclidean spaces and Hilbert spaces. In a wide class of spaces the validity of the weak four-point property or of the Pythagorean theorem is shown to ensure the euclidean character of the metric. Important notions in the theory of imbedding are the notion of congruence order and of congruence indices of a metric space with respect to some class of metric spaces. A detailed study is made of the congruence order of some euclidean subsets, mostly with respect to the class of subspaces of a euclidean space (E_n). The congruence order of E_n with respect to metric spaces is shown to be $n+3$; this means that a metric space is congruently contained in E_n if every $n+3$ of its points have that property. This leads to the study of a pseudo-euclidean space S, for which each set of $n+2$ points is congruently imbeddable in E_n, without S being congruent with a subset of E_n.

Part III of the book is devoted to the imbedding and characterization theorems of spherical spaces, elliptic spaces, and hyperbolic spaces. Compared with the E_n no essentially different methods are needed for hyperbolic spaces and the theorems obtained are quite analogous. The theory of spherical spaces contains some new elements (diametrical points). One of the principal differences is the existence of pseudo-sets of more than $n+3$ points. These sets are completely investigated. The metric behaviour of elliptic spaces, as compared with euclidean spaces, is still more different and much more complicated. One of the peculiarities of elliptic space is the distinction between congruence and superposability. An intensive comparative study of these two notions is made. Another difference in metric character is shown by the theorems about the congruence order. The congruence order of the elliptic plane is 7 and not 5, whereas the congruence orders of elliptic spaces of more than two dimensions are not yet known.

The distance relations which enter into many metric characterizations of subsets of euclidean and non-euclidean spaces lead to a number of determinant theorems formulated in Chapter XIII as applications of distance geometry. As a second application the author presents a geometrization of the theory of linear inequalities. The final chapter deals with metric methods in the theory of normed lattices and autometricized Boolean algebras. Among the topics not included are the applications in the calculus of variations.

The book is clearly written and self-contained. The reader is well provided with exercises of various degrees of difficulty. The bibliography is almost complete, and the index adequate. In my opinion the author would have improved the usefulness of the book as a textbook if he had restricted the number of special problems treated in the text. On the other hand, this work is the only exposition of distance geometry available at this moment and it is certainly useful to have a survey of the results obtained so far in the geometric study of metric spaces.

J. HAANTJES

Exterior ballistics. By E. J. McShane, J. L. Kelley, and F. V. Reno. The University of Denver Press, 1953. 834 pp., 15 figures, 24 plates. $12.00.

This book forms a welcome addition to the very limited number of works on exterior ballistics; had there been many more competitors I suspect its welcome would have been equally warm.

The first chapter, roughly one-sixth of the book, is devoted to mathematical and physical preliminaries and is intended "to make the book intelligible to anyone who has had a reasonably good undergraduate course either in mathematics or physics." This chapter discusses vectors, the equations of rigid body motion, dimensional analysis (including an original proof of the Buckingham Π theorem) and appropriate parts of statistics. I feel that it is impossible in a single chapter, even a chapter as long as this, to provide an adequate background to exterior ballistics and am of the opinion that the authors were over-ambitious to attempt it. Any complete background picture must contain the elements of aerodynamics as well as of dynamics, and any such inclusion would no doubt have made the space required entirely prohibitive. Leaving aside the desirability of such a chapter and its sins of omission I found the presentation of the material selected very satisfying and feel that the authors are to be congratulated on it, save on one point. This point concerns the equations of motion of a rigid body which are proved on the sweeping assumption that the internal forces between any two points of the body lie in the line joining them (for a discussion of this point see, for example, Jeffreys and Jeffreys, *Methods of mathematical physics*, Cambridge University Press, pp. 76 and 294).

Coming now to exterior ballistics proper the treatment starts in Chapter II by a discussion of the aerodynamic forces acting on the projectile; two force coefficients (K_{XF} the Magnus cross force due to cross spin and K_S the cross force due to cross spin) and a moment coefficient (K_{XT} the Magnus cross-torque due to cross spin) are added to "complete" the system considered in the classical treatment of Fowler, Gallop, Lock and Richmond (Philos. Trans. Roy. Soc. London vol. 221 (1921) p. 295). The equations of motion (both C.G. and yawing) are then derived relative to axes fixed in the projectile—one along the projectile axis; the equations in the two directions perpendicular to the axis are combined, as is customary, into a single complex equation. Chapter III is devoted to methods of determining drag from velocity and retardation measurements, to wind tunnel methods of determining drag, lift, and overturning moment coefficients (K_D, K_L, and K_M) and to damping experiments in the wind tunnel for providing the yawing moment due to yawing (coefficient K_H) which by variation of the axis of oscillation can be made to give the cross force due to cross spin (coefficient K_S).

Chapters IV to IX are devoted to the "normal" equations of the trajectory, their solution and their differential effects. It is perhaps here that the time which has elapsed since the book was first projected is most manifest. There is no doubt in my mind that the advent of digital computing machines has vastly affected this side of the problem and that we are rapidly approaching a state where a trajectory and its differential effects can and will be obtained by brute machine force. In these circumstances the space devoted to these aspects of the problem appears unduly large but there is no doubt that the chapters do provide a most thorough survey of the whole field.

Apart from a short chapter on bomb-sights and an historical appendix, the rest of the book is effectively devoted to the angular motion of projectiles spun and unspun, and its effect on the C.G. motion; it is here that the book will, I think, capture most attention. I found it a most stimulating account of the subject and would single out for special mention the treatment of stability. As a finale to this part of the work the analysis of Spark Range data for the spun projectile is shown to lead theoretically to the determination of all the force and moment coefficients except K_{XT} (the Magnus cross torque due to cross spin). Practically, the accuracy with which the coefficients can be determined is severely restricted—the authors quote probable errors of 5% for K_L (the lift coefficient), 30% for K_F (Magnus cross force due to cross velocity), and over 50% for both K_{XF} (Magnus cross force due to cross spin) and K_S (cross force due to cross spin). This means that with existing methods of analysis and measurement, of the three additions to the force system used by Fowler et al., one (K_{XT}) is undetermined and two (K_{XF} and K_S) have probable errors in their determination of over 50%. Though one cannot help feeling therefore that, since the Spark Range measurements represent the most refined measurements available, the Fowler system is still adequate for many practical purposes, one cannot fail to see the need for further research. If the analysis presented here were to provide, as one hopes it will, a stimulus towards methods of measurement which will define the additional coefficients more precisely and this were the only outcome of the book, it would indeed have served its purpose.

L. HOWARTH

A first course in functions of a complex variable. By W. Kaplan. Cambridge, Mass., Addison-Wesley, 1953. 8+485–620 pp. $3.50.

The publishing of this text represents a new step by the publishing industry, taken presumably to help combat the sharply increasing price of textbooks. The author's *Advanced calculus* (A) published in 1952 contained a long chapter on complex variables, and it is this chapter now in separate binding that is offered at $5.00 less than the price of A. The only alteration, aside from the correction of misprints, is the addition of an appropriate index, a subset of the index of A.

The author has an attractive style of writing, the material is well organized and in a natural order, and despite the limited space available the author covers an amazing amount of material, including even a brief account of the applications of conformal mapping to the theory of elasticity. Answers are provided for all numerical exercises, a remarkable and refreshing innovation for advanced textbooks. On the debit side the treatment of the Riemann sphere and the linear fractional transformation is too brief, and the cross ratio is relegated to the exercise list.

Although the book has considerable merit and would certainly be a satisfactory text for a first course, the central question to be considered is the publisher's decision to reprint one chapter of A. Despite the author's assertion that "Outside of a few references to earlier chapters of A this book is essentially self-contained," there are a rather large number of places where the student who does not possess a copy of A will be at a distinct disadvantage. To cite only one such case, the author gives the Cauchy-Hadamard formula for the radius of convergence of a power series, and then for a proof states that "Since the ratio test, root test, and M-test all carry over to complex variables · · · , the proof given in Section 6–15 can be repeated without change." But Section 6–15 is not in the complex variable chapter.

Although recent years have seen a substantial increase in the number of texts on complex variables available, there is still a need for a good elementary one. The author has here an excellent start, but the reviewer hopes he will continue the present book in both directions and expand it in the middle so that on the one hand it will really be self-contained, while on the other hand the student who wishes to pursue the subject further can do so.

A. W. GOODMAN

Bernstein polynomials. By G. G. Lorentz. (Mathematical Expositions, no. 8.) University of Toronto Press, 1953. 10+130 pp. $5.75.

For a function $f(x)$ defined for $0 \leqq x \leqq 1$ the formula

$$B_n(x) = \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right)\binom{n}{\nu} x^\nu (1-x)^{n-\nu}$$

defines the Bernstein polynomial of order n of $f(x)$. Evidently $B_n(x)$ is a polynomial in x of order less than or equal to n. S. Bernstein, who introduced these polynomials, proved that if $f(x)$ is continuous for $0 \leqq x \leqq 1$, then

$$\lim_{n\to\infty} B_n(x) = f(x)$$

uniformly for $0 \leqq x \leqq 1$, thus obtaining a particularly simple and elegant proof of the Weierstrass approximation theorem. Bernstein polynomials are of interest in themselves, and in addition play an important role in several other mathematical topics, notably in the finite moment problem and in the related theory of Hausdorff summability.

Chapter I deals with the approximation of continuous functions by Bernstein polynomials. A great many results are proved of which the following may serve as an example: if $f(x)$ satisfies a Lipschitz condition of order α, $0 < \alpha < 1$, then

$$|f(x) - B_n(x)| = O(n^{-\alpha/2}) \qquad (n \to \infty),$$

uniformly for $0 \leqq x \leqq 1$. Care is taken to explain the relation of these results to the general theory of polynomial approximation.

Chapter II treats the changes which are necessary if $f(x)$ is no longer continuous, but merely integrable, or even only measurable. For example if $f(x)$ is integrable $B_n(x)$ may be replaced by

$$P_n(x) = \sum_{\nu=0}^{n}\binom{n}{\nu} x^\nu(1-x)^{n-\nu}\left[(n+1)\int_{\nu/(n+1)}^{(\nu+1)/(n+1)} f(t)dt\right].$$

It is shown that

$$\lim_{n\to\infty} P_n(x) = f(x)$$

almost everywhere for $0 \leq x \leq 1$. Also considered are a number of modifications and generalizations of Bernstein polynomials. For example, if the basic interval is $(0, b)$ rather than $(0, 1)$ so that $B_n(x)$ is replaced by

$$B_n(x, b) = \sum_{\nu=0}^{n} f\left(\frac{b\nu}{n}\right)\binom{n}{\nu}\left(\frac{x}{b}\right)^{\nu}\left(1 - \frac{x}{b}\right)^{n-\nu}$$

and if $f(x)$ is continuous and bounded for $0 \leq x < \infty$, then

$$\lim_{n\to\infty} B_n(x, b_n) = f(x) \qquad (0 \leq x < \infty)$$

if $b_n \to \infty$, $b_n = o(n)$, as $n \to \infty$. Also contained in this chapter are certain results on summability.

Chapter III is devoted to the finite moment problem. A sequence $\{\mu_k\}_0^\infty$ is said to be a moment sequence if there exists a function $\alpha(x)$ of bounded variation in $0 \leq x \leq 1$ such that

$$\mu_k = \int_0^1 x^k d\alpha(x).$$

The "moment problem" is to determine necessary and sufficient conditions that a given sequence be a moment sequence with $\alpha(x)$ of some prescribed form. Such conditions are most simply expressed in terms of the polynomial operator $\mathfrak{M}$ defined by

$$\mathfrak{M}[c_0 + c_1x + \cdots + c_nx^n] = c_0\mu_0 + c_1\mu_1 + \cdots + c_n\mu_n.$$

For example $\{\mu_k\}_0^\infty$ is a moment sequence with $\alpha(x)$ nondecreasing if and only if

$$\mathfrak{M}\left[\binom{n}{\nu}x^\nu(1 - x)^{n-\nu}\right] \geq 0 \qquad (\nu = 0, \cdots, n; n = 0, 1, \cdots).$$

Other theorems prescribe necessary and sufficient conditions that $\{\mu_k\}_0^\infty$ be a moment sequence with $\alpha(x) = \int_0^x a(u)du$, where $a(u)$ belongs to some linear space of functions over $0 \leq u \leq 1$. In this connection the author introduces the Banach spaces $\Lambda(\phi, p)$ and $M(\phi, p)$. Here $1 < p$, and $\phi(x)$ is a positive nonincreasing function normalized by the condition $\int_0^1 \phi(t)dt = 1$. A function $f(x)$ $(0 \leq x \leq 1)$ belongs to $\Lambda(\phi, p)$ if $\|f\|_\Lambda$ is finite where

$$\|f\|_\Lambda = \underset{\phi^f}{\text{l.u.b.}} \left\{\int_0^1 |f(x)|^p \phi^f(x)dx\right\}^{1/p},$$

ϕ^f ranging over all measurable rearrangements of $\phi(x)$. Similarly

$f(x)$ belongs to $M(\phi, p)$ only if $\|f\|_M$ is finite where

$$\|f\|_M = \operatorname*{l.u.b.}_{e} \left\{ \int_e |f(x)|^p dx \Big/ \int_0^{me} \phi(x)dx \right\}^{1/p},$$

e ranging over all measurable subsets of $(0 \leq x \leq 1)$ for which $me > 0$, where me is the measure of e. Many properties are established for these spaces. The chapter concludes with a discussion of Hausdorff summability.

Chapter IV presents the quite difficult theory of the behavior of Bernstein polynomials in the complex domain. The results here are too complicated to admit simple description or illustration.

The exposition of the theory of Bernstein polynomials which this volume contains is quite complete. In collecting together this material, much of it from widely scattered and not easily available sources, the author has performed a valuable service.

I. I. HIRSCHMAN, JR.

An introduction to abstract harmonic analysis. By L. H. Loomis. New York, van Nostrand, 1953. 10+190 pp. $5.00.

Abstract harmonic analysis is a branch of mathematics based on the concept of the Fourier-Lebesgue integral transform on the real line, replacing that line by a locally compact topological group G, and the functions $e^{i\lambda x}$ by functions arising in representations of G (e.g. characters). The appeal of this rather new and growing doctrine is largely derived from its providing a field of application for concepts and techniques (in point set topology, abstract integration, linear spaces, operators in Hilbert space, and so forth) which had not yet found applications commensurate with their purely aesthetic value.

In any case, this doctrine provides an eagerly desired framework on which may be fastened, for contemplative or expository purposes, many of the more striking ideas of spectral theory and functional analysis. The structural members of this framework are more or less purely algebraic concepts from the theory of groups and their representations, algebras and their ideal theory.

The present work (an outgrowth of courses given at Harvard by G. W. Mackey and later the author), in tacit agreement with this point of view, is built around one of the simplest and yet most valuable topologico-algebraic concepts: the commutative Banach algebra.

The first chapter contains point set theoretic preliminaries, topological products of compact spaces, and Stone's generalization of

Weierstrass' theorem on approximating continuous functions by polynomials.

Banach spaces are considered in the second chapter. The theorem that an operation with closed graph is continuous, as well as the Hahn-Banach theorem even for the complex case, is proved.

The chapter on integration (III) is based on the method devised by Percy John Daniell (1889–1946) in 1917, for elaborating a linear functional ("elementary integral") defined in a vector lattice of functions on a set S. In the definition of the L^p spaces, technical convenience is attained by admitting only Baire functions. Finally, the case of locally compact S is considered, and integration over product spaces (Fubini's theorem) is established for that case.

The introduction to Banach algebras (Chap. IV) requires close attention of the reader. Although the Banach algebras really needed later are commutative (except for the H^*-algebras), the discussion leaves the way open to wider applications. Thorough use is made particularly of the device of considering algebras of complex functions on arbitrary sets, generalizing the algebra of functions on the maximal-ideal space of the commutative case. The broad approach provides occasions for making variant definitions of essentially the same concept; in some cases the variations are apprently made tacitly. For example the reader must guess what Δ on p. 55 is, and amplify the propositions there.

Banach algebras without units are treated in Segal's manner, with regular ideals, adverses, etc. The adjunction of a unit element is considered but regarded as impractical. (In any case, the theorem about the enlarged ring on p. 59 is inaccurate, and should be limited to maximal ideals or commutative algebras.) Segal's treatment of the kernel (of a set of maximal ideals) and the hull (-set of an ideal) is included. The uniqueness of norm, spectral radius, application of analytic mappings within the algebra, are also presented in this chapter.

The succeeding one takes up commutative semi-simple Banach algebras which are *regular* (where this word now means that the sets $\{\delta;\ \delta(x)\neq 0\}$, x in A, form a basis for the space Δ of continuous homomorphisms of A on the complex numbers). It is characteristic of the arrangement of the book that a Tauberian-type theorem (about frontiers of hulls) is next proved rather than after the introduction of the L^1 algebra (which is the main application). Of course, suitable forward-looking references are always made at such points. The full functional representation theorem for commutative C^*-algebras and the spectral theorem come next. Then for *self-adjoint* algebras (e.g., if there is a suitable involution) there is proved a Herglotz-

Bochner-Weil-Raikov theorem on positive functionals, and a Plancherel theorem (after Godement). Thus the later proofs of the more familiar form of these theorems for group algebras become spectacularly abrupt.

Haar measure (with a neat treatment of quotient measures) and the Banach algebra of L^1 under convolution over a locally compact group having been disposed of in Chapter VI, the next chapter gives that satisfactory treatment of the theory of characters for the abelian case which the concept of maximal ideals makes possible. Pontrjagin's duality theorem itself is rather ignominiously embedded in a section on miscellaneous theorems, and provides an occasion for remarking that the important structural considerations on which this duality theorem was based in pre-normed ring days are not (as contrasted with A. Weil's book) here presented.

Compact groups and almost periodic functions are next treated by means of Ambrose's H^*-algebras, which were (of course) set up in an earlier chapter. The work closes with a stimulating chapter on further developments. A paper of H. J. Reiter, whose place of publication was indefinite at the time, is in the Trans. Amer. Math. Soc. vol. 73 (1952) pp. 401–427 and further references of interest are in R. Godement's paper, pp. 496–556 in the same volume.

The author is certainly to be thanked for his efforts in producing, in this field, a text which will take its place next to Pontrjagin's and A. Weil's in the literature of locally compact groups. Aesthetic considerations certainly played a large part in shaping the exposition. It is to be hoped that the book will be widely used as a text for graduate courses. I would guess that there was material for about 75 lectures of the usual sort. The instructor will have to consider each lecture carefully in advance, think about adding to the index (an index of symbols would help), and provide alternative motivations with detailed examples. The student will understand that this monograph is not a treatise on the fields partly covered by the chapters on prerequisites (Banach algebras, operator algebras, topological groups, etc.). Nevertheless, after careful study of this book, he should be able to decide whether he shares the ability of some, and the enthusiasm of many, for research in these fields.

RICHARD ARENS

Conformal representation. By C. Carathéodory. 2d ed. Cambridge University Press, 1952. 10+115 pp. $2.50.

Except for the addition of a chapter on the uniformization theorem, this is an exact reprint of the original 1931 edition (which, by an oversight, was never reviewed in this Bulletin).

It is a tribute to both the author's powers as an expositor and to his sure feeling for what is important and what is not, that despite the very considerable progress made in the theory of conformal mapping in the intervening years the book has retained its old freshness. There is much that could be added to the book, but there is hardly anything that should have been omitted. The style is concise without being obscure—a combination not often found in present-day mathematical writing—and the presentation is rigorous without being tedious or overly symbol-laden. In addition to still being a valuable introduction to the subject, the book is a model of mathematical exposition.

It goes without saying that the book is not an adequate introduction to the present state of the theory of conformal mapping. For example, the mapping of multiply-connected domains has been left out altogether (except in the doubly-connected case). Curiously enough, no mention is made even of what was known in this field at the time the first edition was written. The author must have shared the view—rather widely held at that time—that since the universal covering surfaces are simply-connected, the study of simply-connected domains would eventually take care of the general case.

The first two chapters contain a detailed discussion of the linear transformation and of the Klein-Poincaré non-Euclidean geometry based on the group of linear transformations of a circle onto itself. Chapter III considers some elementary mappings, and Chapter IV discusses Schwarz' lemma and some of its ramifications. Chapter V deals with the theory of normal families and includes the by now classical proof of the Riemann mapping theorem based on an extremal property of the mapping function. Chapter VI is devoted to the investigation of the boundary behavior of conformal mappings, while Chapter VII studies the mapping of closed surfaces. Chapter VIII, the only one not contained in the first edition of the book, reproduces the elegant and astonishingly short proof of the general uniformization theorem given by van der Waerden in 1941.

Z. NEHARI

Ideal theory. By D. G. Northcott. Cambridge University Press, 1953. 8+111 pp. 12s. 6d.

This is an excellent exposition of the basic facts about Noetherian rings, that is, commutative rings with unit element and ascending chain condition for ideals. Starting from scratch (no knowledge of modern algebra is assumed) the author proceeds clearly and efficiently up to the deeper parts of ideal theory that are used today in algebraic geometry. The transparency of style makes the book excellent for use as a text or for the practicing mathematician who wishes to absorb the essentials of the theory quickly and painlessly. The expert will be especially grateful for the unified presentation of results which, except for the more elementary parts already available in several texts, have thus far appeared only in Ergebnisse-style monographs or in the many widely scattered original papers of the last 25 years.

The chapter headings are: I. The primary decomposition. II. Residue rings and rings of quotients. III. Some fundamental properties of Noetherian rings. (This includes Krull's intersection theorem, symbolic powers of prime ideals, composition series for primary ideals, and dimension theory.) IV. The algebraic theory of local rings. (Including accounts of regular local rings and the quasi-gleichheit theory.) V. The analytic theory of local rings. (The existence of a local ring's completion and some transition properties.) The book ends with brief notes indicating the history and applications of the theory. The applications indicated are of course to algebraic geometry, but the author quite naturally refrains from identifying the two subjects.

Polynomial rings appear only incidentally or as examples and no mention is made of their unique factorization. Very little is said about rings of dimension one and even the theory of Dedekind rings, whose full development would have required about one extra page at the appropriate place, is omitted. Field theory, valuation theory, the structure theory of complete local rings, the important relations existing between the ideals of a ring and an integrally dependent over-ring, the Hilbert function, and unmixedness theorems are either missing entirely (presumably for reasons of space, connectedness and essentiality) or are mentioned in passing, in the notes at the end. There is a brief bibliography, most of whose items are made mainly of historical interest by the present tract. A brief list of the best (as opposed to the earliest) references would have been of more help to the unguided reader wishing to pursue the subject further.

MAXWELL ROSENLICHT

Complexes linéaires. Faisceaux de complexes linéaires. Suites et cycles de complexes linéaires conjugués. By A. Charrueau. Paris, Gauthier-Villars, 1952. 84 pp.

This memoir, as far as new contributions to the theory of linear complexes are concerned, is based primarily on a series of notes of the author appearing in the *Comptes Rendus* (Paris) during 1948 and 1949. These notes introduced the concept of a sequence C_i of linear complexes in which C_{j+2} is the reciprocal polar of C_j with respect to C_{j+1} for all j; and also the concept of a cycle of linear complexes which is defined as a closed sequence, that is, $C_{n+1} = C_1$, $C_{n+2} = C_2$, where n is the number of complexes in the cycle.

The first part of the work is devoted to a review of the classical theory of linear complexes. The second part is concerned with the study of pencils of linear complexes. This again is mostly a résumé of known facts. There is a discussion of the locus of the poles of planes passing through a fixed straight line with respect to all the complexes of a pencil; also a discussion of the straight lines conjugate to a fixed straight line with respect to all the complexes of a pencil. A good choice of coordinate system, used extensively by the author, results in simplifying the equation of any pencil of linear complexes and the equations of geometric configurations associated with the pencil.

The third part of the memoir is devoted to the study of sequences and cycles of linear complexes. In every pencil of linear complexes, $\Phi = \phi_1 + \lambda\phi_2 = 0$, where the base complexes have real coefficients, and at least one of these is not special, there exists a sequence (or cycle) of linear complexes. Let λ_1 and λ_2 be the values of λ corresponding to the special complexes of Φ, and let $\lambda_1 \neq \lambda_2$. If μ_j is the value of λ corresponding to C_j, and if $\alpha_j = (\mu_j - \lambda_1)/(\mu_j - \lambda_2)$, then $\alpha_j\alpha_{j+2} = \alpha_{j+1}^2$ so that the α_j form a geometric progression with the common ratio α_2/α_1. A necessary and sufficient condition that a sequence be a cycle of length p is that this common ratio be a primitive pth root of unity. In case $\lambda_1 = \lambda_2$, no cycle in a pencil is possible.

The most general collineation or dualistic transformation transforms a linear complex into a linear complex and a pencil of linear complexes into a pencil of linear complexes. Consider products of transformations by reciprocal polars relative to members of a pencil of linear complexes, $\Phi = \phi_1 + \lambda\phi_2 = 0$, where ϕ_1 and ϕ_2 are nonspecial and their equations have real coefficients. If $\tau_{\mu_1}, \tau_{\mu_2}, \cdots, \tau_{\mu_n}, \cdots$ are the transformations by reciprocal polars relative to the nonspecial complexes $\mu_1, \mu_2, \cdots, \mu_n, \cdots$ of Φ corresponding to the values $\mu_1, \mu_2, \cdots, \mu_n, \cdots$ of λ, and if $T_n = \tau_{\mu_1}\tau_{\mu_2} \cdots \tau_{\mu_n}$ is the product of the transformations $\tau_{\mu_1}, \tau_{\mu_2}, \cdots, \tau_{\mu_n}$, then T_n is a collineation if n is even or a dualistic transformation if n is odd. If $\lambda_1 \neq \lambda_2$, the transformation T_n for n odd and >1 is equal to the transformation τ_{k_n}, relative to the nonspecial complex k_n where k_n is defined by $(k_n - \lambda_1)/(k_n - \lambda_2) = \alpha_1\alpha_3\alpha_5 \cdots \alpha_n/\alpha_2\alpha_4 \cdots \alpha_{n-1}$. The collineation T_n depends only on $a_n = \alpha_1\alpha_3 \cdots \alpha_{n-1}/\alpha_2\alpha_4 \cdots \alpha_n$ for n even. The author is able to give a geometrical interpretation for a_n.

The case $\lambda_1 = \lambda_2$ is discussed in a similar manner.

There is a brief discussion of the case where the parameters of all the members of a pencil are equal, the case where the axes of the base complexes of a pencil are parallel, and the case where the axes of all members of a pencil are identical.

The memoir is well printed with only a few minor misprints. It seems to this reviewer that the first two parts of the memoir are unnecessarily repetitive. The third part on cycles and sequences is interesting and is more compactly written.

ALICE T. SCHAFER

NEW JOURNAL

Journal d'Analyse Mathématique. Vols 1, 2. Jerusalem, 1951, 1952–53. Vol. 1, 8+386+26 pp.; vol. 2, 8+381+23 pp. $20 per volume; 40% discount to individual members of mathematical societies.

Although there have been and still are a number of mathematical journals specializing in restricted fields, this is, I believe, the first to take analysis in general as its avowed specialty. As such, it is welcome to analysts, and should be welcome to other specialists at least for its effect in decreasing the pressure on other journals. The average length of the papers is appreciable: the two volumes contain 15 and 17 papers, respectively. Indeed, in a day when most authors are under pressure to emulate Landau in compression, although infrequently in clarity, of mathematical exposition, this journal is one of the last refuges of the long paper and the typographically complicated formula. One must be grateful to the editor, B. A. Amirà, for his courage, and to the organizations and individuals which have subsidized the publication so far. The journal is evidently going to be one of the indispensable components of any serious general mathematical library.

This is intended to be an international journal, and it has an international editorial board. In the first two volumes, an overwhelming majority of the papers originated in the U.S.A. or France. Both well-established names and relative newcomers appear in the list of authors. The languages of the journal are English and French; the extra pages at the ends of the volumes contain summaries in Hebrew. The typography of the formulas is excellent, but the pages have an unconventional look which can be traced to the wide spacing between the lines of the text and the use of spaced-out roman instead of italic type for statements of theorems and the like.

R. P. BOAS, JR.

BRIEF MENTION

Mécanique générale. By J. Pérès. Paris, Masson, 1953. 8+408 pp. 2,545 fr.

This book is written as a text for an intermediate course in mechanics. Its purposes are on the one hand to develop "the qualities of order, precision, and initiative"; on the other, "to bring into evidence the insufficiency and the limitations of classical mechanics" arising from the fact that the principles themselves furnish only an underdetermined system and have to be supplemented by constitutive equations "whose origin is empirical, which are sometimes very crude, and which fail to recognize the complexity of real bodies and the interdependence of phenomena relegated to the different domains of our science."

Although there is much material on mass-point dynamics and some on continuum mechanics, the author's principal interest is in rigid bodies. He emphasizes general theorems, clearly stated and succinctly derived, often in generalized forms due to the French mathematicians of the last century. Applications of these theorems, usually interesting ones, are given in nearly every case. On the other hand, in keeping with the expressed purpose of the book, there is no attempt at a systematic exposition of the entire field. Elastic and inelastic impact, friction, stability, holonomic and non-holonomic constraints are discussed in detail with many worked out examples.

The book follows the French tradition of good writing and is typical of the better French textbooks; it reflects also the recent tendency to discuss the range of application of particular results to physical experience. Students of mechanics will profit from this thoughtful presentation by a connoisseur who gives evidence of detailed knowledge and sincere love for the subject.

C. TRUESDELL

Die Laplace-Transformation und ihre Anwendung. By P. Funk, H. Sagan, and F. Selig. Vienna, Deuticke, 1953. 8+106 pp. $2.40.

This little book is designed to help engineers and teachers of engineers. It is based on a series of lectures given by the first named author in the *Ausseninstitut der Technischen Hochschule* in Vienna and was prepared by the other two. The plan of the book is perhaps best described by a paragraph from the introduction (freely translated): "The present book seeks to provide a link between the purely mathematical and the purely technical literature of the field and as such has not the character of a systematic text book but rather of a short introduction. It is aimed primarily at engineers and physicists and caters to their particular needs."

The book begins with a rapid description of the properties of and the operational manipulations of the one-sided Laplace transform. There is a deliberate omission of details in order to meet the desire of the engineer to arrive *in medias res* as quickly as possible. Such tools as the Riemann-Lebesgue theorem are dismissed with a reference. On the other hand the authors give an exposition of the Fourier double integral (with a Lipschitz condition for local restriction), feeling that this tool is so near to the heart of the subject as to be indispensable. The applications include: differential systems involving electrical circuits, Abel's integral equation, heat conduction, Thomson's cable. No table of transforms is included since extensive tables are now easily available elsewhere. The book concludes with a brief historical appendix not intended to be systematic but rather to shed light on particular aspects of the theory.

The authors frequently use an illuminating intuitive approach that ought to be very valuable for the class of readers expected. In the present reviewer's opinion the book could have been more useful if a careful statement of results (even if unproved) had been added. For, must not the applied scientist know the range of validity of his results?

D. V. WIDDER

A course of geometry. By R. N. Sen. Calcutta University Press, 1953. 36+311 pp. 12 rupees.

The book has an outlook similar to Graustein's *Higher geometry.* The content in terms of chapters is as follows: An algebraic introduction called Basic algebra, then eleven chapters on plane geometry entitled: Vectors and angles; Cross ratio; Rigid motions; Conics; Transformations of symmetry and similarity; The circle (includes inversion); Affinity; Involution (of pencils of points or lines); Geometry in the extended Cartesian plane; Collineation and correlation; Geometry in the projective plane with an appendix on Trilinear and areal coordinates. The remaining eight chapters deal with space geometry and are entitled: The Euclidean space; Projective space (here we find for the first time an independent definition of projective space, as contrasted to an extension of the Euclidean space); groups of transformations and classification of geometries (deals with Klein's Erlanger Programm); Projective theory of quadrics; Polarity; Geometry in the extended Cartesian space; Orthogonal transformation and affinity; Quadrics in Euclidean space with an appendix on the Law of Inertia for quadratic forms.

The treatment is quite predominantly algebraic. The book is easy to read and very clear in the small; however, the non-initiated will not recognize a clear overall structure. Also, the fundamental principles are not presented as poignantly as one might wish. A typical example is the statement that Brianchon's Theorem is proved by dualizing the proof of Pascal's Theorem, instead of emphasizing that the duality principle makes a separate proof unnecessary.

H. BUSEMANN

Vorlesungen über die Theorie der Integralgleichungen. By I. G. Petrovskij. Würzburg, Physica-Verlag, 1953. 100 pp. 7.80 DM.

This is a clear and concise exposition in less than one hundred pages of the classical theory of integral equations. Unfortunately while the book undoubtedly fills a need for Russian speaking students—it was originally written in that language—a German edition seems to me redundant. There is little that cannot be found in the opening chapters of Courant-Hilbert.

The book contains a useful table of analogies between finite-dimensional space and Hilbert space (as a function space), and an appendix in which the author shows the advantage of Lebesgue integration in developing the theory of symmetric kernels. The Fredholm theory is done, not by the famous method of Fredholm determinants, but by E. Schmidt's method of approximation by degenerate kernels.

There is little in the way of illustrative material or exercises.

HARRY POLLARD

Atti del Quarto Congresso dell' Unione Matematica Italiana tenuto in Taormina nei giorni 25–31 *Ottobre* 1951. Rome, Perrella, 1953. Vol. 1, 9+324 pp.; vol. 2, 2+684 pp. 9000 lire.

The first volume contains addresses and the second contains abstracts of papers presented to the Congress.

Opere. By L. Bianchi. Ed. by the Unione Matematica Italiana with the assistance of the Consiglio Nazionale delle Ricerche. Vol. 1, part 2. Rome, Perrella, 1953. 4+278 pp. 2500 lire.

For part 1 see this Bulletin, vol. 59, p. 416. Part 2 contains papers on elliptic normal curves, continuous groups, and partial differential equations; and biographical articles.

Opere. By U. Dini. Ed. by the Unione Matematica Italiana with the assistance of the Consiglio Nazionale delle Ricerche. Vol. 1. Rome, Perrella, 1953. 6+699 pp. 600 lire.

Dini's works are to appear in 3 volumes. This one contains papers on algebra and differential geometry.

RESEARCH PROBLEMS

8. Andrew Sobczyk: *Projections in Banach spaces.*

Does there exist any infinitely-dimensional, separable, closed linear subspace X of the nonseparable space (m) of all bounded sequences, such that there is a continuous projection of (m) onto X? Phillips and Sobczyk have shown that there is no continuous projection of (m) onto (c_0), the subspace of all sequences convergent to zero. Sobczyk has shown that for any separable closed linear subspace W, $W \supset (c_0)$, there is a projection of bound 2 of W onto (c_0), and therefore no continuous projection of (m) onto W. A lemma of Murray states that there is a continuous projection onto a closed linear subspace Y if and only if there is a complementary closed linear subspace Z. For any Banach space $U \supset (m)$, there is a projection of bound 1 onto (m). Does there exist a pair of complementary closed linear subspaces Y, Z for (m), such that neither Y nor Z is separable or isomorphic with (m)? Similar questions may be asked concerning the existence of closed projections. References: D. B. Goodner, Trans. Amer. Math. Soc. vol. 69 (1950) pp. 89–108. F. J. Murray, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 76–93. R. S. Phillips, Trans. Amer. Math. Soc. vol. 48 (1940) pp. 516–541. A. Sobczyk, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 938–947; Duke Math. J. vol. 8 (1941) pp. 78–106; Trans. Amer. Math. Soc. vol. 55 (1944) pp. 153–169. (Received February 11, 1954.)

9. Lowell J. Paige: *Elements of odd order in a finite group.*

Let G be a finite group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \cdot 2^s$, where $p_1, p_2, \cdots, p_k$ are distinct odd primes. Let P be a Sylow 2-subgroup of G and let S be the set of all elements of G satisfying the equation $x^r = 1$, where $r = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. For the coset expansion of G by P,

$$G = g_1P + g_2P + \cdots + g_rP,$$

is there an element of S in each coset $\{g_iP\}$ $(i = 1, 2, \cdots, r)$? Note that the problem is trivial if P is normal or if P is its own normalizer in G and the intersection of P with each of its conjugates is the identity. (Received February 15, 1954.)

10. Casper Goffman: *The group of similarity transformations of a simply ordered set.*

Let S be a simply ordered set. A similarity transformation of S is a one-to-one correspondence between S and itself which preserves order. The similarity transformations of S form a group $G(S)$. For a well ordered set S, $G(S)$ consists of a single element, but there are other ordered sets with this property. The problem is the following: (a) characterize the ordered sets S for which $G(S)$ consists of one element; (b) characterize the ordered sets S for which $G(S)$ is abelian; (c) characterize the groups G for which there is an ordered set S such that $G = G(S)$. (Received February 17, 1954.)

11. O. Taussky: *Multiply monotonic sequences.*

Szegö (Duke Math. J. vol. 8 (1941) pp. 559–564) investigated the following theorem of Fejér: Let the sequence $\{a_n\}$ be monotonic of order 4. Then the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is regular and univalent for $|z| < 1$. Szegö proved that this theorem remains true for monotonic sequences of order 3, but is not true for monotonic sequences of order 2. Investigate what happens to monotonic sequences of order α with $2<\alpha<3$. (For multiply monotonic sequences of non-integral order see K. Knopp, Math. Zeit. vol. 22 (1925) pp. 75–85). (Received March 1, 1954.)

12. O. Taussky: *The Hilbert matrix.*

Denote by H the infinite matrix

$$\left(\frac{1}{i+k}\right).$$

Let $x=(x_1, x_2, \cdots)$ be a vector with $x'x=\sum x_i^2<\infty$. Hilbert proved that $x'Hx/x'x<\pi$. From this fact it follows at once that the system of equations $Hx'=\pi x'$ cannot be solved. It seems of interest to find out whether $Hx'=\pi x'$ can be solved if x is an arbitrary vector as long as the product Hx' exists. (For relevant recent literature see, e.g., W. Magnus, Amer. J. Math. vol. 72 (1950) pp. 699–704, Archiv d. Math. vol. 2 (1951) pp. 405–412; O. Taussky, Quart. J. Math. Oxford Ser. vol. 20 (1949) pp. 80–83.) (Received March 1, 1954.)

13. L. C. Young: *Parallel polyhedral surfaces.*

The two-dimensional oriented polyhedra π_1 and π_2 situated in Euclidean n-space are termed *parallel* if they can be decomposed into finite sums $\pi_1=\sum T_{1n}$, $\pi_2=\sum T_{2n}$ where T_{1n} and T_{2n} are triangles derivable from one another by translation. More generally, π_1 and π_2 are parallel outside area ϵ if they can be expressed in the form $\pi_1=\pi_1'+\pi_1''$, $\pi_2=\pi_2'+\pi_2''$, where π_1' and π_2' are parallel and $\pi_1''+\pi_2''$ has area $\leqq\epsilon$. The following question seems to have an important bearing on surface-integral problems of the calculus of variations: If we suppose π_1 closed and $\epsilon>0$ given, does there always exist a corresponding π_2 *of the type of the sphere*, such that π_1 and π_2 are parallel outside area ϵ? (Received March 18, 1954.)

14. P. L. Butzer: *Tauberian conditions.*

In the theory of divergent series, Tauberian theorems assert that any sequence which is summable by a definite method of summation and which satisfies an appropriate additional condition (τ) is necessarily convergent. The condition (τ) is said to be the *Tauberian condition* for the method of summation in question. *Conjecture*: There exist conditions (τ) which are Tauberian conditions for the Cesàro but not the Abel method. Likewise we may ask whether there are conditions (τ) which are Tauberian for the Lambert (see G. H. Hardy, *Divergent series*, Oxford, 1949, p. 372) but not the Abel method. (Received March 24, 1954.)

15. Richard Bellman: *Stability theory.*

It is known that if all the solutions of the vector-matrix equation $dy/dt=A(t)y$ are bounded as $t\to\infty$, then all the solutions of the perturbed equation $dz/dt=(A(t)+B(t))z$ are bounded as $t\to\infty$, provided that $\int^\infty\|B(t)\|dt<\infty$, in the two cases where $A(t)$ is constant or periodic. Does the result hold if $A(t)$ is merely restricted to be almost-periodic, or, in particular, have as elements finite trigonometric sums? (Received March 24, 1954.)

16. Richard Bellman: *Number theory.*

Let the integer n be written in the dyadic scale, $n=2^{k_1}+2^{k_2}+\cdots+2^{k_r}$, with $k_1>k_2>\cdots>k_r\geqq 0$, and define the number-theoretic function $\alpha(n)=r$. Is it true that there exist infinitely many primes for which $\alpha(n)$ is less than some fixed integer? (Received March 24, 1954.)

17. Richard Bellman: *Matrix theory.*

Let $\{A_i\}$, $i=1, 2, \cdots, k$, be a finite set of positive square matrices, and let $S_N=\{\prod_i B_i\}$ be the set of k^N matrices obtained by taking all possible products $B_1B_2 \cdots B_N$ where each B_i is an A_j. For any positive matrix X, let $\phi(X)$ denote the characteristic root of X of largest absolute value. A classical result of Perron asserts that $\phi(X)$ is positive. Let C_N be a matrix in the set S_N for which this root is a maximum. It is easy to show that $\lambda=\lim_{N\to\infty}\phi(C_N)^{1/N}$ exists. Let M_N denote the smallest majorant of the set S_N, that is, the ijth element of M_N is the maximum of the k^N ijth elements of matrices in S_N. Then again it is easy to show that $\mu=\lim_{N\to\infty} \phi(M_N)^{1/N}$ exists. Is it true that $\lambda=\mu$? (Received March 29, 1954.)

18. Richard Bellman: *Systems of renewal equations.*

Let $\{\phi_{ij}(t)\}$, $i, j=1, 2, \cdots, N$, be a matrix of non-negative functions, all of whose Laplace transforms possess a finite abscissa of convergence, with the additional condition that $\int_0^\infty \phi_{ii}\,dt>1$ for some i. It has been shown by Bohnenblust that the root of $\left|\int_0^\infty e^{-st}\phi_{ij}\,dt-I\right|=0$ with largest real part is positive. That it is also simple is trivial for $N=1$, and readily demonstrated for $N=2$. Is this root simple for $N\geq 3$? (Received March 29, 1954.)

NOTES

The following two hundred and thirty-seven doctorates, with mathematics, mathematical physics, or statistics as a major subject, were conferred during 1953 in universities in the United States and Canada. The university, month in which degree was conferred, minor subjects (other than mathematics), and title of the dissertation are given in each case if available.

Julián Adem, Brown, June, *On the axially-symmetric steady wave propagation in elastic circular rods.*

O. P. Aggarwal, Stanford, June, *Bayes and minimax procedures in sampling from finite populations.*

Bernard Altschuler, New York, February, *Nonlinear buckling of a spherical shell.*

V. L. Anderson, Iowa State, June, *A model for the study of quantitative inheritance.*

F. C. Andrews, California, Berkeley, September, *Asymptotic behavior of some rank tests for analysis of variance.*

Robert Baer, Illinois, October, minor in physics, *Strong ordering in self adjoint operator space.*

R. B. Barrar, Michigan, February, *Some estimates for the solutions of linear parabolic equations.*

J. D. Baum, Yale, June, *Asymptoticity in topological dynamics.*

Jack Bazer, New York, February, *Propagation of plane electromagnetic waves past a shore line.*

V. N. Behrns, Buffalo, June, *Some mathematical aspects of operations research.*

C. B. Bell, Jr., Notre Dame, August, *Structure of measure spaces.*

J. S. Bendat, Southern California, August, *On monotone and convex operator functions.*

S. E. Benesch, Illinois, June, minor in physics, *Theory of nonlinear integral equations with complex-valued singular kernels.*

Paul Berg, New York, June, *On cavitational flow of gases.*

Jerome Berkowitz, New York, June, *Spectral theory of singular differential operators.*

Kurt Bing, Harvard, March, *Definability in the theories of integers and of natural numbers.*

J. F. Blackburn, North Carolina, August, *Steady state temperature distribution between solids and gases under nonlinear boundary conditions.*

R. B. Blake, Florida, January, minor in physics, *The solution of* *certain problems of plane strain in laminated orthotropic structures by means of polynomials.*

W. W. Bledsoe, California, Berkeley, June, *Separative measures for topological spaces.*

J. R. Blum, California, Berkeley, June, *Strong consistency of stochastic approximation methods.*

J. W. Brace, Cornell, June, *Transformations on Banach spaces.*

W. H. Bradford, Texas, May, minor in physics, *Sub-biharmonic functions.*

Joseph Bram, Chicago, August, *On subnormal operators.*

R. D. Branstetter, Iowa State, June, *A round-off theory for scalar products.*

W. E. Briggs, Colorado, June, minors in physics and education, *Two new proofs of theorems concerning the representation of prime numbers by binary quadratic forms.*

John Brown, Jr., Brown, June, *On transient heat transfer through a laminar boundary layer.*

M. D. Burrow, McGill, September, *Generalisation of method of Young-tableaus in the theory of group representation.*

H. S. Butts, Ohio State, May, *Corresponding residue systems in algebraic number fields.*

G. C. Byers, Duke, June, minor in physics, *Class number formulas for quadratic forms over* $GF[q, x]$.

J. R. Byrne, Washington, Seattle, August, *n-tuple rings and n-ideals.*

C. E. Capel, Tulane, June, *Inverse limit spaces.*

Buchanan Cargal, Iowa State, June, minor in physics, *Generalizations of continuity.*

R. C. Carson, Wisconsin, August, *On functions analytic in an annulus.*

T. Y. Chow, Cornell, February, minor in aerodynamics, *A study of extremal value problems of functions regular in an annulus.*

H. E. Chrestenson, Oregon, June, *A class of generalized Walsh functions.*

C. J. Clark, Oklahoma A and M, June, *Introduction to C-R algebras.*

M. H. Clarkson, Texas, May, *A second-order theory for three dimensional wings in supersonic flow.*

K. G. Clemans, Oregon, June, *Limiting distributions of certain statistics of the Kolmogorov-Smirnov type.*

D. R. Clutterham, Illinois, June, minor in physics, *Interaction of curved shocks.*

Margaret F. Conroy, Brown, June, Part I: *Plastic-rigid analysis of* *long beams under transverse impact loading.* Part II: *Dynamic transverse loading of beams of a material exhibiting linear strain-hardening.*

Mischa Cotlar, Chicago, March, *On the theory of Hilbert transforms.*

G. C. Cree, Washington, St. Louis, June, *A partial differential equation arising in the theory of quasi conformal mapping.*

Frederic Cunningham, Jr., Harvard, June, *L^1-structure in Banach spaces.*

W. F. Darsow, Chicago, August, *On positive definite functions on locally compact groups and LH^* algebras.*

Anne Davis, California, Berkeley, January, *A study in the arithmetic of order types.*

R. B. Dawson, Jr., Harvard, June, *Unbiased tests, unbiased estimators, and randomized similar regions.*

R. B. Deal, Jr., Oklahoma, August, minor in physics, *Union differentiation and union correspondence in metric differential geometry.*

Richard Dean, Ohio State, December, *On the word problems for lattices and related topics.*

W. E. Deskins, Wisconsin, January, *The role of the parastrophic matrices in the theory of linear associative algebras.*

Alice B. Dickinson, Michigan, February, *Compactness conditions and uniform structures.*

R. C. DiPrima, Carnegie, June, *Vibrations of twisted beams.*

B. M. Drucker, North Carolina, August, *Existence and approximations of solutions of ordinary differential equations.*

D. W. Dubois, Oklahoma, June, minor in physics, *Structure of Archimedean partially ordered fields.*

D. W. Dunn, Massachusetts Institute of Technology, June, minor in aeronautical engineering, *Stability of the laminar boundary layer in a compressible fluid.*

H. P. Edmundson, California, Los Angeles, June, *Statistical estimation of matrix quantities by means of a class of discrete Markov chains.*

M. S. Edwards, Ohio State, August, *The representation problem for surfaces.*

Leon Ehrenpreis, Columbia, April, *Theory of distributions in locally compact spaces.* Part I.

Gertrude Ehrlich, Tennessee, August, *The structure of continuous rings.*

Robert Ellis, Pennsylvania, June, *Continuity and homeomorphism groups.*

M. P. Epstein, Columbia, May, *On the theory of Picard-Vessiot extensions.*

D. H. Evans, Brown, June, *Two problems concerning the measurement of ultrasonic attenuation.*

R. E. Fagen, Minnesota, December, *Certain probability limit theorems and transformations of stochastic processes.*

Jean B. Feidner, Buffalo, June, *Topological properties of a linear neighborhood space.*

H. C. Filgo, Jr., Rice, June, *Uniformizing functions for certain classes of symmetric Riemann surfaces.*

A. B. Finkelstein, New York, February, *The initial value problem for transient water waves.*

Irwin Fischer, Harvard, June, *The totality of curves of genus g.*

R. C. Fisher, Kansas, June, minor in physics, *A study of line congruence by means of a modified tensor analysis.*

J. E. Flanagan, Illinois, June, minor in philosophy, *Topics in information theory.*

L. R. Ford, Jr., Illinois, June, minor in physics, *Transitive homeomorphism groups.*

William Forman, New York, February, *Abstract prime number theorems.*

H. H. Fox, Illinois, June, minor in chemistry, *On approximation of analytic functions.*

K. D. Fox, Brown, June, *On the stability of some flows of an ideal fluid with free surfaces.*

Joel Franklin, Stanford, June, *On the method of successive approximations.*

H. D. Friedman, Pennsylvania State, June, *Topologies on a space of continuous functions.*

C. S. Gardner, New York, June, *Properties of the wave equation related to Huyghen's principle.*

G. H. F. Gardner, Princeton, June, *The concept of a rigid body in special relativity.*

D. C. Gerneth, Oklahoma A and M, June, *On lattice valued metrics.*

Dean Gillette, California, Berkeley, June, *Representable infinite games.*

Leonard Gillman, Columbia, February, *On intervals of ordered sets.*

Seymour Ginsburg, Michigan, February, *Order types and similarity transformations.*

R. D. Glauz, Brown, June, Part I: *Transient wave analysis in linear time dependent material.* Part II: *The viscoelastic vibrating reed.*

Harry Gonshor, Harvard, June, *Spectral theory for a class of non-normal operators.*

S. T. Gormsen, Florida, January, minor in electrical engineering, *Maps of certain algebraic curves invariant under cyclic involutions of periods three, five, and seven.*

Lillian Gough, Buffalo, June, *Logical relations between pairs of prop-*

erties used to define endpoint.

H. C. Griffith, Tennessee, June, *The enclosing of cells in 3-space by simple closed surfaces.*

J. K. Hale, Purdue, May, minor in physics, *On the asymptotic behavior of the solutions of systems of differential equations.*

K. D. C. Haley, Stanford, January, *Estimation of the dosage mortality relationship when the dose is subject to error.*

J. F. Hannan, North Carolina, June, *Asymptotic solutions of compound decision problems.*

H. A. Hanson, Michigan State, June, *Some relations between various types of normality of numbers.*

J. F. Hart, Toronto, June, minor in quantum electrodynamics, *Idealism and realism in the philosophy of science; theory of spin effects in the quantum mechanics of a many-electron atom with special reference to* PbIII.

Joseph Hashisaki, Illinois, June, minor in physics, *On group algebras of prime power groups.*

C. S. Herz, Princeton, June, *Bessel functions of matrix argument.*

J. G. Hocking, Michigan, June, *On approximations to monotone mappings on two-dimensional manifolds.*

W. C. Hoffman, California, Los Angeles, August, *A statistic associated with the joint distribution of N successive amplitudes.*

J. C. Holladay, Yale, June, *Boundary conditions for algebras of continuous functions.*

D. G. Horvitz, Iowa State, June, *Ratio method of estimation in sample surveys.*

L. N. Howard, Princeton, June, *Constant speed flows.*

W. W. Hoy, Ohio State, June, *The estimation of parameters in the Ornstein-Uhlenbeck process.*

J. P. Hoyt, George Washington, May, *Estimates and asymptotic distributions of certain statistics in information theory.*

K. C. Hsu, Kansas, June, minor in physics, *On the Haar functions.*

G. A. Hufford, Princeton, June, *Banach spaces and the perturbation of ordinary differential equations.*

Jack Indritz, Minnesota, August, minor in physics, *Applications of the Rayleigh-Ritz method to the solutions of partial differential equations.*

Shigeru Ishii, Illinois, June, minor in philosophy, *Convergence of open mappings of locally compact spaces.*

H. G. Jacob, Jr., Yale, June, *Lie automorphisms of rings.*

W. E. Johnson, Wisconsin, January, *Asymptotic solutions of a linear second order differential equation with two turning points.*

T. L. Jordan, Vanderbilt, August, minor in physics, *Generalized* *metric lattices and lattices of subgroups.*

Irving Kanter, Brown, June, *The response of a strong oblique shock wave to an arbitrary disturbance in the downstream pressure field.*

R. H. Kasriel, Virginia, June, *A study of multiplicity functions.*

Irwin Kay, New York, February, *Direct method for pulse wave diffraction problems.*

N. D. Kazarinoff, Wisconsin, August, *Asymptotic forms for the Whittaker functions of large complex order M.*

D. C. Kleinecke, California, Berkeley, September, *On a class of large perturbations.*

H. A. Knapp, Jr., Massachusetts Institute of Technology, June, minor in physics, *Potentials uniquely determined from their phase shifts in a Schrödinger equation.*

S. G. Kneale, Harvard, June, *Axially symmetric Stokes flows.*

R. J. Koch, Tulane, June, *Topological semigroups.*

R. M. Kozelka, Harvard, February, *On some special order statistics from the multinomial distribution.*

Martin Krakowski, Carnegie, June, *Solution of the Stokes equation for lubrication flows.*

Saul Kravetz, Harvard, March, *On the geometry of Teichmueller manifolds and the structure of the mapping-class group.*

S. H. Lachenbruch, Stanford, January, minor in physics, *Symmetrization in N dimensions.*

E. B. Leach, Massachusetts Institute of Technology, September, minor in physics, *Cohomology theory on complex analytic manifolds.*

T. H. Lee, North Carolina, June, *Matrices with generalized quaternions as elements.*

C. E. Lemke, Carnegie, June, *Extremal problems in linear inequalities.*

J. J. Levin, Massachusetts Institute of Technology, June, minor in physics, *Singular perturbations of non-linear systems of differential equations.*

G. J. Lieberman, Stanford, June, minor in industrial engineering, *Contributions to sampling inspection.*

W. D. Lindstrom, Iowa, August, *Some aspects of an arithmetic theory of algebraic differential equations of first order.*

V. Linis-Linins, McGill, May, *Analysis in nonarchimedean spaces.*

R. F. Link, Princeton, October, *Statistical techniques useful for estimating the mean life of a radioactive source.*

Oscar Litoff, Chicago, August, *The general linear group.*

E. F. Low, Jr., Florida, June, minor in engineering mechanics, *Vertical loading on a straight boundary of an orthotropic plate.*

B. H. McCandless, Indiana, February, minor in physics, *Test spaces for dimension n.*

J. H. McKay, Washington, Seattle, December, *Group extensions by left loops.*

J. D. McKnight, Purdue, January, minor in physics, *On the characterization of rings of functions.*

K. M. McMillin, Cincinnati, June, *Abel summability of the conjugate of the double Fourier series.*

R. W. MacDowell, Michigan, February, *On spaces and algebras of continuous functions.*

S. F. Mack, California, Berkeley, June, *Discontinuities of second derivatives of single-layer potentials.*

Arne Magnus, Washington, St. Louis, January, *Volume preserving transformations in several complex variables.*

Pauline C. Mann, Brown, June, *The interaction of an acoustic wave and an elastic spherical shell.*

Jean-Marie Maranda, McGill, May, *On P-adic integral representations of finite groups.*

J. M. Marr, Tennessee, August, *Continuous images of transfinite intervals.*

A. D. Martin, Washington, St. Louis, June, *Singular quadratic functionals and linear differential equations.*

G. W. Medlin, North Carolina, June, *Some new results on the characteristic roots of matrices.*

Leroy Meyers, Syracuse, June, *Certain transformations of Hermitian functionals.*

Luna Mishoe, New York, June, *On the expansion of an arbitrary function in terms of eigenfunctions of a non-selfadjoint differential systems.*

O. B. Moan, Purdue, May, minor in economics, *The simultaneous distribution of the mean and range in small samples.*

Mabel D. Montgomery, Buffalo, June, *Axioms of topology.*

C. B. Moore, Kentucky, June, *On regression for a compound bivariate surface.*

Marian A. Moore, Purdue, May, *Approximation of ϕ-integrals by Riemann and Darboux sums, and other contributions to the theory of ϕ-integrals in general spaces.*

R. A. Moore, Washington, St. Louis, June, *The behavior of solutions of linear self-adjoint differential equations of the second order.*

P. B. Moranda, Ohio State, June, *Estimation of parameters of the Ornstein-Uhlenbeck and related stochastic processes.*

G. W. Morganthaler, Chicago, June, *The central limit theorem for* *orthonormal systems—the Walsh functions.*

W. P. Morse, Florida, August, minor in education, *An analysis of the development of celestial navigation.*

P. S. Mostert, Purdue, August, *Fiberable spaces.*

Shu-Teh S. Moy, Michigan, February, *Applications of conditional expectation.*

Leon Nemesever, New York, February, *On the convergence of finite difference approximations to the solution of systems of quasilinear hyperbolic partial differential equations with singular coefficients.*

J. A. Nohel, Massachusetts Institute of Technology, June, minor in physics, *Stability of perturbed periodic motions.*

Katsumi Nomizu, Chicago, June, *Invariant affine connections on homogeneous spaces.*

Bernard Noonan, McGill, May, *Induced representations of Lie algebras.*

E. S. Northam, Michigan State, June, *Topology in lattices.*

O. T. O'Meara, Princeton, June, *Quadratic forms over local fields.*

Steven Orey, Cornell, September, *Formal development of ordinal number theory and applications to proofs of consistency.*

James Pachares, North Carolina, August, *Distribution of quadratic forms.*

Emanuel Parzen, California, Berkeley, June, *On uniform convergence of families of sequences of random variables.*

Philip Parzen, New York, June, *Propagation of electromagnetic waves in electric beams.*

G. W. Peglar, Iowa, June, *Some results in the theory of multiply homogeneous polynomials.*

E. J. Pellicciaro, North Carolina, June, *A study of critical sets of functions.*

T. S. Pitcher, Massachusetts Institute of Technology, June, minor in humanities, *Sets of positive functions in H-systems.*

R. L. Plunkett, Virginia, June, *Representatives of homotopy classes of mappings into spheres.*

Joseph Putter, California, Berkeley, June, *Contributions to sampling theory and nonparametric hypothesis testing.*

F. D. Quigley, Chicago, December, *On Schubert's formula.*

G. N. Raney, Columbia, October, *Completely distributive complete lattices.*

D. B. Ray, Cornell, June, *Spectra of second order differential operators.*

John Riner, Notre Dame, June, *Topological groups associated with separated uniform and completely regular spaces.*

T. J. Rivlin, Harvard, February, *Overconvergent Taylor series and the zeros of related polynomials.*

R. K. Ritt, Columbia, May, *Spectral theory: rings of functions.*

Louis Robinson, Syracuse, June, *Extension of the Kneser inequality to complexes and a generalization of the degree of mapping.*

M. E. Rose, New York, February, *On the solution of hyperbolic equations by difference methods.*

Lawrence Rosenfeld, Harvard, March, *On distortion at the boundary of a conformal mapping.*

S. L. Ross, Boston, June, *The Cauchy problem for a fourth order parabolic equation by difference methods.*

Hanan Rubin, New York, February, *The dock of finite extent.*

M. F. Ruchte. Wisconsin, January, *Number theoretic properties of the measure density.*

T. L. Saaty, Yale, June, *On the Bessel-Tricomi equation.*

H. E. Salzer, Columbia, February, *A generalization of divided differences with applications.*

D. E. Sanderson, Wisconsin, August, *Isotopic deformations in 3-manifolds.*

R. L. SanSoucie, Wisconsin, May, *Right alternative division rings of characteristic two.*

Anne E. Scheerer, Pennsylvania, June, *Brownian motion and the Green's function—the plane case.*

Harry Schiff, McGill, May, *Theoretical calculations of electron capture cross sections.*

Seymour Shuster, Pennsylvania State, June, *Pencils of polarities in projective space.*

Sol Schwartzman, Yale, June, *On transformation groups.*

R. G. Selfridge, Oregon, June, *Generalized Walsh transforms.*

D. H. Shaffer, Carnegie, June, *Two applications of Bessel functions.*

L. S. Shapley, Princeton, June, *Additive and non-additive set functions.*

R. T. Sharp, McGill, May, *On the diamagnetism of graphite.*

K. S. Shih, Illinois, June, minor in philosophy, *Cohomology of associative algebras and spectral sequences.*

J. R. Shoenfield, Michigan, February, *Models of formal systems.*

J. W. Siry, Maryland, June, minors in electrical engineering and physics, *Chromatic polynomials of large maps.*

Rosedith Sitgreaves, Columbia, March, *Contribution to the problem of classification.*

H. T. Slaby, Wisconsin, October, *Central nilpotency of commutative*

Moufang loops.

J. M. Slye, Texas, May, minor in physics, *Flat spaces for which the Jordan curve theorem holds true.*

W. K. Smith, Michigan, June, *The Banach algebras of continuous mappings from a compact Hausdorff space to a Banach algebra.*

P. N. Somerville, North Carolina, June, *A topic in the sequential design of experiments.*

G. L. Spencer, Michigan, June, *The compressible flow about a pointed body of revolution of curved profile with attached shock wave.*

F. L. Spitzer, Michigan, February, *On the theory of stochastic processes which appear in the description of two dimensional Brownian motion by polar coordinates.*

H. D. Sprinkle, Florida, January, minor in philosophy, *Topology in B-metric spaces.*

V. R. Staknis, Boston, June, *A theorem on rings of continuous functions.*

R. D. Stalley, Oregon, June, *A modified Schnirelmann density.*

S. K. B. Stein, Columbia, January, *The homology of the two-fold symmetric product.*

P. H. Sterbenz, Ohio State, June, *Some topics in cohomology theory.*

E. J. Stewart, Rice, June, *Some properties of a harmonic function in the neighborhood of an isolated singularity.*

F. D. Stockton, Brown, June, *Experimental evidence on non-linearity in plastic stress-strain relations.*

W. E. Strimling, Minnesota, August, minors in chemistry and electrical engineering, *Solution of a linear, second order, partial differential equation in three variables by Weiner integration.*

Irving Sussman, California, Berkeley, June, *Theory of associate rings.*

G. L. Thompson, Michigan, June, *Projective relations in modular lattices.*

R. N. Tompson, Brown, June, *Areas of k dimensional non-parametric surfaces in k+1 space.*

B. B. Townsend, Texas, May, minor in chemistry, *Certain metric extensors.*

R. G. Van Nostrand, North Carolina, August, minor in geology, *The theory of direct current prospecting in the presence of curved boundary surfaces.*

Lila P. Walker, North Carolina, August, *The reduction of a matrix to a rational canonical form.*

A. J. Wang, Brown, June, *Plastic flow in a deeply notched bar.*

C. Y. Wang, Minnesota, June, I. *Polynomial approximations to* *solutions of variational problems in the complex domain.* II. *On the mean convergence of mapping functions in conformal mapping of regions bounded by nearly smooth curves.*

L. E. Ward, Jr., Tulane, June, *Partially ordered spaces.*

F. C. Warner, Buffalo, June, *Operations research applied to some problems in education.*

W. H. Warner, Carnegie, June, *Plastic flow laws and their effects on structural stability.*

Edwin Weiss, Massachusetts Institute of Technology, June, minor in humanities, *Bounded topological rings.*

Lionel Weiss, Columbia, January, *On the use of moments in approximation distribution function and expectations.*

Mortimer Weitz, New York, February, *Reflection of water waves.*

Chien Wenjen, California, Los Angeles, August, *A generalization of certain properties of Banach algebras.*

D. G. Wertheim, Toronto, November, *The forms of the linear, continuous functional on certain abstract spaces.*

J. R. Wesson, Vanderbilt, June, minor in physics, *Finite plane projective geometries.*

A. B. Willcox, Yale, June, *Some structure theorems for a class of Banach algebras.*

R. J. Wisner, Washington, Seattle, June, *Rings with given module*

W. D. Wood, Purdue, January, *On functions defined by certain infinite series which converge in a half-plane.*

D. H. Wright, George Washington, April, *Survival probability.*

F. B. Wright, Jr., Chicago, December, *Ideals in operator algebras.*

F. M. Wright, Northwestern, August, *On the backward extension of moment sequences.*

W. S. Wunch, Stanford, June, *The reproduction of an arbitrary function of time by discontinuous control.*

John Zickel, Brown, June, *Pretwisted beams and columns.*

R. E. Zink, Minnesota, August, minor in physics, *Direct unions of measure spaces.*

Michael Yanowitch, New York, February, *Nonlinear buckling of circular elastic plates.*

Aboulghassem Zirakzadeh, Oklahoma A and M, June, *A mapping of Euclidean N-space on the plane.*

The following doctorate was conferred in 1952, but was not included in the list in the preceding volume of this Bulletin (vol. 59, pp. 286–296):

D. D. Strebe, Buffalo, June, *Certain applications of irreducibility to connected Hausdorff spaces.*

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VARIATION OF DOMAIN FUNCTIONALS[1]

M. SCHIFFER

1. Introduction. Let D be a domain in an m-dimensional space and $\phi[D]$ be a number which depends on the domain. We shall call $\phi[D]$ a domain functional. Insofar as D is also characterized by its boundary C or its complement $\tilde{D}$, we may sometimes consider the functional $\phi[D]$ as dependent on C or $\tilde{D}$ instead of D. The problem with which we are occupied is to establish a calculus for the functional with respect to changes of its variable; in other words, a calculus of variations for $\phi[D]$.

We give the following examples of domain functionals: The volume $V[D]$ of the domain and the surface area $A[C]$ of its boundary which are elementary geometric concepts. Let ∇^2 be the Laplace operator and consider the eigenvalue problem $\nabla^2 u + \lambda u = 0$ with the boundary condition $u = 0$ on C. The eigenvalues $\lambda_\nu[D]$ of this problem are often studied domain functionals. The corresponding eigenfunctions $u_\nu(x)$ taken at a fixed point x may be considered likewise as domain functionals. Finally, we mention the Green's function of D with respect to Laplace's equation $G(x, \xi)$. If we consider both its argument points x and ξ as fixed parameters, it becomes a functional of D.

The first domain functionals of nonelementary character to be studied seem to have been the eigenvalues $\lambda_\nu(D)$ of a plane domain [17] and the electrostatic capacity of a three-dimensional conductor [15]. Since both these quantities are closely related to the Green's function of the domains considered with respect to Laplace's equation, their theory can be reduced to that of the Green's function. Hadamard treated $G(x, \xi)$ from the general point of view of functional analysis and gave for the first time a systematic basis to the whole problem complex [7].

In order to study the dependence of a functional on its variable, we have, in the spirit of classical analysis, to investigate how the functional changes under an infinitesimal change of the domain. Hadamard assumed that the domain D is bounded by a closed smooth surface C. He deforms C into a surface C^* by pushing each point ζ on C by an amount δn in the direction of the exterior normal. Under

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An address delivered before the Pasadena meeting of the Society on November 28, 1953, by invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings; received by the editors December 15, 1953.

[1] This work was done under a contract with the Office of Naval Research.

certain restrictive assumptions on δn, we can assert that C^* is also a smooth surface which bounds a domain D^* with a Green's function $G^*(x, \xi)$. If $\delta n > 0$, $G^*(x, \xi)$ is well-defined in D and, by Green's identity, we find

$$(1.1)\quad -\iint_C \left[G^*(\zeta, x) \frac{\partial G(\zeta, \xi)}{\partial n} - G(\zeta, \xi) \frac{\partial G^*(\zeta, x)}{\partial n} \right] d\sigma = G^*(x, \xi) - G(x, \xi).$$

Now, $G(\zeta, \xi) = 0$ on C and

$$G^*(\zeta + \delta\zeta, \xi) = G^*(\zeta, \xi) + (\partial G^*/\partial n)\delta n + \cdots = 0.$$

Hence obviously

$$(1.2)\qquad \delta G(x, \xi) = \iint_C \frac{\partial G(\zeta, x)}{\partial n} \frac{\partial G(\zeta, \xi)}{\partial n} \delta n d\sigma.$$

This is Hadamard's variational formula for the Green's function.

Though the above derivation is purely formal, it can be justified if the surface C is regular enough and many interesting applications can be made. Since $\partial G/\partial n < 0$ on C, we recognize, for example, that $\delta G > 0$ if $\delta n > 0$, that is the monotony of the Green's function in dependence of the domain. Our formula would, at first, guarantee this monotonicity only for domains with sufficiently regular boundary. But it is easy to extend this result to the most general domains, since every domain can be approximated arbitrarily by domains whose boundary is even analytic.

Much deeper results can be derived by similar considerations; each time when we have a functional of D which is a combination of Green's functions and various partial derivatives of Green's functions such that the coefficient of $\delta n d\sigma$ in the variational formula is positive, we arrive at a monotonicity theorem for this functional.

Similar variational formulas can be easily established for the Green's functions of other partial differential equations of elliptic type, for higher orders, and for different boundary conditions. Hadamard studied, in particular, the Green's functions for the biharmonic equations which play a central role in the theory of elasticity of plates. But while the theory of boundary value problems in partial differential equations proceeded to more general domains, D, the variational method was inherently tied to a very restricted class of regular domains. Even very simple extremum problems for domain functionals could not be solved by variational methods, since one is not sure, a priori, that the sought extremum domain is regular enough to admit a variational formula. It is clear that the difficulty lies in the kinematics of our variation. Since we describe the transition from one domain to its neighbor by a normal shift, it is obvious that the normal at the varied points of C will play a role in the variational formula, and that we shall, a priori, exclude all domains D from the theory whose boundary C does not possess a normal at each point.

2. **Theory of univalent functions.** The theory of domain functionals was applied in a precise and successful manner at first in the theory of conformal mapping. Let Δ be a domain in the complex z-plane which contains the point at infinity. We call a domain D in the complex w-plane conformally equivalent to Δ if there exists in Δ a univalent analytic function which has near $z = \infty$ the development

$$w = f(z) = z + a_0 + \frac{a_1}{z} + \cdots \tag{2.1}$$

and maps Δ onto D. Since D determines $f(z)$ in a unique way, we may consider all coefficients $a_\nu[D]$ as domain functionals of D and pose various extremum problems with respect to them. de Possel [16] asked for a domain D which is conformally equivalent to Δ and for which

$$\mathrm{Re}\,\{a_1[D]\} = \max. \tag{2.2}$$

The existence of such a domain is insured by theorems on the normality of the family of univalent functions with the development (2.1) at infinity. There is only the question of characterizing the extremum function. de Possel showed that the extremum domain D consists of the whole w-plane slit along rectilinear segments parallel to the real axis. Since the existence of the extremum domain is sure, this constitutes an existence proof for a canonical conformal mapping. Since de Possel's proof many other canonical mappings have been derived in the same way. The preceding reasoning shows the great value of the variational method for the existence proof in complicated boundary value problems. The method is to be clearly distinguished from extremum methods as, for example, Dirichlet's principle. There we work with a fixed domain and a very wide class of admissible functions, while in our case we are dealing only with analytic univalent functions but varying domains.

Another classical problem in conformal mapping deals with the family of all functions

$$f(z) = z + a_2z^2 + \cdots + a_nz^n + \cdots \tag{2.3}$$

which are univalent in the unit circle $|z|<1$. This family plays an important role in the general theory of conformal mapping and uniformization. It is a normal family which implies again that each sensible extremum problem has a solution, that is, at least one extremum function. The classical problem of the theory, as yet unsolved, is the question of max $|a_\nu|$ for the class $\mathfrak{F}$. Bieberbach's conjecture

$$(2.4) \qquad |a_\nu| \leqq \nu$$

which is proved for $\nu=2, 3$ attracted the attention of many mathematicians because of its great simplicity.

Each function $f(z)\subset\mathfrak{F}$ determines a domain D in the complex plane which may be considered as the graphical representation of $f(z)$ and, conversely, $f(z)$ and its coefficients a_ν may be conceived as functionals of D. Let now $\phi(z)$ be the inverse function of $f(z)$ which maps D onto the unit circle; then

$$(2.5) \qquad -\frac{1}{2\pi}\log|\phi(z)| = G(z, 0)$$

is easily seen to be the harmonic Green's function of D with the source point at the origin. Thus, the coefficient problem for univalent functions may be considered as an extremum problem for the Green's function of a domain D. That this point of view is natural is seen from the following fact. The function $\phi(z)$ which is nearer to the Green's function than its inverse $f(z)$ also gives rise to a coefficient problem. This problem is simpler than the preceding one and was solved completely by Löwner in 1923 [11]. Thus, the problem closer to the Green's function problem seems for this reason to be easier to handle.

In his classical paper of 1923 Löwner considered a sequence of varying domains which depend on a parameter t as follows. Let Γ be a Jordan arc $z(t)$ which runs from $z(0)$ to infinity as t runs from zero to infinity. Let Γ_t be the subarc of Γ between $z(t)$ and infinity and let

$$(2.6) \qquad f(z, t) = \sum_{\nu=1}^{\infty} a_\nu(t)z^\nu$$

be the function which maps the unit circle onto the complex plane slit along the arc Γ_t. Then, $f(z, t)$ is differentiable in t and satisfies a simple and elegant first order partial differential equation in z and t. On the other hand, Julia [9] transformed Hadamard's variational formula for the Green's function into a variational formula for the corresponding mapping function $f(z)$ and Biernacki [1] showed that

Löwner's differential equation follows from Julia's formula by a proper passage to the limit. Thus, one of the most successful methods in the theory of univalent functions was tied to Hadamard's variational method.

In the special case of the Laplace equation in two variables there exists a simple device in order to extend Hadamard's formula to general domains and thus to make it applicable to extremum problems. This device is based on the fact that a harmonic function in two variables remains harmonic under conformal mapping; if $u(z)$ is harmonic in x, y ($z=x+iy$), then $u[f(\zeta)]$ is harmonic in ξ, η ($\zeta=\xi+i\eta$). Let now D be a domain in the z-plane with finitely many analytic boundary curves C; let Δ be a subdomain interior to D with finitely many analytic boundary curves Γ. Let $f(z)$ be regular analytic outside of Δ; then $z^*=z+\epsilon f(z)$ will be regular analytic on C and, for small enough ϵ, even univalent. Hence, the curve system C will be mapped onto a new curve system C^* which determines a new domain D^* with the Green's function $G^*(z;\zeta)$. Clearly

$$g_\epsilon(z,\zeta)=G^*[z+\epsilon f(z),\zeta+\epsilon f(\zeta)] \tag{2.7}$$

is harmonic in $D-\Delta$ for $z\neq\zeta$ and vanishes for z or ζ on C. We assume that z, ζ lie in $D-\Delta$ and apply Green's identity:

$$\begin{aligned} &g_\epsilon(z,\zeta)-G(z,\zeta)\\ &\quad=\int_{C+\Gamma}\left[g_\epsilon(t,\zeta)\frac{\partial G(t,z)}{\partial n}-G(t,z)\frac{\partial g_\epsilon(t,\zeta)}{\partial n}\right]ds. \end{aligned} \tag{2.8}$$

Since $G(t,z)$ as well as $g(t,\zeta)$ vanishes on C, there remains only the integration over the curve Γ, interior to D.

Formula (2.8) was derived for a domain D with analytic boundary C; but now we may extend this formula to the most general case C by approximating C by analytic curves. Using theorems on the uniform convergence of the corresponding Green's functions in a closed subdomain of D, we can justify the above result for the general case.

The law of deformation

$$z^*=z+\epsilon f(z) \tag{2.9}$$

defines to the most general boundary C of a domain D a new boundary C^* if C lies outside of Δ and if ϵ is small enough. For most applications it is sufficient to choose $f(z)$ in a very special way, namely,

$$f(z)=\frac{1}{z-z_0},\qquad z_0\in D. \tag{2.10}$$

Using variations of this type, numerous extremum problems in the theory of univalent functions could be treated [6; 19; 20; 21]. One of the simplest and most important results states that the extremum functions of the coefficient problem are analytic on the periphery of the unit circle with the exception of finitely many points.

More precisely, a function $f(z)$ belonging to a maximal value of $|a_n|$ satisfies an ordinary differential equation

$$f'(z)^2 P_n[f(z)^{-1}] = z^{-(n+1)} Q_n(z) \tag{2.11}$$

where P_n and Q_n are polynomials whose coefficients depend in a specified manner on the first n coefficients of $f(z)$ itself. It can be shown that the function

$$f(z) = \frac{z}{(1-z)^2} = z + 2z^2 + \cdots + nz^n + \cdots, \tag{2.12}$$

which is the conjectured extremum function for all values of n, satisfies the above functional-differential equation.

Teichmüller has proved a converse of the above theorem [23]. Namely, suppose that a univalent function $f(z)$ satisfies a differential equation of the type (2.11) with any polynomial

$$P_n(x) = \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_{n-1} x^{n-1}, \quad \alpha_{n-1} = 1; \tag{2.13}$$

then Re $\{a_n\}$ will be maximal for this function with respect to all functions of the family $\mathfrak{F}$ which have the same first $n-1$ coefficients as $f(z)$. This result shows that every univalent solution of (2.11) is in some sense an extremum function. However, in order to decide whether $f(z)$ is the extremum function of some specific extremum problem, more powerful methods seem to be needed.

It is natural to expect additional information from a theory of the second variation for the functionals considered. Since the vanishing of the first variation ensures already the analyticity of the boundary of the extremum domain, it seems sufficient to establish a formula for the second variation in the Hadamard kinematics. However, the computational difficulties in deriving the second variation for the Green's function are considerable; the formula was established only very recently and will be given in §4.

3. **General interior variations.** The variational kinematics used in the preceding section can be easily applied to much more general domain functionals. In particular, it permits us to deal with functionals which are connected with boundary value problems for self-adjoint elliptic differential equations. For the sake of simplicity, we

shall illustrate the method by treating Laplace's equation in three-space.

We define in the space with coordinates x_i ($i=1, 2, 3$) a vector field $S_i(x)$ which is twice continuously differentiable and consider the transformation of the whole space

$$x_i^* = x_i + \epsilon S_i(x) \tag{3.1}$$

which depends on the real parameter ϵ. If ϵ is small enough, this transformation will be univalent in a given sphere of radius R. Every domain D in this sphere is mapped topologically onto a domain D^* and we denote the corresponding harmonic Green's functions by $G(x, \xi)$ and $G^*(x, \xi)$, respectively. Our problem is to express $G^*(x, \xi)$ in terms of $G(x, \xi)$ and of the transformation vector field. We shall call the deformation of a domain by means of such a vector field an interior variation.

In order to study $G^*(x, \xi)$, we consider the function

$$g(x, \xi; \epsilon) = G^*(x^*(x), \xi^*(\xi)) \tag{3.2}$$

which is defined in the original domain D, is twice continuously differentiable there if $x \neq \xi$, and which vanishes if x or ξ lies on the boundary C of D. In the case of harmonic functions in the plane, we utilized a deformation vector field $S_i(x)$ of particular type, namely, we chose S_i so that $f(z) = S_1 + iS_2$ was an analytic function of $z = x_1 + ix_2$ outside of some subregion Δ. This particular choice had the advantage that $g(x, \xi; \epsilon)$ was still harmonic in D outside of Δ for arbitrary choice of ϵ. Now, the situation is more difficult; we cannot preserve the harmonicity of $G(x, \xi)$ under the deformation, but we are led to a new differential equation for $g(x, \xi; \epsilon)$ which expresses that $G^*(x^*, \xi^*)$ is harmonic in D^*. In fact, a simple calculation shows that Laplace's equation in D^* is translated into

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\sum_{k=1}^{3} A_{ik} \frac{\partial}{\partial x_k} g(x, \xi; \epsilon)\right) = 0 \tag{3.3}$$

with

$$A_{ik} = A_{ki} = \theta \sum_{j=1}^{3} \frac{\partial x_i}{\partial x_j^*}\,\frac{\partial x_k}{\partial x_j^*}, \qquad \theta = \frac{\partial(x_1^*, x_2^*, x_3^*)}{\partial(x_1, x_2, x_3)}. \tag{3.3'}$$

We observe that the coefficients of the new differential equation depend analytically on ϵ.

It is clear that $g(x, \xi; \epsilon)$ and $G^*(x, \xi)$ are equivalent functions and that the knowledge of one leads to the other in an elementary way.

We have expressed the Green's function $G^*(x, \xi)$ for the fixed Laplace equation and a variable domain D^* in terms of the Green's function $g(x, \xi; \epsilon)$ for a varying differential equation (3.3) but with respect to the original fixed domain D. This was achieved by transplanting $G^*(x, \xi)$ back into D by use of the inverse transformation of (3.1).

The dependence of the solution of a partial differential equation $L_\epsilon[u]=0$ with fixed boundary value on the parameter ϵ occurring in the coefficients of the equation has been frequently investigated. Hilbert applied the parametrix method successfully for this problem [8]. It can be shown [3] that in our particular case the Green's function $g(x, \xi; \epsilon)$ depends analytically on the parameter ϵ.

We may develop g into a power series in ϵ and calculate without difficulty the coefficients of this development. These are clearly closely related to the variations of the respective orders of the Green's function $G(x, \xi)$. We find, for example:

$$(3.4) \qquad \frac{\partial}{\partial \epsilon} g(x, \xi; \epsilon)\Big|_{\epsilon=0} = \iiint_D \sum_{i,k=1}^{3} T_{ik}(\eta; x, \xi) \frac{\partial S_i(\eta)}{\partial \eta_k} d\eta_1 d\eta_2 d\eta_3$$

with

$$(3.4') \qquad \begin{aligned} T_{ik}(\eta; x, \xi) = {} & \frac{\partial G(\eta, x)}{\partial \eta_i} \frac{\partial G(\eta, \xi)}{\partial \eta_k} + \frac{\partial G(\eta, x)}{\partial \eta_k} \frac{\partial G(\eta, \xi)}{\partial \eta_i} \\ & - \delta_{ik}(\nabla_\eta G(\eta, x) \cdot \nabla_\eta G(\eta, \xi)). \end{aligned}$$

The tensor $T_{ik}(\eta; x, \xi)$ plays a central role in the variational theory of the Green's function. It may be called the Maxwell tensor of D; if we put $x=\xi$, we see in fact that $T_{ik}(\eta; x, x)$ is the classical Maxwell tensor of the electrostatic field created in the domain D with grounded conducting walls C by a unit charge at the point $x \in D$.

The tensor T_{ik} is obviously symmetric in its indices as well as in x and ξ. It satisfies the divergence condition

$$(3.5) \qquad \sum_{k=1}^{3} \frac{\partial}{\partial \eta_k} T_{ik}(\eta; x, \xi) = 0, \qquad i = 1, 2, 3.$$

This identity indicates that formula (3.4) can be simplified considerably by use of the divergence theorem; we can reduce the triple integral into an integral over the surface C of D provided that C is regular enough to admit the application of the divergence theorem. We have to observe in the transformation of the integral that T_{ik} becomes infinite for $\eta=x$ and $\eta=\xi$ and have to consider the residues from these singular points. Carrying out all the indicated steps, we arrive at

$$\frac{\partial}{\partial\epsilon}G^*(x,\xi)\Big|_{\epsilon=0}=\iint_\sigma\frac{\partial G(\eta,x)}{\partial n}\,\frac{\partial G(\eta,\xi)}{\partial n}\,(S\cdot n)d\sigma \tag{3.6}$$

which is exactly Hadamard's variational formula (1.2). We obtain in this way a new and precise proof for Hadamard's classical formula under the weakest assumptions. On the other hand, (3.4) appears now clearly as the generalization of this formula to the case of arbitrarily bounded domains D.

A completely analogous formula holds also in the case of two-dimensional harmonic Green's functions. In this case it happens that the trace $\sum_i T_{ii}$ of the Maxwell tensor vanishes identically. Hence, T_{ik} has only two essential components, and we have

$$T_{11}=-T_{22},\qquad T_{12}=T_{21}. \tag{3.7}$$

If we choose now the vector field $S_i(x)$ in such a way that

$$\frac{\partial S_1}{\partial x_1}=\frac{\partial S_2}{\partial x_2},\qquad \frac{\partial S_1}{\partial x_2}=-\frac{\partial S_2}{\partial x_1} \tag{3.8}$$

we see that the integrand in (3.4) vanishes identically. This means, of course, the invariance of the Green's function under conformal mapping. It is now clear why the method of interior variation is particularly successful in the two-dimensional case; by choosing S to be an analytic vector field in the whole plane except for isolated singular points, we can express the variation of the Green's function in terms of its values and the values of its derivatives at these distinguished critical points. Extremum conditions on the Green's function which have, in general, the form of integro-differential equations reduce in this case to ordinary differential equations.

We illustrate the general variational method in a relatively simple case. We consider a plane, finite, and simply-connected domain D with boundary C and the eigenvalue problem

$$\nabla^2u+\lambda u=0,\qquad u=0\text{ on }C. \tag{3.9}$$

Let z_0 be the complex coordinate of a point in the z-plane but not on C. We use the deformation

$$z^*=z+\frac{\epsilon}{z-z_0}, \tag{3.10}$$

with complex parameter ϵ, which is univalent in $|z-z_0|>|\epsilon|^{1/2}$ and which maps therefore, for small enough ϵ, D into a domain D^*. From the general variational procedure, we find that each eigenvalue $\lambda(D)$

which is nondegenerate corresponds to an eigenvalue $\lambda(D^*)$ by the formula [4]

$$(3.11)\qquad \begin{aligned} \lambda(D^*) = \lambda(D) - \operatorname{Re}\Bigg\{ & 8\pi\epsilon\delta(z_0)\left(\frac{\partial u(z_0)}{\partial z_0}\right)^2 \\ & - 2\lambda(D)\epsilon \iint_D \frac{u(z)^2}{(z-z_0)^2}\, d\omega\Bigg\} + o(\epsilon). \end{aligned}$$

Here $\delta(z_0) = 0$ or 1 if z_0 lies in the exterior or interior of D, respectively, and

$$(3.11')\qquad \frac{\partial}{\partial z_0} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} - i\frac{\partial}{\partial y_0}\right).$$

$o(\epsilon)$ can be estimated uniformly for all domains D whose boundaries stay outside of a fixed circle $|z-z_0| = \alpha$. The above formula applies, in particular, always to the first eigenvalue $\lambda_1(D)$ which is known to be nondegenerate.

Under the same variation (3.10) the area $A(D)$ is transformed into

$$(3.12)\qquad A(D^*) = A(D) - \operatorname{Re}\left\{2\epsilon \iint \frac{d\omega}{(z-z_0)^2}\right\} + o(\epsilon).$$

We may now treat Rayleigh's problem to find the minimum for the product $A\cdot\lambda_1$. The existence of a minimum domain can be derived by use of the conformal mapping function which carries D into the unit circle and applying the normality of the family of univalent functions. We can characterize the extremum domain by the fact that the first variation of $\lambda_1 A$ has to vanish for each choice of ϵ and $z_0 \notin C$. This leads to the functional equation

$$(3.13)\qquad \delta(z_0)\left(\frac{\partial u(z_0)}{\partial z_0}\right)^2 = \frac{\lambda_1(D)}{4\pi A(D)} \iint_D \frac{A(D)u(z)^2 - 1}{(z-z_0)^2}\, d\omega.$$

By a careful analysis of this condition one can show that D has an analytic boundary curve C along which $\partial u/\partial n = \text{const}$. We obtain by the variational method less than from the more elementary method of symmetrization which has been developed in great elegance by Pólya-Szegö [13]. However, the same variational technique can be applied in much more general problems where the extremum domain will not happen to be symmetric and the method will work as well.

In §2, we pointed out that the variational theory of domain functionals leads often to important existence theorems. We want to discuss here a problem which plays a role in applications. Let B be a body in three-space, considered immersed in an incompressible nonviscous fluid of density 1. Suppose that the fluid stream past the body is a stationary irrotational flow with velocity one at infinity and direction parallel to the x-axis. We denote the velocity potential of the flow by $\phi(x, y, z)$; it is a harmonic function in the domain D outside of B except at infinity and is at each fixed point in D a domain functional of B (or D). If we split off from $\phi(x, y, z)$ the singular term due to the source of the flow at infinity, we can write it in the form

$$\phi(x, y, z) = -x + \varphi(x, y, z) \tag{3.14}$$

where φ is regular harmonic in D. The quantity

$$M = \iiint_D (\nabla\varphi)^2 dx dy dz \tag{3.15}$$

is called the virtual mass of B and plays a central role in the hydromechanics of B; M is also a functional of B.

Suppose now that we deform the surface C of B into a surface C^* of B^* by a normal shift δn applied to all its points. It can be shown [22] that

$$\delta M = \iint_C (\nabla\phi)^2 \delta n d\sigma - \delta V, \tag{3.16}$$

where at the same time the volume $V(B)$ varies according to

$$\delta V = \iint_C \delta n d\sigma. \tag{3.17}$$

We apply these results to an important problem in fluid dynamics, namely, the free boundary problem. Helmholtz was the first to point out that the flow past a body B is by no means uniquely determined. The moving fluid may bypass the body B and leave also a part of the fluid itself adjacent to B at rest. The part of the fluid at rest together with the body B forms a larger body B' whose velocity potential $\phi'(x, y, z)$ determines the correct velocity field of the flow. However, since the hydrostatic pressure in the resting fluid must be constant, it is easily seen that along the surface dividing between resting and moving fluid, the so-called free boundary, the velocity has to be constant, that is, $(\nabla\phi')^2 = \text{const}$. Hence, under a δn-deformation which affects only the free boundary of the body B', we should have, by (3.16) and (3.17), $\delta M' = \mu\delta V'$. Thus, a free boundary leads to a stationary value for $M - \mu V$. One can now invert the reasoning and pose the extremum problem for $M - \mu V$ [18]. If one can show that a solution exists and leads to a regular surface, one arrives at an existence proof for an interesting flow pattern. Such existence proof could be carried out indeed by interior variations in the case of plane flow [5] and of axially-symmetric flow [2]. The general case of a flow in three dimensions has not yet been solved satisfactorily.

The alternative approach to the free boundary problem for plane flows due to Leray [10] is based on fixed point theorems in function spaces and utilizes therefore also higher methods of functional analysis.

4. The second variation. The importance of the second variation for the theory of domain functionals is obvious. Consider a class of domains D and a functional $\phi[D]$. Suppose now that any two domains D_0 and D_1 of the class can be connected by a continuously varying set of domains D_t of the class with $0 \leqq t \leqq 1$. Then $\phi[D_t]$ becomes a function of the variable t. Suppose further that we can assert $d^2\phi_t/dt^2 > 0$ if $D_0 \neq D_1$. In this case, it is clear that there can exist only one domain in the class for which $\delta\phi = 0$ for all admissible variations and which yields, consequently, a minimum of ϕ. In many cases, it is more important to know that the functional equation implied by $\delta\phi \equiv 0$ has a unique solution than to know the exact value of the minimum. This is, for example, the case if the minimum problem has been set up artificially in order to obtain an existence proof as in the free boundary problem of the preceding section. In such cases, the knowledge of the second variation is most useful. Often it will even be sufficient to show that $d^2\phi_t/dt^2 > 0$ only in the case that $\delta\phi = 0$ in order to make analogous deductions.

As indicated already in §3, it is often easy to find the higher order variations for functionals if we use the method of interior variations. Returning to the special case of the three-dimensional harmonic Green's function, we know already that $g(x, \xi; \epsilon)$ is analytic in ϵ and that we can derive formally all coefficients in the power series development in ϵ, obtaining finally all desired variations for the Green's function. We may express, for example, the second variation of $G(x, \xi)$ in terms of a triple integral over the domain D involving the vector field $S_i(x)$ and the Green's function itself. We know, however, *a priori* that the variation of G must depend only on the shift of the boundary C of D and not on the interior values of the field; for the deformation of C alone determines the varied domain D^* with the Green's function G^*. Thus, it must be possible to express the second variation of the Green's function in the form of a surface integral over C, provided only that the boundary surface C of D is smooth enough. Similar arguments hold, of course, also in the case of variations of higher order.

We can give an elegant formula for the second variation of $G(x, \xi)$ in the case that the boundary surface C is three times continuously differentiable. We consider a fixed vector field S_t and denote by $G_\epsilon(x, \xi)$ the Green's function of the domain D_ϵ which arises from $D = D_0$ through the variation (3.1). We define $G(x, \xi) = G_0(x, \xi)$. Let

$$N = S \cdot n; \tag{4.1}$$

N is defined on C and n is the exterior normal on C with respect to D. By Hadamard's formula (3.6), we have

$$H(x, \xi) = \frac{\partial}{\partial \epsilon} G_\epsilon(x, \xi)\Big|_{\epsilon=0} = \iint_C \frac{\partial G(\eta, x)}{\partial n} \frac{\partial G(\eta, \xi)}{\partial n} N(\eta) d\sigma. \tag{4.2}$$

Here, $H(x, \xi)$ is regular harmonic in D and a linear functional of the vector field S on the boundary C. We can prove [3]:

$$\begin{aligned} \frac{\partial^2}{\partial \epsilon^2} G_\epsilon(x, \xi)\Big|_{\epsilon=0} &= -2 \iiint_D (\nabla_\eta H(\eta, x) \cdot \nabla_\eta H(\eta, \xi)) d\eta_1 d\eta_2 d\eta_3 \\ &\quad - \iint_C \frac{\partial G(\eta, x)}{\partial n} \frac{\partial G(\eta, \xi)}{\partial n} k(\eta) N^2(\eta) d\sigma, \end{aligned} \tag{4.3}$$

where $k(\eta) = 1/\rho_1 + 1/\rho_2$ is the mean curvature of C at the point η. This formula holds under the additional assumption that the vector field S has on C exact normal direction; otherwise some additional terms will come in depending on the tangential components of S. Such effect of the tangential deformation was to be expected in the theory of the second variation; for, if we displace each boundary point in direction of the tangent plane, we do not affect the domain in the first order, but we must expect an influence of the second order variation.

The above formula for the second variation can be simplified considerably in important special cases. Consider, for example, a closed surface T which encloses the boundary C of D. Let $V(x)$ be harmonic in the shell between C and T and let $V = 0$ on C, $V = 1$ on T. In other words, $V(x)$ is the potential of the conductor made up of the surfaces C and T. The level surfaces

4.4)
$$V(x) = t, \qquad 0 \leq t \leq 1,$$

are a continuously varying sequence of surfaces C_t between C and T. Let $G_t(x, \xi)$ be the Green's function of the domain D_t interior to C_t. It is then easily shown that

$$(4.5)\qquad \frac{\partial^2}{\partial t^2} G_t(x, \xi) = -2\iiint_{D_t} \left(\nabla_\eta \frac{\partial}{\partial t} G_t(\eta, x)\cdot\nabla_\eta \frac{\partial}{\partial t} G_t(\eta, \xi)\right) d\eta_1 d\eta_2 d\eta_3.$$

We observe that

$$(4.6)\qquad G_t(x, \xi) = \frac{1}{4\pi r(x, \xi)} - \Gamma_t(x, \xi), \qquad r = \left(\sum_{i=1}^{3} (x_i - \xi_i)^2\right)^{1/2},$$

where $\Gamma_t(x, \xi)$ is regular harmonic in the entire domain D. Since the singularity term is independent of the domain,

$$(4.7)\qquad \frac{\partial G_t}{\partial t} = -\frac{\partial \Gamma_t}{\partial t}, \qquad \frac{\partial^2 G_t}{\partial t^2} = -\frac{\partial^2 \Gamma_t}{\partial t^2}.$$

Hence, we find

$$(4.8)\qquad \frac{\partial^2}{\partial t^2}\left[\sum_{i,k=1}^{N} \alpha_i\alpha_k\Gamma_t(x^{(i)}, x^{(k)})\right] = 2\iiint_{D_t} \left(\nabla_\eta \sum_{i=1}^{N} \alpha_i \frac{\partial \Gamma_t(\eta, x^{(i)})}{\partial t}\right)^2 d\eta_1 d\eta_2 d\eta_3 \geqq 0$$

for any choice of the constants α_i and of the points $x^{(i)}$ in D. We arrive thus at a very general convexity theorem in potential theory and the applications indicated at the beginning of this section become possible.

Sometimes we can utilize the formula for the second variation more effectively than by just using the fact that it is positive. Take, for example, the functional $K[B]$, the electrostatic capacity of the body B; it is defined as follows: Let $\psi(x)$ be harmonic in the outside D of B and have the boundary value 1 on C. Then, $\psi(x)$ is called the conductor potential of B. It has at infinity the behavior

$$(4.9)\qquad \psi = \frac{K[B]}{r} + o(1/r), \qquad r^2 = \sum_{i=1}^{3} x_i^2$$

and this serves to define the electrostatic capacity $K[B]$.

Let now C_t be a sequence of surfaces connecting continuously the surface C of B with another enclosing surface T and being the level lines (4.4) of a harmonic function $V(x)$. The C_t bound bodies B_t with exterior D_t, capacity K_t, and conductor potential ψ_t. We can show from (4.5) that

$$\frac{\partial^2}{\partial t^2} K_t = \frac{1}{2\pi} \iiint_{D_t} \left(\nabla \frac{\partial \psi_t}{\partial t}\right)^2 dx_1 dx_2 dx_3. \tag{4.10}$$

This shows that K_t is convex from below as a function of t. But we can estimate the right-hand integral better if we observe that $h(x) = \partial\psi_t/\partial t$ is harmonic in D_t. Then, we have by Green's identity

$$\iiint_{D_t} (\nabla h \cdot \nabla \psi_t) dx_1 dx_2 dx_3 = -\iint_{\sigma_t} \frac{\partial h}{\partial n} d\sigma = 4\pi a \tag{4.11}$$

if $\lim_{r\to\infty} hr = a$. Hence, by Schwarz's inequality

$$(4\pi a)^2 \leqq \iiint_{D_t} (\nabla h)^2 dx_1 dx_2 dx_3 \cdot \iiint_{D_t} (\nabla \psi_t)^2 dx_1 dx_2 dx_3. \tag{4.12}$$

Finally, by applying (4.11) for the case $h = \psi_t$ and using (4.9), we find

$$\iiint_{D_t} (\nabla \psi_t)^2 dx_1 dx_2 dx_3 = 4\pi K_t. \tag{4.13}$$

From (4.9) we deduce that $\lim_{r\to\infty} r(\partial\psi_t/\partial t) = (\partial K_t/\partial t)$ and hence (4.12), (4.13) yield

$$\iiint_{D_t} \left(\nabla \frac{\partial \psi_t}{\partial t}\right)^2 dx_1 dx_2 dx_3 \geqq \frac{4\pi}{K_t}\left(\frac{\partial K_t}{\partial t}\right)^2. \tag{4.14}$$

Hence, (4.10) leads to the differential inequality

$$K_t\, \partial^2 K_t/\partial t^2 \geqq 2\,(\partial K_t/\partial t) \tag{4.15}$$

which means that K_t^{-1} is convex from above as a function of t.

This result is the best possible; for in the case that T is a level surface $\psi(x) = \text{const.}$ of the conductor potential of B, it is easily seen that K_t^{-1} is linear in t.

As another application, we consider two convex curves C_0 and C_1 in the plane which are described by their supporting functions $p_0(\phi)$ and $p_1(\phi)$. We can connect both curves by a continuous sequence of convex curves C_t which are determined by the supporting functions,

$$p_t(\phi) = (1 - t)p_0(\phi) + tp_1(\phi). \tag{4.16}$$

In this case, the theory of the second variation yields the formula

$$\begin{aligned} &\frac{\partial^2}{\partial t^2} G_t(x, \xi) \\ &\leqq -2 \iiint_{D_t} \left(\nabla_\eta \frac{\partial}{\partial t} G_t(\eta, x) \cdot \nabla_\eta \frac{\partial}{\partial t} G_t(\eta, \xi)\right) d\eta_1 d\eta_2 d\eta_3 \end{aligned} \tag{4.17}$$

where G_t is the Green's function of the interior D_t of C_t. This result leads again to numerous applications and convexity theorems in conformal mapping.

A main problem in applying the theory of the second variation is to find a convenient parametrization which permits us to connect any two domains of the class considered and which leads, on the other hand, to a simple second derivative of the functional in question with respect to that parameter.

5. **Transplantation.** Till now, we dealt mostly with functionals which arise from certain boundary value problems in partial differential equations. We consider now the important subclass of functionals which are defined as the minimum value of a given positive-definite integral extended over the domain considered and depending on functions of a prescribed function class with respect to this domain. The fact that the functional is an actual minimum allows certain estimates by use of competing functions which lead to interesting inequalities and even to variational formulas.

We explain the method for the case of the electrostatic capacity $K[B]$ of a body B with boundary surface C and exterior D. We can characterize $K[B]$ by the following extremum properties [13; 14]:

1. *Dirichlet's principle*: Consider all functions $U(x)$ which have continuous derivatives in D, are of order $O(1/r)$ at infinity, and have on C the boundary value one. Then

$$K[B] = \min \frac{1}{4\pi} \int\int\int_D (\nabla U)^2 dx_1 dx_2 dx_3 \tag{5.1}$$

and the minimum is attained for the conductor potential $\psi(x)$ of B.

Dirichlet's principle leads to easy upper bounds for $K[B]$ since we may choose any permissible function $U(x)$ and find an inequality for $K[B]$.

2. *Thomson's principle*: Consider all vector fields $q(x)$ in D with continuously differentiable components, which are solenoidal, that is, satisfy $\nabla q = 0$, are normalized by

$$\int\int_C q \cdot n d\sigma = 1, \tag{5.2}$$

and vanish at infinity such that $|q| = O(r^{-2})$ there. Then

$$\frac{1}{4\pi K[B]} = \min \int\int\int_D q^2 dx_1 dx_2 dx_3 \tag{5.3}$$

and the minimum is attained by the vector field $q = (1/4\pi K)\nabla\psi$.

Thomson's principle leads to lower bounds for $K[B]$ and is very useful in combination with the preceding Dirichlet principle.

3. *Gauss' principle*: Consider all functions $\mu(P)$ defined on the boundary surface C with the normalization

$$\iint_C \mu d\sigma = 1. \tag{5.4}$$

Then

$$\frac{1}{K} = \min \iint_C \iint_C \mu(x)\mu(\xi)r(x, \xi)^{-1}d\sigma_x d\sigma_\xi \tag{5.5}$$

and the minimum value is attained for $\mu(x) = -(1/4\pi K)\partial\psi/\partial n$.

Gauss' principle stresses the fact that the capacity is ultimately a functional of the surface C only. In physical interpretation, it expresses the capacity in terms of electric charges rather than in terms of electric fields.

We start now with a body $B = B(1)$ and consider the transformation of the x-space

$$x_1' = tx_1, \qquad x_2' = x_2, \qquad x_3' = x_3; \tag{5.6}$$

this is a stretching in the x_1-direction in the ratio t:1. The body $B(1)$ is deformed continuously through a sequence of bodies $B(t)$ with exteriors $D(t)$, boundaries $C(t)$, conductor potentials $\psi_t(x)$, and capacities $K(t)$. Since $\psi_{t_0}(x)$ is defined in $D(t_0)$, clearly

$$U(x) = \psi_{t_0}\left(\frac{t_0}{t} x_1, x_2, x_3\right) \tag{5.7}$$

is well-defined in $D(t)$. It has the boundary value 1 on $C(t)$ and the correct continuity and asymptotic behavior at infinity in order to be admissible in the Dirichlet principle (5.1) with respect to $D(t)$. Thus, we obtain

$$\begin{aligned} K(t) \leqq \frac{1}{4\pi} \iiint_{D(t)} \Bigg[\left(\frac{t_0}{t}\right)^2 \psi_{t_0;1}\left(\frac{t_0}{t} x_1, x_2, x_3\right)^2 \\ + \psi_{t_0;2}^2 + \psi_{t_0;3}^2 \Bigg] dx_1 dx_2 dx_3 \end{aligned} \tag{5.8}$$

where $\psi_{t_0;i}$ denotes the partial derivative of ψ_{t_0} with respect to its ith variable. We may refer back the integration to $D(t_0)$ by replacing $(t_0/t)x_1$ by x_1. This will change also the volume element in the integral and (5.8) obtains the form

$$(5.9)\quad t^{-1}K(t) \leqq \frac{1}{4\pi t_0}\iiint_{D(t_0)}\left[\frac{t_0^2}{t^2}\psi_{t_0;1}^2+\psi_{t_0;2}^2+\psi_{t_0;3}^2\right]dx_1dx_2dx_3$$

where the arguments of all $\psi_{t_0;i}$ are now x_1, x_2, x_3. Thus, we may put this inequality into the form

$$(5.9')\qquad t^{-1}K(t) \leqq t^{-2}A + B$$

where A and B depend on t_0 but not on t. Moreover

$$(5.9'')\qquad t_0^{-1}K(t_0) = t_0^{-2}A + B.$$

Hence, if we plot $t^{-1}K(t)$ versus $\tau = t^{-2}$, we see that the straight line $a\tau + B$ touches this curve at the point $\tau_0 = t_0^{-2}$ but lies elsewhere always above it. Hence, the curve $t^{-1}K(t)$ possesses at τ_0 a supporting line from above. Since t_0 is arbitrary, we proved that $t^{-1}K(t)$ is convex from above as a function of t^{-2}. If we did not know already from the general theory that $K(t)$ is analytic in t, we could deduce at least from the convexity result that $K(t)$ has a derivative in t almost everywhere. We can easily derive

$$(5.10)\qquad K'(t) = \frac{1}{4\pi t}\iiint_{D(t)}[\psi_{t2}^2+\psi_{t3}^2-\psi_{t1}^2]dx_1dx_2dx_3;$$

and the differential inequality stating the above convexity

$$(5.11)\qquad t^2K''(t) + tK'(t) \leqq K(t).$$

The basic idea in the above procedure is the transplantation of the correct extremum function from its domain $D(t_0)$ into another domain $D(t)$ where it serves as a comparison function in the extremum principle and leads thus easily to an inequality between the functionals of $D(t_0)$ and $D(t)$. This method of transplanting the extremum function has been used extensively in the following form. One uses one distinguished domain D_0, which is suspected to be the extremum domain, and its extremum function. Transplating this extremum function into all admissible domains D of the class considered one shows that the functional $\phi[D]$ by its minimum property is less than the value achieved by the transplanted function and moreover that this latter value is less than $\phi[D_0]$. This shows that $\phi[D_0] > \phi[D]$ and establishes the extremum property of D_0 [13].

The difference between this well known and useful device and the above method is that in the latter we transplant the extremum function of each domain into each other domain; in this way, we obtain convexity statements and differential inequalities [14].

We can repeat the procedure in the case of the Thomson principle. Here, we have to deal with an extremum vector field instead of an extremum function. Let $q \equiv (q_1, q_2, q_3)$ be the extremum vector field for the exterior $D(t_0)$ of the body $B(t_0)$. Then

$$(5.12) \qquad q_t = \left(q_1\left(\frac{t_0}{t}x_1, x_2, x_3\right), \frac{t_0}{t}q_2\left(\frac{t_0}{t}x_1, x_2, x_3\right), \frac{t_0}{t}q_3\left(\frac{t_0}{t}x_1, x_2, x_3\right)\right)$$

will be defined in $D(t)$ and represent a solenoidal vector field there. It is easily seen that the normalization (5.2) is transplanted also. Hence, we may use q_t in order to obtain a lower bound for $K(t)$. An easy calculation shows that $tK(t)^{-1}$ is convex from above as a function of t^2. This leads to another differential inequality which combines with (5.11) to estimate $K''(t)$ from above and from below. We obtain the inequality

$$(5.9''') \qquad (at^2 + b)^{-1} \leqq t^{-1}K(t) \leqq At^{-2} + B$$

where equality holds on both sides for $t = t_0$. Thus, the curve $t^{-1}K(t)$ lies between two differentiable curves which touch for $t = t_0$. Clearly, $t^{-1}K(t)$ must be also differentiable there and since t_0 is quite arbitrary, we have proved that $K(t)$ is differentiable in t everywhere. Thus, combining the Dirichlet with the Thomson principle we obtained a much stronger result than could have been obtained from each of them.

Finally, we may utilize the Gauss principle by transplanting the charge density $\mu d\sigma$ in the deformation. This leads to the result that $tK(t)^{-1}$ is convex from above as a function of t. This result is an improvement on the convexity statement derived from the Thomson principle. It is also easy to read off from the Gauss principle that $K(t)$ increases with t; in fact, under transplantation corresponding charges are pulled away from each other and the energy of the charge distribution is clearly decreased and the capacity K is, consequently, increased. On the other hand, $K'(t)$ is given by (5.10) and it is by no means clear that this integral must be non-negative. But the combination of the two results leads to the inequality

$$(5.13) \qquad \iiint_D \psi_{x_1}^2 dx_1 dx_2 dx_3 \leqq \iiint_D [\psi_{x_2}^2 + \psi_{x_3}^2] dx_1 dx_2 dx_3$$

valid for every conductor potential and arbitrary choice of the coordinate system. Equality holds in (5.13) if the body B is a plate in a plane $x_1 = \text{const}$. In fact, such a plate is not affected by a stretching in the x_1-direction and hence $K'(t) \equiv 0$.

It seems that the transformation by stretching is a very special kind of deformation and that the results obtained in this way are rather restricted. Before discussing other and more general deformations which admit a similar treatment, we want to show a slight generalization of the stretching deformation which leads to rather important applications. We consider the transformation

$$x_1' = \begin{cases} tx_1 & \text{if} \quad x_1 \geqq 0, \quad t > 0, \\ x_1 & \text{if} \quad x_1 \leqq 0, \end{cases} \qquad x_2' = x_2,\ x_3' = x_3. \tag{5.14}$$

This is a continuous transformation of the x-space, but it is no longer continuously differentiable. We call it a partial stretching in the x_1-direction.

If we want to apply the transplantation method to the capacity $K[B]$ in the case of a partial stretching, we have to observe, at first, that the transplanted functions and vector fields have discontinuities or discontinuous derivatives along the plane $x_1 = 0$. However, it can be seen that Dirichlet's and Thomson's principles still hold for extended classes of functions or vector fields, which include the transplanted fields obtained. Hence, it can be shown, as before, that $t^{-1}K(t)$ is convex from above in t^{-2} while $tK(t)^{-1}$ is convex from above in t^2, even in the case of a partial stretching. The derivative of $K(t)$ with respect to t can be shown to be

$$K'(t) = \frac{1}{4\pi t} \int\int\int_{D(t)^+} [\psi_{t2}^2 + \psi_{t3}^2 - \psi_{t1}^2]\, dx_1 dx_2 dx_3 \tag{5.15}$$

where $D(t)^+$ is the intersect of $D(t)$ with the half-space $x_1 \geqq 0$.

Since we may put the axes of reference arbitrarily with respect to the body considered, we obtain a great variety of deformations of the original body under partial stretching. In particular, we may put the plane $x_1 = 0$ in such a way that it cuts off only a very small part of the body B so that the partial stretching becomes a variation of a surface element of B. Using our above results, and in particular (5.15), we obtain a new approach to variational formulas of the Hadamard type.

Other interesting applications of the transplantation method arise in eigenvalue problems. In the membrane equation

$$\nabla^2 u + \lambda u = 0, \qquad u = 0 \text{ on } C \tag{5.16}$$

we can characterize the eigenvalues by well known extremum principles. We restrict ourselves here to the lowest eigenvalue $\lambda_1(D)$ of a domain D in the (x, y)-plane. Let $u(x, y)$ be any continuously differentiable function in D which vanishes on the boundary C; then we have

$$\lambda_1 = \min \frac{\iint_D (\nabla u)^2 dxdy}{\iint_D u^2 dxdy}. \tag{5.17}$$

The minimum is attained by the eigenfunction of (5.16) to the eigenvalue λ_1.

It is easy to find upper bounds for the eigenvalue λ_1 because of its minimum definition; it is, in general, not quite as easy to derive lower bounds. Here, convexity statements for the eigenvalue in dependence on some parameter t for a sequence of domains $D(t)$ may be of great value.

We can easily show that under a stretching

$$x' = tx, \qquad y' = y \tag{5.18}$$

we obtain a sequence of domains $D(t)$ with the nth eigenvalue $\lambda_n(t)$ such that $\sum_{\nu=1}^{N} \lambda_\nu(t)$ is a monotonic function of t^{-2}, convex from above; this holds for every choice of the integer N. Observe that the sum considered is not necessarily continuously differentiable in t; it may have discontinuous derivatives for the values of the parameter t for which $\lambda_N(t)$ is a degenerate eigenvalue. The same result holds for the eigenvalues of the more general differential system

$$\nabla^2 u + \lambda u = 0, \qquad \frac{\partial u}{\partial n} + hu = 0 \text{ on } C. \tag{5.19}$$

These convexity results have been applied to estimate the eigenvalues for isosceles triangles from the known eigenvalues for the equilateral triangle and their asymptotic behavior for $t \to 0$ and $t \to \infty$ [12].

Consider next the sequence of domains $D(t)$ which are obtained from a given domain $D = D(0)$ by the conformal mappings

$$z^* = z + tf(z) \tag{5.20}$$

where $f(z)$ is analytic in $D(0)$. It is not necessary that all domains $D(t)$ lie schlicht over the complex plane; we may define their eigenvalues $\lambda_n(t)$ of the membrane problem even if they lie over a Riemann surface.

If $u(x, y)$ is a correct first eigenfunction of the membrane problem for the domain $D(0)$, we may consider $u(x, y)$ referred by $z(z^*)$ to $D(t)$ as a comparison function for the extremum problem. We may assume that u was normalized in $D(0)$ by the condition

$$\iint_{D(0)} u^2 dxdy = 1. \tag{5.21}$$

Observe that the Dirichlet integral in the numerator of (5.17) is invariant under conformal transformation. Thus, (5.17) applied to $D(t)$ leads to

$$\begin{aligned} \lambda_1(t)^{-1} &\geqq \lambda_1(0)^{-1} \iint_{D(0)} u^2(x, y) \mid 1 + tf'(z) \mid^2 dxdy \\ &\geqq \lambda_1(0)^{-1} + \lambda_1(0)^{-1} t \cdot 2 \operatorname{Re} \left\{ \iint_{D(0)} u^2 f'(z) dxdy \right\}. \end{aligned} \tag{5.22}$$

This shows that $\lambda_1(t)^{-1}$ is convex from below as a function of t, since we can deduce in the same way the existence of a supporting line for every value of t.

Using the extremum definition for the higher eigenvalues and the same transplantation argument, we can prove that $\sum_{\nu=1}^{N} \lambda_\nu^{-1}(t)$ is convex from below as a function of t, for every choice of the integer N. This result leads to interesting inequalities between the eigenvalues of the membrane problem and conformal moduli of the domain D [14].

An extension of the preceding convexity result to the more general system (5.19) seems not easy. The boundary conditions are only conformally invariant if $k = 0$ and, in this case, we have $\lambda_1 = 0$. Hence, no corresponding result is known in the more general case (5.19).

We do not want to multiply examples, but mention that the transplantation method proved also useful in the theory of torsional rigidity, virtual mass, conformal radius, etc. [14].

It may be of interest to quote a result from the theory of differential equations of higher than the second order. Consider the differential equation

$$\nabla^4 u = \lambda u, \qquad u = \frac{\partial u}{\partial n} = 0 \text{ on } C, \tag{5.23}$$

for a plane domain D with boundary C. Then, it can be shown that under a stretching (5.18) of D, the functional $\lambda(D)$ is convex from above as a function of t^{-4}.

6. **General method.** In the case that $\phi[D]$ is a functional defined

as the minimum value of a positive-definite Dirichlet integral $Q[u]$ which is quadratic in its argument function u, we can develop a very simple variational theory. We assume that the minimum in $Q[u]$ is understood with respect to a well-defined function class $\mathfrak{F}$ characterized by certain continuity and boundary conditions with respect to D. Suppose that a transformation

$$x_i^* = x_i + \epsilon S_i(x) \tag{6.1}$$

is considered which carries $D = D(0)$ into a sequence of domains $D(\epsilon)$; we assume that the functions of the class $\mathfrak{F}$ are transformed by transplantation into $D(\epsilon)$ into functions of the corresponding function class $\mathfrak{F}_\epsilon$ of the new domain and vice versa. Thus, each admissible function $u^*(x^*)$ in $D(\epsilon)$ becomes admissible in $D(0)$ after transplantation.

The Dirichlet integrals $Q_\epsilon[u^*]$ for the various domains $D(\epsilon)$ become after transplantation into $D(0)$ well-defined Dirichlet integrals $Q[u;\ \epsilon]$ and are to be considered over the function class $\mathfrak{F}$; clearly $Q[u;\ 0] = Q[u]$.

Let $u(x;\ \epsilon)$ be the extremum function which yields the minimum

$$\phi(\epsilon) = Q[u(x;\ \epsilon);\ \epsilon] \leq Q[u_0;\ \epsilon], \qquad u_0 \in \mathfrak{F}, \tag{6.2}$$

within the class $\mathfrak{F}$. Obviously

$$\phi(\epsilon) = \phi[D(\epsilon)] \tag{6.3}$$

and the study of $\phi(\epsilon)$ in its dependence on the parameter ϵ leads to the complete variational theory of the functional $\phi[D]$.

Observe now that

$$\begin{aligned} \phi(\epsilon) - \phi(0) &= Q[u(x;\ \epsilon);\ \epsilon] - Q[u(x;\ \epsilon);\ 0] \\ &\quad + Q[u(x;\ \epsilon);\ 0] - Q[u(x;\ 0);\ 0] \\ &= Q[u(x;\ \epsilon);\ \epsilon] - Q[u(x;\ 0);\ \epsilon] \\ &\quad + Q[u(x;\ 0);\ \epsilon] - Q[u(x;\ 0);\ 0]. \end{aligned} \tag{6.4}$$

Since $u(x;\ \epsilon)$ and $u(x;\ 0)$ both belong to the class $\mathfrak{F}$, we have by the minimum properties (6.2)

$$Q[u(x;\ \epsilon);\ \epsilon] \leq Q[u(x;\ 0);\ \epsilon]; \quad Q[u(x;\ 0);\ 0] \leq Q[u(x;\ \epsilon);\ 0]. \tag{6.5}$$

Thus, (6.4) leads to the inequalities

$$\begin{aligned} &Q[u(x;\ \epsilon);\ \epsilon] - Q[u(x;\ \epsilon);\ 0] \\ &\qquad \leq \phi(\epsilon) - \phi(0) \leq Q[u(x;\ 0);\ \epsilon] - Q[u(x;\ 0);\ 0]. \end{aligned} \tag{6.6}$$

We have, until now, not made any assumption regarding the character of the quadratic Dirichlet integral $Q[u]$. We suppose that $Q[u;\epsilon]$ has for fixed $u(x)$ a continuous derivative $Q_1[u;\epsilon]$ with respect to ϵ which is again a quadratic functional of u. We have then by the mean value theorem

$$Q[u;\epsilon_2] - Q[u;\epsilon_1] = (\epsilon_2 - \epsilon_1)Q_1[u;\eta] \tag{6.7}$$

where η is an intermediate value between ϵ_1 and ϵ_2. We restrict ourselves to the interval $|\epsilon| \leqq 1$ and assume further that $Q_1[u;\epsilon]$ is there uniformly continuous as a functional of u in the $Q[u]$-metric and that there exist positive constants a, b, α, β such that

$$aQ[u] \leqq Q[u;\epsilon] \leqq bQ[u]; \qquad \alpha Q[u] \leqq |Q_1[u;\epsilon]| \leqq \beta Q[u]. \tag{6.8}$$

In many important cases, it is possible to verify that all these assumptions are fulfilled.

We derive now from (6.6) and (6.7) the inequality

$$Q_1[u(x;\epsilon);\eta_1]\epsilon \leqq \phi(\epsilon) - \phi(0) \leqq Q_1[u(x;0);\eta_2]\epsilon \tag{6.9}$$

where η_1 and η_2 lie between 0 and ϵ. From our assumptions (6.8) follows clearly that

$$\phi(\epsilon) - \phi(0) = O(\epsilon), \tag{6.10}$$

that is, in particular, the continuity of $\phi(\epsilon)$ at $\epsilon=0$.

Next, we observe that

$$\begin{aligned} &Q[u(x;\epsilon)] - Q[u(x;0)] \\ &\qquad = \phi(\epsilon) - \phi(0) - (Q[u(x;\epsilon);\epsilon] - Q[u(x;\epsilon);0]) \end{aligned} \tag{6.11}$$

which leads by (6.7), (6.8), and (6.10) to

$$Q[u(x;\epsilon)] - Q[u(x;0)] = O(\epsilon). \tag{6.12}$$

But from the quadratic character of $Q[u]$ and the minimum property of $u(x;0)$ with respect to this Dirichlet integral, we have

$$Q[u(x;\epsilon)] - Q[u(x;0)] = Q[u(x;\epsilon) - u(x;0)] = O(\epsilon). \tag{6.13}$$

Thus, $u(x;\epsilon)$ converges to $u(x;0)$ in the Q-metric and, consequently, in view of the assumed continuity properties of $Q_1[u;\epsilon]$, we find from (6.9)

$$\lim_{\epsilon\to 0} \frac{\phi(\epsilon) - \phi(0)}{\epsilon} = Q_1[u(x;0);0]. \tag{6.14}$$

This proves the differentiability of the functional and yields an explicit variational formula.

It is interesting to observe the role which the actual minimum character of $\phi[D]$ plays in our reasoning. Terms like

$$\frac{1}{\epsilon}(Q[u(x;\epsilon);0]-Q[u(x;0);0]),$$

which occur in (6.4) and are easily estimated from the minimum property, require a detailed investigation of the dependence of $u(x;\epsilon)$ on ϵ if the minimum property does not hold.

In important cases the value $u(x;\epsilon)$ itself can be characterized by a minimum problem of the above type. In this case, the above method leads to the derivative of $u(x;\epsilon)$ with respect to ϵ which, in turn, leads to the second derivative of $\phi[\epsilon]$. In the case of the harmonic Green's function, for example, such reasoning leads to the derivatives of all orders of $G_\epsilon(x,\xi)$ with respect to the parameter.

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21. M. Schiffer and D. C. Spencer, *Functionals of finite Riemann surfaces*, Princeton, 1954.

22. M. Schiffer and G. Szegö, *Virtual mass and polarization*, Trans. Amer. Math. Soc. vol. 67 (1949) pp. 130–205.

23. O. Teichmüller, *Ungleichungen zwischen den Koeffizienten schlichter Funktionen*, Preuss. Akad. Wiss. Sitzungsber. Phys.-math. Klasse (1938) pp. 363–375.

STANFORD UNIVERSITY

THE APRIL MEETING IN NEW YORK

The five hundred first meeting of the American Mathematical Society was held at Columbia University in New York City on Friday and Saturday, April 23–24, 1954. About 375 persons registered, including 335 members of the Society.

In this meeting the Society associated itself with the theme of Columbia's Bicentennial Celebration—Man's Right to Knowledge and the Free Use Thereof. In celebration of this anniversary a special session was held Friday evening in the McMillin Theatre at which Professor John von Neumann delivered an address entitled *The mathematical method* upon joint invitation of the University and the Society. President Whyburn, who presided, presented a plaque to Columbia felicitating the University on its two centuries of devotion to the furtherance of wisdom and the enlightenment of mankind.

By invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings, Professor Harry Pollard of Cornell University addressed the Society Friday afternoon on *Fundamental sets of functions* and Professor F. I. Mautner of the Johns Hopkins University addressed the Society Saturday afternoon on *Fourier analysis and the theory of groups*, at sessions presided over by Professor J. F. Randolph and Professor J. L. Walsh respectively.

Sessions for contributed papers were held Friday afternoon and Saturday morning and afternoon, Professors C. E. Rickart, Eugene Lukacs, Y. W. Chen, R. D. Schafer, I. M. Sheffer, and T. W. Moore presiding.

The abstracts of contributed papers presented at the meeting follow. Those with "*t*" after the abstract number were presented by title. Where a paper with joint authorship was presented in person, (p) follows the name of the author presenting it. Dr. Moser and Mr. Shenitzer were introduced by Professor Wilhelm Magnus, Dr. Gaier by Professor J. L. Walsh, Dr. Peyerimhoff and Dr. Jurkat by Professor C. N. Moore, Mr. Maximon by Professor G. W. Morgan, Dr. Burrow by Professor Edward Rosenthal, Mr. McAuley by Professor F. B. Jones, Mr. Linis by Dr. G. H. M. Thomas, Dr. Hintikka by Dr. Hartley Rogers, and Miss Heller by Dr. Isidor Heller.

ALGEBRA AND THEORY OF NUMBERS

376*t*. H. W. Becker: *Automedian tetrahedrons*. Preliminary report.

Let $p^2+q^2=2z^2$, $p^2+r^2=2y^2$, $q^2+r^2=2x^2$. Euler's (2), p. 507, Dickson's *History* II, is provisional solution, depending further on (F) $\Delta_f=(a^2-2b^2)(c^2-2d^2)\{(a^2-2b^2)$

$\cdot(c^2-2d^2)+8abcd\}=\square$. Then $f/g=[(bc+ad)(ac-2bd)\pm\Delta_f^{1/2}]/[(bc+ad)^2-(ac+2bd)^2]$; a/b and c/d are the same under a, b and $c, d\leftrightarrow f, g$. Thus each solution has 3 conjugates, each in turn having 2 new conjs., etc. ad inf. Erratum: in Euler (iii$_2$) the + and − signs should be interchanged in the formula for q, this solution being the same as Euler (i). Also Cunliffe's solution, p. 508, is the same as Euler (ii). (F) is satisfied trivially by $a, b, c, d, f, g=t, s, t, s, (t^2-2s^2)(t^2\mp 2ts+2s^2), \pm(t^4+s^4)$; $2t, s, s, -t, \pm 2(2t^2\mp 2ts+s^2), 2t^2-s^2$, etc., such solution giving in general $x=y$ and $p=q$ and in particular the equilateral tet. Euler (iii$_2$) is self-conj. as to Δ_a and $a, b, c, d, f, g=t, -s, t+2s, t+s, (3t^2+8ts+6s^2)\cdot(t^2+2s^2), -2(t^2-2s^2)\cdot(t^2+4ts+2s^2)$ or 0. In Euler (ii) $a, b, c, d, f, g=t, s, t^4-2t^2s^2+4s^4, 2ts(t^2-2s^2), t^6\mp t^5s-8t^4s^2\pm 14t^3s^3\mp ts^5+8s^6, (s^2-t^2)\cdot(t^4\mp 4t^3s+6t^2s^2-4s^4)$. There are also 9 Pythagorean parameters: x, $x_1=k_x(t_x^2\pm s_x^2)$; $x_0=2k_xt_xs_x$, etc., such that $p=y_1+y_0=z_0+z_1$, $q=x_0+x_1=z_0-z_1$, $r=y_1-y_0=x_0-x_1$. In Euler (iii$_2$) $k_x, t_x, s_x, k_y, t_y, s_y, k_z, t_z, s_z=1$, $t^3+3t^2s+2ts^2-2s^3$, $s(5t^2+8ts+2s^2)$, 1, $2t^3+5t^2s+4ts^2+2s^3$, $(t+s)\cdot(t^2-6s^2)$, t^2-2s^2, $2(t^2+2ts+2s^2)$, $t^2+4ts+2s^2$; in Cunliffe's solution, 1, $t(4t^4+8t^3s+2t^2s^2+s^4)$, $4t^4(t+s)+2t^2s^2(t-s)-3ts^4-s^5$, 1, $(t+s)\cdot(4t^4+2t^2s^2-s^4)$, $t(4t^4-6t^2s^2-4ts^3-s^4)$, $2t^2-s^2$, $2t(t+s)(2t^2+s^2)+s^4$, $2ts(2t^2+2ts+s^2)$. (Received March 10, 1954.)

377t. H. W. Becker: *Cunliffe's parametric solution for rational cuboids.*

In Dickson's *History* II, p. 499, Cunliffe[9] has a procedure 8 (or 4) times as prolific of solutions of $x^2+y^2=z^2$, $y^2+z^2=v^2$, $x^2+z^2=w^2$ as the Saunderson[1] (or Lenhart[13] = Rignaux[24a]). Cunliffe's forms C are of the 10th degree, but are the Euler[3] transforms of octics K: u, $x=\epsilon\zeta(\eta^2\pm\theta^2)$; v, $z=\eta\theta(\epsilon^2\pm\zeta^2)$; $y=2\epsilon\zeta\eta\theta$; $\epsilon=\eta=2r^3-2r^2s+2rs^2-s^3$, $\theta=2(r+s)(r-s)^2$, $\zeta=(r-s)(2r^2-4rs+s^2)$; w, $x=s(4r-3s)(\gamma^2\pm\delta^2)$; $\gamma=(2r-s)(r^2-s^2)$, $\delta=r(2r^2-3rs+2s^2)$. Cunliffe's forms cover number 3 of Kraitchik's table, Scripta Mathematica vol. XI (1945) p. 326; and the octics K cover numbers 1, 4, 5, 11, 13, and 22, reducing the number of mavericks to 36. The octics K have the 3 pairs of O'Riordan[22] parameters: $c=r^2-s^2$, $d=r^2-4rs+2s^2$; $e=2r^2-4rs+s^2$, $f=2c$; $a=c(2r-s)^2$, $b=r^2s(4r-3s)$, whose conjugates are: $a'=(r^2-s^2)(2r-5s)^2$, $b'=s(4r-3s)\cdot(3r-4s)^2$; $e'=e(2r^4+4r^3s-6r^2s^2+s^4)^2$, $f'=f(2r^4-12r^3s+14r^2s^2-6rs^3+s^4)^2$; c', $d'=$12-ics. The first O'Riordan conjugate OK is another octic, whose Pythagorean parameters are: $\gamma=(2r-5s)(r^2-s^2)$, $\delta=\zeta=s(4r-3s)\cdot(3r-4s)$, $\epsilon=(2r-5s)(2r^2-3rs+2s^2)$, $\eta=2(r+s)\cdot(r^2-s^2)$, $\theta=2r^3-14r^2s+22rs^2-11s^3$. The Pythagorean parameters are the same in any rational cuboid and its Euler transform. In all known basic cuboids, 2 or 3 of the component Pythagorean ▲ are nonprimitive; the multipliers are the same in any cuboid and its O'Riordan conjugates. But in the Lebesgue transform (Kraitchik ibid. or Bull. Amer. Math. Soc. Abstract 59-6-559) only one Δ is nonprimitive. (Received March 10, 1954.)

378t. A. T. Brauer: *A note on the primitive roots* (mod p^n).

Maxfield and Maxfield [Bull. Amer. Math. Soc. vol. 58 (1952) p. 564] announced that they proved the existence of primitive roots (mod p^n) which are less than p. Without using this result it is proved in this paper that there exist at least $\phi(p-1)/2$ primitive roots (mod p^n) which are less than p. If p is of the form $4k+1$ and greater than 5, then there exist at least $3\phi(p-1)/4$ such primitive roots. (Received March 10, 1954.)

379. A. T. Brauer (p) and H. T. LaBorde: *Numerical computation*

of the characteristic roots of a matrix and of the error in the approximate solution of linear equations.

Let A be a numerically given matrix with nonvanishing determinant. H. Wittmeyer [Zeitschrift für angewandte Mathematik und Mechanik vol. 16 (1936) pp. 287–300] published a method which gives estimates for the error in the approximate solution of a system of inhomogeneous linear equations with the coefficient matrix A. But this method can be used only if it is possible to determine a positive lower bound for the characteristic roots of the matrix A^*A where A^* is the conjugate transpose of A. In this paper a method for the determination of such a bound is obtained. For this purpose the results of Wittmeyer are extended to systems of homogeneous linear equations with vanishing determinant in order to obtain bounds for the components of a characteristic vector belonging to a characteristic root ω. It is not necessary that ω be numerically given. Using this result the characteristic roots of any numerically given matrix can be computed as exactly as necessary. (Received March 9, 1954.)

380*t*. J. L. Brenner and K. L. Cooke: *Hadamard matrices.*

An n by n matrix A with complex elements a_{ij} is said to have *dominant principal diagonal*, or to be an *Hadamard matrix*, if its elements satisfy the conditions $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$ $(i=1, \cdots, n)$. Such a matrix is known to be nonsingular, and the problem of delimiting the regions within which its characteristic roots must lie has been extensively discussed (cf. M. Parodi, *Mémorial des Sciences Mathématiques*, no. 118, Paris, Gauthier-Villars, 1952). M. Müller (Math. Zeit. vol. 51 (1948) p. 291) has pointed out the fact, of interest in discussing the location of the characteristic roots of an arbitrary nonsingular matrix, that if A is any nonsingular matrix, then there exists a nonsingular matrix B such that the matrix AB has dominant principal diagonal. In this article the authors show that B can always be taken to be the product of some permutation matrix by a triangular matrix T, and that, moreover, the first two (but in general not more than two) elements of the principal diagonal of T can be taken as 1. Furthermore, if one removes the restriction that two diagonal elements of T be 1's, T can always be chosen so that AB is a *triangular* matrix with dominant principal diagonal. (Received February 8, 1954.)

381. M. D. Burrow: *A generalization of the Young operator and its application to $GL(2, q)$.*

In the first part of this paper the author proves a sufficient condition for the construction of a primitive idempotent of a group algebra from the linear representations of two subgroups. It is a generalization of the condition of J. von Neumann on which is based Young's method for the representations of the symmetric group. The new condition is not confined to the symmetric group. In addition it is shown that the condition is equivalent to the following: the representations induced in the group by the linear representations of the two subgroups have a single irreducible component in common, and neither induced representation contains this irreducible component more than once. The second part of the paper is an application to the group $GL(2, q)$. Complete bases for all the irreducible representations of this group of degrees 1, q, and $q+1$ are given explicitly. (Received March 8, 1954.)

382*t*. Leonard Carlitz: *Congruences for generalized Bell and Stirling numbers.*

If we put $E_1(x) = e^x - 1$, $E_k(x) = E_1(E_{k-1}(x))$ and set $E_1(uE_{k-1}(x)) = \sum_{r=1}^{\infty} (x^r/r!)$

$\cdot \sum_{s=1}^{r} u^s S(c, r, s)$, $E_s(x) = \sum_{r=1}^{\infty} x^r B(r, s)/s!$, then $S(c, r, s)$ and $B(r, s)$ are the generalized Stirling and Bell numbers in the notation of Becker and Riordan (Amer. J. Math. vol. 70 (1948) pp. 385–394). In their paper periodicity properties (mod p) of these numbers are obtained. In the present paper like results are obtained (mod p^k). Indeed the method applies to the coefficients of the functions $F_0(x)$, $F_k(x) = F_{k-1}(f_k(x))$, where each $f_i(x)$ satisfies a relation of the form $f'(x) = 1 + \sum_1^\infty a_m f^m(x)$ with integral a_m. (Received March 10, 1954.)

383*t*. Leonard Carlitz: *Some arithmetic properties of the Olivier functions.*

Let $k \geqq 1$ and put $\Phi_k(x) = \sum_{m=0}^{\infty} x^{mk}/(mk)!$; also set $(\Phi_k(x))^{-1} = \sum_0^\infty a_{mk} x^{mk}/(mk)!$. It is proved that the coefficients a_m satisfy congruences of Kummer's type. More generally the results apply to the coefficients of $(\Phi_k(x))^{-\lambda}$ as well as certain more general functions. (Received March 10, 1954.)

384*t*. Leonard Carlitz: *The coefficients of the reciprocal of* $J_0(x)$.

Put $f(z) = J_0(2z^{1/2}) = \sum_0^\infty (-1)^m z^m/(m!)^2$ and $(f(z))^{-1} = \sum_0^\infty \omega_m z^m/(m!)^2$. Congruences (mod p) are obtained for the ω_m and for the polynomials $\omega_m(x)$ defined by $f(xz)/f(z) = \sum_0^\infty \omega_m(x) z^m/(m!)^2$. The following more general result may be cited. Let $\{F_m\}$, $\{G_m\}$, $\{H_m\}$ be sets of polynomials with integral coefficients such that $H_m = \sum_0^m \binom{m}{r}^2 F_r G_{m-r}$. Then if two of the sets of polynomials satisfy $F_m \equiv F_{a_0} F_{a_1}^p F_{a_2}^{p^2} \cdots$ (mod p), where $m = \sum a_i p^i$, $0 \leqq a_i \leqq p-1$, the same is true of the third set. (Received March 10, 1954.)

385*t*. A. L. Foster: *On a unique subdirect factorization in universal algebras and their characterization by means of their identities.*

This communication extends such structure results as the following (Foster, *Generalized "Boolean" theory of universal algebras*, etc. Parts I and II, Math. Zeit. vols. 58, 59 (1953)), itself a generalization of familiar special cases (N. McCoy, G. Birkhoff, and others). *Theorem* 1. *Let A be a universal algebra which is functionally (strictly) complete. A sufficient (and necessary) condition for a universal algebra B to be isomorphic with a subdirect-power (=-sum or -multiple) of A is that B satisfy all identities satisfied by A*. Among further results we prove: *Theorem* 2. *Let* $\tilde{A} = \{A', A'', \cdots\}$ *be an independent class of functionally complete algebras. A sufficient (and necessary) condition for a given algebra B to be expressible as a subdirect product of subdirect powers of a finite number of (distinct)* $A^{(i_1)}, A^{(i_2)}, \cdots, A^{(i_n)} \in \tilde{A}$ *is that B satisfy all identities common to* $A^{(i_1)}, \cdots, A^{(i_n)}$. *Theorem* 3. *In the foregoing notation, an algebra B which is representable in* $\tilde{A}$ *as in Theorem* 2, *is uniquely so representable, i.e., the "primary divisors"* $A^{(i_1)}, \cdots, A^{(i_n)}$ *are unique in* $\tilde{A}$. In Theorem 2 an *independent* class requires any two $A^{(i)} \in \tilde{A}$ to be *cofunctional*, and the $A^{(i)} \in \tilde{A}$ to be *uniformly coframal*. These concepts are too long for definition here. It is shown, however, that the conditions of Theorem 2 are widely realizable. (Received March 11, 1954.)

386*t*. Karl Goldberg: Log ($e^x e^y$) *in a free associative ring.*

Let x and y be elements of a free associative ring, and e^x and log $(1+x)$ be abbreviations for the formal power series which usually define these functions. Then let $z = \log(1+u)$ where $u = e^x e^y - 1$. The problem of computing the coefficients in the expression for z as a formal power series in x and y is due to Baker, Campbell, and Hausdorff, and is attacked in this paper. The primary result is a general formula for the coefficients which can be applied easily to numerical computation. Simplifications of this formula are obtained for special cases including the result $z(s, t) = (-1)^t \cdot [\sum_{i=1}^{s} \binom{s}{i} B_{s+t-i}]/s!t!$ where $z(s, t)$ is the coefficient of $x^s y^t$ in z, and B_k is the kth Bernoulli number in the usual notation. Preliminary studies of the computation of these coefficients were made on the National Bureau of Standards Eastern Automatic Computer. (Received February 22, 1954.)

387*t*. I. N. Herstein: *On the Lie ring of a division ring.*

If A is any associative ring, we can attach to it a Lie ring by introducing the new product $[a, b] = ab - ba$. If $[A, A]$ is the additive subgroup of A generated by all $xy - yx$ then $[A, A]$ is also a Lie ring under this product. It is proved that if A is a division ring of characteristic not 2, then any Lie ideal of $[A, A]$ must be contained in the center of A. (Received March 16, 1954.)

388*t*. D. G. Higman: *Modules with a group as operators.*

Let G be a group, S a subgroup of finite index. If M is a (unitary) G-module, M_S will denote the induced S-module. If m is a (unitary) S-module, $I_S(m, G)$ will denote the induced G-module. A G-module H will be called an extension of M if M is a G-submodule of H such that M_S is a direct summand of H_S. M will be called a quotient of H if there exists a G-submodule $\overline{M}$ of H such that $H/\overline{M}$ is isomorphic with M and M_S is a direct summand of H_S. The following generalizes theorems of W. Gaschütz (Math. Zeit. vol. 56 (1952) pp. 376–387) and B. Eckmann (Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 35–42). For a G-module M, the following conditions imply each other: (a) M is a direct summand of $I_S(M_S, G)$, (b) M is a direct summand of every extension, (c) M is a direct summand of every G-module of which it is a quotient, (d) there exists an S-endomorphism α of M such that $\sum x\alpha x^{-1} = 1$, where the summation extends over a set of left representatives x for G over S. Corollary: Let G be a finite group, S a p-Sylow subgroup, and F a field of characteristic p. Then a representation submodule M of a representation module H for G in F is a direct summand of H if and only if M_S is a direct summand of H_S. It is also proved that if S is noncyclic, then G has indecomposable representations of arbitrarily high degree in F, which, combined with a result given in Abstract 59-4-339 enables one to state that G has cyclic p-Sylow subgroups if and only if the degrees of the indecomposable representations of G at characteristic p are bounded. (Received March 2, 1954.)

389*t*. D. G. Higman: *Induced and produced modules.* I.

Let S be a subring of an (associative) ring A. A module M will be called a (right) A-S-module if it has S as a ring of left and A as a ring of right operators such that $s \cdot ua = su \cdot a$ (s in S, u in M, a in A). To each (two-sided) S-module M there corresponds a pair $(I_S(M, A), \kappa)$ consisting of an A-S-module $I_S(M, A)$ and an S-homomorphism $\kappa: M \rightarrow I_S(M, A)$ determined uniquely up to an A-S-isomorphism by the property that for any A-S-module M' and any S-homomorphism $\delta: M \rightarrow M'$ there exists one and only one A-S-homomorphism $\bar{\delta}: I_S(M, A) \rightarrow M'$ such that $\delta = \kappa\bar{\delta}$; dually, there corresponds to M a pair $(P_S(A, M), \pi)$ consisting of an A-S-module $P_S(A, M)$ and an S-homomorphism $\pi: P_S(A, M) \rightarrow M$ determined uniquely up to an A-S-isomorphism by the property that for any A-S-module M' and any S-homomorphism $\delta: M' \rightarrow M$ there exists one and only one A-S-homomorphism $\bar{\delta}: M' \rightarrow P_S(A, M)$ such that $\delta = \bar{\delta}\pi$. The existence is shown by exhibiting constructions. If A has an identity element 1 such that 1 is in S and 1 is the identity operator on the right of M then: $I_S(M, A) =$ the tensor product $M \otimes_S A$ turned into an A-S-module by

$s(u\otimes a)=su\otimes a$, $(u\otimes 1)b=u\otimes ab$, with $\kappa: u\to u\otimes 1$; $P_S(A, M)$ the module of all additive homomorphisms $f: A\to M$ such that $f(as)=(fa)s$, turned into an A-S-module by $(f*a)b=f(ab)$, with $\pi: f\to f1$. $I_S(M, A)$ will be said to be induced by M if κ is an isomorphism; $P_S(A, M)$ will be said to be produced by M if π is onto. The former holds if and only if there exists an A-S-module generated by M; the dual condition is necessary and sufficient for the latter. (Received March 2, 1954.)

390*t*. D. G. Higman: *Induced and produced modules*. II.

The notation is that of the author's abstract immediately preceding. An A-S-module M will be called a quotient of (M', ι, σ), or a quotient of M', if M' is an A-S-module, ι an S-isomorphism of M into M', and σ an A-S-homomorphism of M' onto M such that $\iota\sigma=1$; the concept of an extension of M is defined dually. If M is an A-S-module: $I_S(M, A)$ is induced by M and there exists one and only one A-S-homomorphism t of $I_S(M, A)$ onto M such that M is a quotient of $(I_S(M, A), \kappa, t)$; $P_S(A, M)$ is produced by M and there exists one and only one A-S-isomorphism $j: M\to P_S(A, M)$ such that $(P_S(A, M), \pi, j)$ is an extension of M. The following conditions imply each other: (a) M is a direct summand of $I_S(M, A)$, (b) M is a direct summand of every A-S-module of which it is a quotient; dually, the following conditions imply each other: (a′) M is a direct summand of $P_S(A, M)$, (b′) M is a direct summand of every extension. Suppose that A has an identity element 1, 1 in S, and that A has finite bases as a left and right S-module. If, as an S-module, $A\,[\cong I_S(S, A)]\cong P_S(A, S)$—e.g., A a Frobenius algebra over a field S, A a finite group ring, S the subring spanned by a subgroup—then to every left S-basis $a_1, \cdots, a_n$ of A there corresponds a right S-basis $\bar{a}_1, \cdots, \bar{a}_n$ such that $\sum a_\nu \bar{a}_\nu$ is a center element of A, and $I_S(M, A)\cong P_S(A, M)$ for every S-module M having 1 as the identity operator on the right. In this case, the conditions (a), (b), (a′), (b′) are all equivalent to $(*)$ there exists an S-endomorphism α of M such that $\sum a_\nu \alpha \bar{a}_\nu=1$. This generalizes the author's theorem stated in Abstract 60-4-388. (Received March 2, 1954.)

391*t*. W. J. Klimczak: *A geometric solution of a system of linear inequalities*. Preliminary report.

This paper is concerned with a method of solutions of a system of linear inequalities of the form (A)$L_i(x)=\sum_{j=1}^{n} a_{i,j}x_j<c_i$ $(i=1, 2, \cdots, n)$ which is analogous to the Gaussian elimination method for solving systems of linear equations. The $a_{i,j}$ and c_i $(i, j=1, 2, \cdots, n)$ are known real constants and the solution of (A) is defined to be the totality of points $x=(x_1, x_2, \cdots, x_n)$ in n-dimensional Euclidean space E_n for which (A) is satisfied. Let H_i be the hyperplane in E_n which has $L_i(x)=c_i$ as its equation, and let R_i be the set of all points x of E_n such that $L_i(x)<c_i$. The method of solution of (A) consists of applying a sequence of n linear, nonsingular transformations to E_n. Let T_i $(i=1, 2, \cdots, n)$ denote the matrices of these transformations and $x^{(n)}=xT_1T_2\cdots T_n$, where $x^{(n)}=(x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)})$. The sequence consists of n-dimensional shears and is defined so that it transforms H_i into the hyperplane $H_i^{(n)}$ such that $H_i^{(n)}$ is orthogonal to the $x_i^{(n)}$-axis in the $x_1^{(n)}x_2^{(n)}\cdots x_n^{(n)}$-system. (A) is then reduced to a simpler system of the form (B) $f(a_{i,j})x_i^{(n)}<c_i$ $(i, j=1, 2, \cdots, n)$, each inequality of which defines a set of points $R_i^{(n)}$, the transform of R_i. The inverse image of the intersection of the point sets $R_i^{(n)}$ $(i=1, 2, \cdots, n)$ is the solution of (A). Thus the solution of (A) is given by $x=x^{(n)}T_n^{-1}T_{n-1}^{-1}\cdots T_1^{-1}$, where $x^{(n)}$ is determined by (B). The method may be used to solve the system of equations $L_i(x)=c_i$ $(i=1, 2, \cdots, n)$. (Received March 9, 1954.)

392t. Joachim Lambek: *A necessary condition for arithmetic equivalence between character algebra and center of group ring.*

Let G be a finite group, Z the center of its group algebra over an algebraically closed field of characteristic 0, H its character algebra over the same field (i.e. the basis elements of H are multiplied like the characters of G). It is known that Z and H have the same number s of basis elements, and that their multiplication tables consist of rational integers. When are they arithmetically equivalent, in the sense that the subrings of linear combinations of the basis elements with algebraic or rational integers as coefficients are isomorphic? All isomorphisms between Z and H are constructed explicitly in matrix form. Arithmetic equivalence requires that one of these matrices be unimodular. This becomes: $\prod_{i=1}^{s} h_i = \prod_{j=1}^{s} n_j$, where the h_i are the numbers of elements in the classes of conjugate elements, and the n_j are the degrees of the irreducible representations of G. The condition fails e.g. for the quaternion group of order 8. Does it imply that G is abelian? (Received March 10, 1954.)

393. Karel deLeeuw: *Applications of relative cohomology theory to algebraic number theory.*

This paper takes as point of departure notes of E. Artin (*Algebraic numbers and algebraic functions* II, to appear). There, for every finite group G and G-module A, cohomology groups $H^n(G, A)$ are defined for all integers n. It is proved that if A is the multiplicative group of a local field or is idele classes there is a canonical isomorphism $H^n(G, A) \cong H^{n-2}(G, Z)$ where Z is the integers with trivial action. In this paper, for every subgroup H of G, two kinds of relative cohomology groups are defined and two exact sequences constructed, one containing restriction and the other verlagerung. If A is as above there are canonical isomorphisms between the sequences for A and the sequences for Z which shift the dimension by two. In particular, the relative cohomology of A depends only on G and H and is independent of the arithmetic. In a number field the structure of the quotient group of norms of ideles module norms, in a layer that is not necessarily normal, can be determined in terms of the relative cohomology groups and is found to depend only on the decomposition groups and is otherwise independent of the arithmetic. (Received March 8, 1954.)

394. Arthur Mattuck: *Abelian varieties over p-adic fields.*

Let k be a field of characteristic 0 complete under a non-archimedean absolute value, and let A be an abelian variety of dimension d defined over k. That is, A is a d-dimensional algebraic variety embedded in a projective space over some universal domain containing k, whose points form without exception an algebraic group; k is to be a field of definition both for A and the group composition law. It is proved that the group of points on A rational with respect to k contains a subgroup isomorphic and homeomorphic to the direct sum of d copies of the integers of k. Further, if k is a finite algebraic extension of the rational p-adic field, then this subgroup is of finite index. An identical pair of statements may be made about the group of rational divisor classes of degree 0 of an algebraic function field in one variable over k as constant field: this follows from Chow's construction of an abelian variety defined over k (the Jacobi variety) whose rational points represent these divisor classes in a 1-1 fashion. In this form the case genus 1 was proved by Lutz (J. Reine Angew. Math. vol. 177 (1937)); the general statement was conjectured by Weil. (Received March 9, 1954.)

395*t*. E. V. Schenkman: *The existence of outer automorphisms of some nilpotent groups of class* 2.

Let G be a p-group whose commutator subgroup is in the center and such that for some fixed number p^k the p^kth powers of all the elements of G are in the commutator subgroup. Then the following two statements are valid: (1) G has an outer automorphism (except if G is of order 2). (2) If the order of G is odd, then it divides the order of the automorphism group of G provided G is not Abelian. (Received March 23, 1954.)

396. Abe Shenitzer: *Decomposition of a group with a single defining relation into a free product.*

Let $G(a_1, \cdots, a_n; R(a_1, \cdots, a_n)=1)$ be a group with a single defining relation. Using theorems by Grushko (A. G. Kurosh, *Theory of groups*, Gostekhizdat, 1944) and J. H. G. Whitehead (Ann. of Math. vol. 37 (1936)) it is shown that G cannot be decomposed into a free product if (a) R is a product $R_1 \cdots R_k$ of words $R_1, \cdots, R_k$, each of which is minimal (T) (in the sense of Whitehead) and of length greater than 1, and the sets $S_1, \cdots, S_k$ of generators of G appearing in $R_1, \cdots, R_k$, respectively, are pairwise disjoint or (b) all the exponents of the generators of G appearing in R are greater than or equal to 2. These statements are derived from the theorem that G is decomposable into a free product if and only if there exists a form of R which is minimal (T) and which contains at most $n-1$ of the n generators of G. (Received March 9, 1954.)

397. Morris Weisfeld: *A note on purely inseparable extensions.*

The results of Baer (Sitzungsberichte Heidelberger Akademie (1927) pp. 15–32) are generalized to purely inseparable extensions of finite exponent by means of the concept of a higher derivation due to Hasse-F. K. Schmidt (J. Reine Angew. Math. vol. 177 (1937) pp. 215–237). The main result shows that given a purely inseparable extension K of exponent e over a field C, there exists a higher derivation of rank less than or equal to p^e in K having C as the subfield of constants. (Received March 11, 1954.)

398. A. B. Willcox: *On the group algebra of the direct product of a compact group and a locally compact abelian group.*

Two properties of the group algebra $L^1(G)$ of the direct product G of a compact group and a locally compact abelian group are proved in this note. (1) The structure space of maximal regular ideals of $L^1(G)$ is Hausdorff and so is the structure space of the algebra formed by formally adjoining an identity to $L^1(G)$. The proof is based on results due to Kaplansky [Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949)] and allows techniques due to Silov [Trav. Inst. Math. Stekloff, 1947] and the author [Doctoral dissertation, Yale University, 1953] to be used in studying these algebras. (2) $L^1(G)$ is a cross algebra (in the sense of Schatten [Annals of Mathematics Studies, no 26, 1950] with the obvious extension of the multiplication operation) of the group algebras of the component groups of G (here it is sufficient that the component groups be locally compact). (1) and (2) are used to show that the Wiener Tauberian theorem in its ideal theoretic form holds for this case. The note concludes with one further application of a theorem from the above mentioned doctoral dissertation. The result concerns the spectral resolution of a closed ideal in $L^1(G)$ and says, in part, that if

$\mathbf{k}(\mathbf{h}(I))$, the kernel of the hull of I, is a regular ideal and if the kth derived set of $\mathbf{h}(I)$ is void, then $[\mathbf{k}(\mathbf{h}(I))]^k \subset I \subset \mathbf{k}(\mathbf{h}(I))$. (Received March 11, 1954.)

ANALYSIS

399. W. G. Bade: *A theorem on convergence of projections.*

Let X be a reflexive Banach space and let B be a bounded Boolean algebra of projections in X, i.e. $|E| \leqq M$, $E \in B$. Theorem: The strong and weak operator topologies coincide on B. The theorem is proved by reducing the problem to an integration problem with respect to a vector valued measure on a suitable measure space. (Received March 10, 1954.)

400. J. H. Barrett: *Oscillation and boundedness of solutions of* $y'' + q(x)y = 0$.

This paper is concerned with a modified polar coordinate transformation of the second order differential equation $y'' + q(x)y = 0$, specifically, $y = r(x) \sin \theta(x)$, $y' = w(x)r(x) \cos \theta(x)$. This transformation with $w(x) = 1$ was introduced by Prüfer. Certain conditions for oscillation, nonoscillation, and boundedness of solutions involving $w(x)$ are given. It is noted that certain choices of $w(x)$ give known results and that other choices give new results. Among the new results are: Theorem. If $\int_a^x |q| = O(x)$ as $x \to \infty$; $w(x) > 0$, $w'(x) \leqq 0$, and $Q(x) = \int_a^x |w - q/w|$ for $x \geqq a$, then $\limsup_{x\to\infty} (|y(x)|/w(x)) \exp Q(x)/2 > 0$. This theorem is proved by applying techniques of Levinson (Duke Math. J. vol. 8 (1941) pp. 1–10) to inequalities obtained by this transformation and a specialization of w gives a result of Levinson's as a corollary. Theorem: If there exists $w(x) > 0$ such that $w'(x) \leqq 0$, $w^2(x) \leqq q(x)$ and $\lim_{x\to\infty} [\int_a^x w + \ln w] = \infty$, then all solutions are oscillatory. (Received March 10, 1954.)

401t. Stefan Bergman: *On functions of two complex variables in domains with a distinguished boundary surface.*

Generalizing the investigations on analytic functions of two complex variables in Math. Zeit. vol. 36, p. 177 and vol. 39, p. 76, the author considers domains which are bounded by n analytic hypersurfaces k_S^3. The sum Γ^2 of intersections $i_{KS}^2 = k_K^3 \cap k_S^3$, $K \neq S$, $S, K = 1, 2, \cdots, n$, forms the distinguished boundary surface of M^4. The author introduces functions $\phi_S(Z)$, $Z = (z_1, z_2)$, which are orthonormal in M^4, when integrating over the distinguished boundary surface Γ^2, and shows that $\sum_{S=1}^{\infty} |\phi_S(Z)|^2 < \infty$, for $Z \in M^4$. The proof uses the lemma stating that the minimum of $\sum_{K,S} \int_{i_{KS}^2} |f(z_1, z_2)|^2 [\partial(z_1, z_2)/\partial(\lambda_K, \lambda_S)] d\lambda_K d\lambda_S$ under the condition $f(t_1, t_2) = 1$, $(t_1, t_2) \in M^4$, is uniformly bounded in every closed subdomain of M^4. Here $\{f\}$ denotes the totality of functions which are regular in M^4, continuous in M^4. Using this result the author obtains a development in a domain M^4 in terms of values on the distinguished boundary surface. In the case of three (and more) variables one obtains functions which are orthonormal when integrating over three- and four-dimensional boundary manifolds. Corresponding considerations lead to analogous representations. Finally, the specialization of the considerations to the case of functions satisfying linear differential equations is discussed. (Received April 8, 1954.)

402. P. L. Butzer: *Summability of generalized Bernstein polynomials.*

Let $f(x) \in L[0, 1]$, then the generalized Bernstein polynomials defined as $P_n^f(x)$

$=\sum_{\nu=0}^{n}(n+1)\binom{n}{\nu}x^{\nu}(1-x)^{n-\nu}\int_{\nu/(n+1)}^{(\nu+1)/(n+1)}f(t)dt$ have the property that $\lim_{n\to\infty} P_n'(x) = f(x)$ a.e. in $[0, 1]$. In this note the polynomials $\mathfrak{P}_n'(x)=(n+1)P_n'(x)-nP_{n-1}'(x)$ are considered. It is shown if $f(x)$ is defined on $[0, 1]$, $|f(x)|\leq M$, the sequence $\{P_n'(x)\}$ is summable $(C, -1)$ to $f(x)$, i.e. $\mathfrak{P}_n'(x)\to f(x)$ at every point $x\in[0, 1]$ where $f'(x)$ exists. If $f(x)$ has a modulus of continuity $\omega(\delta)$ in $0\leq x\leq 1$, with $\omega(\delta)=o(\delta^{1/2})$, then $\lim_{n\to\infty}\mathfrak{P}_n'(x)=f(x)$ uniformly in every interval $[\epsilon, 1-\epsilon]$, $0<\epsilon<1/2$. Furthermore, there exist functions continuous on $[0, 1]$ such that the $\mathfrak{P}_n'(x)$ diverge at a given point x, $0<x<1$. The proofs of the latter two results depend on the Lebesgue constants for the $\mathfrak{P}_n'(x)$ which are found to be of order $O(n^{1/2})$. (Received March 8, 1954.)

403. R. H. Cameron (p) and J. M. Shapiro: *Nonlinear integral equations.*

Under suitable conditions on y and F, this paper gives various expressions for the solution $x(t)$ of the nonlinear integral equation $y(t)=x(t)+\int_0^t F(t, s, x(s))ds$ in terms of Wiener integrals involving F and y. These results generalize the main results of two papers by Cameron and Martin [Amer. J. Math. vol. 66 (1944) pp. 281–298, and Ann. of Math. vol. 51 (1950) pp. 629–642] and a paper of Cameron, Lindgren, and Martin [Proc. Amer. Math. Soc. vol. 3 (1952) pp. 138–143]. This research was sponsored by the Office of Ordnance Research, U. S. Army, under contract DA-11-022-ORD-489. (Received March 8, 1954.)

404t. Y. W. Chen: *On a problem of minimal surface with polygonal boundary.*

Let C be a simple closed convex polygon on the x, y plane. The problem is to find a minimal surface solution $z=z(x, y)$ with the following properties: (1) it extends in the exterior of C to infinity, (2) $\partial z/\partial n=0$ on C, (3) at infinity $z_x:z_y:-1=\beta:0:-1$. This problem can be interpreted as a problem of disturbed uniform stream around profile C with given velocity at infinity—using Chaplygin's approximate adiabatic law. The present paper is concerned with the existence of solution and its behavior around the vertices of the polygon. It is known that such a solution can *not* exist, if around the corners the solution behaves in the same manner as in the case of incompressible flow. The existence proof is based on an earlier work of the author (Trans. Amer. Math. Soc. vol. 65), in which minimal surface solutions were obtained in parametric representation by harmonic functions $x(w)$, $y(w)$, and $z(w)$, with $w=u+iv$ and $|w|\leq 1$. A solution $z=z(x, y)$ is obtained if it is shown that the Jacobian $\partial(x, y)/\partial(u, v)\neq 0$. The significant feature in the proof is that the boundary mapping between $|w|=1$ and C is not a one to one point transformation. Except for one or two corners, every other corner has as its antecedent a whole arc on $|w|=1$. The function $z(x, y)$ is therefore discontinuous at these corners, although the parametric representation is analytic and continuous for $|w|\leq 1$. For applications the solution can be represented explicitly by a formula which generalizes the classical formula of Schwarz and Christoffel. (Received March 8, 1954.)

405. Ruth M. Davis: *The regular Cauchy problem for the Euler-Poisson-Darboux equation.*

Let $u(x, t)$ denote $u(x_1, x_2, \cdots, x_m, t)$ and let k be a real parameter. A solution to the equation (1) $L[u]=u_{tt}+kt^{-1}u_t-\Delta u=0$ with $u(x, t_1)=f(x)$, $u_t(x, t_1)=0$ is found for $t_1>0$ by a modified Riesz method. A function (2) $V^{\alpha}(x, t_0; y, t)=As^{\alpha-m-1}(t_1/t)^{k/2}\times F[(1-k/2), k/2; (\alpha+1-m)/2; (-s^2/4tt_1)]$, where A is known explicitly, is obtained, for which $M[V^{\alpha+2}]=V^{\alpha}$ where $M[u]$ is the adjoint operator to $L[u]$. F denotes the hypergeometric series. Applying Green's Theorem to u and $V^{\alpha+2}$, it is found by a limiting process as $\alpha\to 0$ that the solution to (1) is (3) $u(x,t)=a\sum_{j=0}^{\infty}[b_j t^{-k/2-j}t_1^{k/2+j+1}\cdot I^{(2j+2-m)/2}F(s)+c_j t^{-k/2-j}t_1^{k/2-j}s^{1/2}I^{(2j+1-m)/2}F(s)]$ where the coefficients are explicitly known. $I^{\beta}F(s)$ is the Riemann-Liouville integral with $F(s)=M(x,\ s^{1/2};\ f)s^{(m-2)/2}$ where $s=(t-t_1)^2$ and $M(x,\ s^{1/2}; f)$ denotes Poisson's mean value. (Received February 11, 1954.)

406. Albert Edrei: *On the zeros of successive derivatives.*

A recent result of the author [Bull. Amer. Math. Soc. Abstract 60-2-221] yields a number of propositions relative to the limit points of the zeros of the derivatives of an analytic function. Three of the simplest will be stated. (I) If the entire function $f(z)$ is real for real values of z and if it is bounded on the real axis, then every real point x_0 for which $\lim_{n\to\infty}\{f^{(n+1)}(x_0)/f^{(n)}(x_0)\}=\infty$ is a point of accumulation of zeros of the derivatives of $f(z)$. (II) Consider the function $f(z)=\sum a_\nu z^\nu$ and assume that $f(z)$ is not an entire function of exponential type. Then it is possible to choose a sequence of numbers ω_ν, each of which is equal to $+1$ or -1, and such that the origin is a limit point of the zeros of the derivatives of $h(z)=\sum a_\nu\omega_\nu z^\nu$. (III) The zeros of the derivatives of Mittag-Leffler's function $\sum z^n/\Gamma(1+n/2)$ have no finite limit point. The latter proposition settles a question of Pólya [Bull. Amer. Math. Soc. vol. 49 (1943) p. 181, footnote 2]. (Received March 8, 1954.)

407t Herbert Federer: *An analytic characterization of distributions whose partial derivatives are representable by measures.*

All first order partial derivatives of a distribution (in the sense of Laurent Schwartz) in Euclidean n-space are representable by locally finite signed Borel measures if and only if the distribution itself is representable by a locally Lebesgue integrable function f such that, for $j=1, 2, \cdots, n$ and each bounded interval I, the essential variation of $f(y_1, \cdots, y_{j-1}, t, y_j, \cdots, y_{n-1})$ with respect to t on I is locally Lebesgue integrable with respect to y in $(n-1)$-space. (The essential variation of a real-valued Lebesgue measurable function g on an interval I is the supremum of $\sum_{k=1}^{p}|g(t_k)-g(t_{k-1})|$ where $t_0<t_1<\cdots<t_p$ are points of approximate continuity of g in I.) (Received March 8, 1954.)

408t. R. F. Gabriel: *The Schwarzian derivative and convex functions.*

Let $f(z)=1/z+a_1z+a_2z^2+\cdots$ be regular in $0<|z|<1$. Let c be the smallest positive root of the equation $2x^{1/2}-\tan(x^{1/2})=0$. If $\{f(z),f\}$ denotes the Schwarzian derivative of $f(z)$ and if $|\{f(z),f\}|\leq 2c$ for $|z|<1$, then $f(z)$ is univalent in $0<|z|<1$ and maps the interior of each circle $|z|=r<1$ onto the exterior of a convex region. The constant c is a best possible one. Let $g(z)=z+b_2z^2+b_3z^3+\cdots$ be regular in $|z|<1$ and real on the real axis. Let $\Re\{zg'(z),z\}\geq -\pi^2/2$ in $|z|<1$. Then $g(z)$ is univalent in $|z|<1$ and maps the interior of the unit circle into a region which is convex in the direction of the imaginary axis. The constant $-\pi^2/2$ is a best possible one. The results are obtained by a modification of the method employed by Nehari [Bull. Amer. Math. Soc. vol. 55 (1949) pp. 545–551]. (Received March 4, 1954.)

409. Dieter Gaier: *On the equivalence of the two methods of Borel summability.*

If $B(x, s_n)$ and $B'(x, s_n)$ are Borel's exponential and integral transform, respectively, of a series $\sum a_n$, $s_n = \sum_{r=0}^{n} a_r$, then $B(x, s_n) \rightarrow s$ implies $B'(x, s_n) \rightarrow s$ $(x \rightarrow +\infty)$. The converse is not true in general; however, the author shows its validity under the assumption (1) $a_n = O(k^n)$ $(n \rightarrow \infty$, k any fixed constant), thus improving conditions by Garten $[a_n = O(n^{1/2})]$ and Karamata $[a_n = O(e^{n^\rho})$ $(n \rightarrow \infty, \rho < 1/3)]$. The proof uses a simple fact about integral functions $f(z)$ of mean type of order one, namely that $f(x) \rightarrow s$ $(z = x + iy)$ implies $f'(x) \rightarrow 0$ $(x \rightarrow +\infty)$. The condition (1) cannot be improved to $a_n = O(k^n \cdot n^{\epsilon n})$ $(\epsilon > 0$, fixed). There are close relations to the problem, when the change of index for Borel-summable series is allowed [cf. Math. Zeit. vol. 58 (1953) pp. 453–455]. (Received March 2, 1954.)

410. R. E. Gomory: *Critical points at infinity and forced oscillation.*

Let $dx/dt = X(x, y)$, $dy/dt = Y(x, y)$ be a pair of ordinary differential equations. Let s be the sum of the indices of the critical points. Let $X(x, y)$, $Y(x, y)$ be polynomials such that when the equations are extended to the projective plane, (1) there are critical points on the line at infinity $(z = 0)$ and they are simple, (2) there are no two consecutive saddle points on $z = 0$, (3) $s \neq 0$. It is then shown that the equations $dx/dt = X(x, y) + e_1(t)$, $dy/dt = Y(x, y) + e_2(t)$, with $e_1(t)$ and $e_2(t)$ periodic of period T, have a periodic solution. The first order systems which arise from $d^2x/dt^2 + f(x)dx/dt + g(x) = 0$ by setting $dx/dt = y$ generally have complicated critical points at infinity. For these systems, with f and g polynomials, and degree $f \geq$ degree $g > 0$, the critical points on $z = 0$ are completely analyzed, and their nature found to depend only on the leading coefficients a_n and b_m of f and g. Geometric information, such as the existence of limit-cycles in equations like that of van der Pol, is then easily deduced. Finally forced oscillations are shown to exist for the equation $d^2x/dt^2 + f(x)dx/dt + g(x) = e(t)$, $e(t)$ periodic, provided that (1) degree $f \geq$ degree g, (2) degree g is odd; (3) not both $b_m > 0$ and degree f odd hold. Both forced oscillation theorems show the existence of periodic motions for systems not necessarily dissipative for large displacements. (Received March 10, 1954.)

411*t*. E. C. Gras: *On a generalisation of the integrals of Weber and Schafheitlin.*

The following formula generalizing the theory of Weber's integrals is obtained: $\lim y^{-2k-1} \int_0^\infty e^{-xt} J_{r+j}(yt) J_{r+k}(bt) t^{-(1+j+k)} dt = \beta \sum_{i=0}^{j} C_i (b/y)^{2i+r} + (\pi - \beta) \sum_{i=0}^{k} D_i (y/b)^{2i+r}$, where $x \rightarrow 0^+$, and $x^{-1}(y - b) \rightarrow \cot \beta$, and $r > 0$, $1 + j + k \geq 0$, and $C_i = 0$ if $j < 0$, and $D_i = 0$ if $k < 0$. The proof is based on an extension of the methods of generalized axially symmetric potential theory of A. Weinstein (Bull. Amer. Math. Soc. vol. 59 (1953) pp. 20–38). The result includes as a special case the integrals of Weber-Schafheitlin. (Received February 10, 1954.)

412. Sigurdur Helgason: *Banach algebras of almost periodic functions.*

The space of continuous almost periodic functions $f(x) \sim \sum a(\lambda) e^{i\lambda x}$ is considered as a Banach algebra A under convolution-multiplication. The automorphisms T of A are determined, and correspond by $Tf(x) \sim \sum a(\sigma(\lambda)) e^{i\lambda x}$ to a group H of permutations of R. H contains all permutations of the form $\sigma(\lambda + \nu) = \sigma(\lambda) + \sigma(\nu) - \sigma(0)$, and these permutations correspond exactly to the isometric automorphisms of A. A function $q(\lambda)$ is called a multiplier if $\sum q(\lambda) a(\lambda) e^{i\lambda x}$ is an almost periodic Fourier series whenever $\sum a(\lambda) e^{i\lambda x}$ is. The multipliers are precisely the functions of the form $q(\lambda) = q_1(\lambda) - q_2(\lambda) + iq_3(\lambda) - iq_4(\lambda)$ where the $q_j(\lambda)$ are positive definite (in general nonmeasurable) functions. (The continuous multipliers were determined by R. Doss, Ann. of Math. vol. 46 (1945).) A multiplier $q(\lambda)$ induces an operator on A which is isometric if and only if $q(\lambda+\nu)q(0)=q(\lambda)q(\nu)$, $|q(0)|=1$. For the special case where $q(\lambda)$ is the characteristic function of a set $S\neq R$ the following is proved: (1) S has "mean density" 0. (2) If $f_S(x)\sim\sum_{\lambda\in S}(\lambda)e^{i\lambda x}$ is positive whenever $f(x)$ is, then S is a subgroup of R, and all subgroups have this property. The results hold for all abelian groups G with exception of (1) which is proved under the assumption that the character group $\widehat{G}$ is an infinitely divisible group. (Received March 9, 1954.)

413*t*. Sigurdur Helgason: *On the distribution of values of analytic almost periodic functions.*

A normal almost periodic function is a nonconstant analytic almost periodic function whose Laurent-Dirichlet expansion contains a smallest or a largest exponent. The inverse of such a function exists and is again of this type (H. Bohr, Ann. of Math. vol. 32 (1931) pp. 247–260), and the same is proved for the composition of two such functions. The main results of Nevanlinna's theory for the distribution of values of single-valued analytic functions are extended to normal almost periodic functions. An essential tool is the generalization of Jensen's formula, due to B. Jessen (Math. Ann. vol. 108 (1933) pp. 485–516). The characteristic function $T(\sigma)$, the defect $\delta(a)$, and the ramification index $\theta(a)$ can be introduced for any complex number a. $T(\sigma)$ is increasing and convex and describes the asymptotic behavior of the corresponding normal almost periodic function $f(s)$. Two different cases arise: (1) $f(s)$ defined in the whole s-plane. (2) $f(s)$ defined in a half plane, say $(-\infty, 0)$. In the first case $\lim_{\sigma\to\infty}T(\sigma)=\infty$ and $\sum_a(\delta(a)+\theta(a))\leqq 1$, and in the second case the quantity $\kappa=\limsup_{\sigma\to 0}1/(1-e^{\sigma})\cdot T(\sigma)$ is introduced, and the relation $\sum_a(\delta(a)+\theta(a))\leqq 1+2\kappa$ is proved. In particular, if $\kappa=0$, $f(s)$ omits at most one value. (Received March 9, 1954.)

414. Peter Henrici: *On certain series expansions involving Whittaker functions and Jacobi polynomials.*

Substitution of polar coordinates in (*) $\partial^2u/\partial x^2+\partial^2u/\partial y^2+(\mu/x)\partial u/\partial x+(\nu/y)\partial u/\partial y+[\lambda-k^2(x^2+y^2)]u=0$ leads to a set of solutions F_n of (*) expressible in terms of Whittaker functions and Jacobi polynomials. With a view towards deriving some functional relations involving hypergeometric functions, a technique is developed to construct expansions of functions of a certain class E of regular solutions of (*) in terms of the F_n. The method (related to Bergman's integral operator method) consists in setting up a 1-1 correspondence between E and the class of even analytic functions of one complex variable regular at 0 by associating to a $u(x, y)\in E$ the function $u(s/2, -is/2)$. Since F_n is shown to correspond to s^{2n}, the expansion problem in question reduces to the determination of the Taylor expansion of an analytic function of one variable. Application of this technique to some special solutions of (*) gives rise to three expansions involving various types of hypergeometric functions. As special cases and corollaries one obtains, among others, Bateman's addition theorem for Bessel functions, Ramanujan's formula for the product of two ${}_1F_1$, Erdélyi's addition theorem for the product of two M-functions, and Bailey's decomposition formula for Appell's function F_4. (Received March 9, 1953.)

415. W. B. Jurkat: *Elementary Mercerian theorems.*

A Mercerian theorem is generally spoken of as an equivalence theorem between two summability methods, one of them usually is the ordinary convergence. A and B

being regular and normal matrices, I the unit matrix, then there are considered the following Mercerian theorems: $rA+(1-r)I \approx I$ or more general $rA+(1-r)B \approx B$, $0<r<1$. In the first case it is possible to give necessary and sufficient conditions introducing the conception of a mean value theorem for A. Several monotone conditions on the elements of the matrix A, sufficient for the mean value theorem, are established, and each of these conditions leads to a Mercerian theorem of the first kind. In the more general second case similar sufficient conditions can be given. They mean roughly speaking that A should be stronger than B. Some known and also some unknown special cases are included in the theorems established. A special case covered is $rC_k+(1-r)C_i \approx C_i$ with $0<r<1, 0\leqq i\leqq k\leqq 1$, where C_i, C_k denote Cesàro methods of orders i, k respectively. (Received March 9, 1954.)

416t. Viktors Linis: *Power series in fields with nonarchimedean valuation*. Preliminary report.

The case when the valuation is nonarchimedean and nondiscrete is studied. It is shown that for functions defined by convergent power series the maximum modulus principle and Cauchy estimate holds. For entire functions a factorization theorem analogous to that of Weierstrass in complex analysis is established. The proofs are based on the theorems on moduli of polynomials and distribution of their zeros. (Received March 8, 1954.)

417t. C. N. Moore: *On questions of regularity for Nörlund means of double series*.

Suppose that the matrix p_{mn} for the Nörlund mean of a double series yields a regular method. The question arises as to whether or not the individual rows and columns of this matrix yield regular methods for simple series. It can be shown by examples that they do not. Furthermore, it can also be shown by examples that in case the individual rows and columns do not furnish a regular method for simple series, the corresponding double Nörlund means are not necessarily consistent. Hence the requirement of regularity for the rows and columns is an essential supplementary condition in extending Nörlund's theorem to double series. (Received March 8, 1954.)

418. Jürgen Moser: *A singular problem of perturbation theory*.

Consider an eigenvalue problem $(P+\epsilon Q)u(x)=\lambda u(x)$, where P, Q are differential operators with respect to x of even orders $2p$ and $2q$, resp. In $a\leqq x\leqq b$ the coefficients of P, Q are assumed to be real and infinitely differentiable. The coefficients of the highest derivative in $(-1)^pP$ and $(-1)^qQ$ are positive. At $x=a$, $x=b$ the solution has to satisfy q boundary conditions. The essential assumption is $p<q$; hence the order of $P+\epsilon Q$ changes for $\epsilon=0$. The problem is to investigate the dependence of the eigenvalues on the positive parameter ϵ, especially for $\epsilon\rightarrow 0$. Under general assumptions it can be proved that the asymptotic behavior of an eigenvalue $\lambda(\epsilon)$ for $\epsilon\rightarrow 0$ is given by an asymptotic series $\lambda(\epsilon)\sim\lambda_0+\eta\lambda_1+\eta^2\lambda_2+\cdots$, where $\eta=\epsilon^{1/(2q-2p)}$. In general the series diverges, but approximates $\lambda(\epsilon)$ in the sense of Poincaré. Also the eigenfunctions can be described by such asymptotic series. The essential difficulty of the problem lies in the fact that "unperturbed" eigenvalue problem, for $\epsilon=0$, is not completely known. Namely, for $\epsilon=0$ there are too many boundary conditions prescribed. It can be decided (from pure algebraic arguments) which boundary conditions have to be cancelled for $\epsilon=0$. The proof uses the known results concerning the asymptotic behavior of solutions of differential equations containing a small parameter in the highest derivative. (Received March 9, 1954.)

419*t*. Leopoldo Nachbin: *Bornological spaces of continuous functions and cartesian products.*

The vector space of all real-valued continuous functions on a completely regular space T is bornological in the sense of Mackey (cf. Bourbaki, Annales Inst. Fourier vol. 2 (1950) pp. 5–16) with respect to the compact-open topology if and only if T is saturated, i.e. complete under the weakest uniform structure rendering uniformly continuous all real-valued continuous functions on T. Every regular space in which any open covering contains a countable subcovering is saturated. A discrete space is saturated if and only if there is no Ulam measure on it. Let $E=\prod_{\iota\in I}E_\iota$, $E_\iota\neq 0$, be a cartesian product of real vector spaces and, for $J\subset I$, let E_J be the canonical image in E of $\prod_{j\in J}E_j$. Then every convex subset $V\subset E$, containing 0 and absorbing any set $\prod_{\iota\in I}F_\iota$ whose components $F_\iota\subset E_\iota$ are finite, must contain some E_J with $I-J$ finite if and only if the discrete space I is saturated. It results in the known fact that E is bornological if and only if every E_ι is bornological and I is saturated. (Received January 21, 1954.)

420. Alexander Peyerimhoff: *Summability factors for absolute Cesàro summable series.*

By a well known theorem of I. Schur, necessary and sufficient conditions for numbers ϵ_n in order to be (C_α, C_β) summability factors, i.e. to transform every C_α-summable series $\sum a_n$ into a C_β-summable series $\sum a_n\epsilon_n$ $(\alpha\geqq 0, \beta\geqq 0)$, are the following: $\sum\binom{n+\alpha}{n}\left|\Delta^{\alpha+1}\epsilon_n\right|<\infty$, $\epsilon_n=O(n^{\beta-\alpha})$. The report deals with the corresponding problem for absolute Cesàro summability and proves for $(|C_\alpha|, |C_\beta|)$ summability factors the necessary and sufficient conditions $\Delta^\alpha\epsilon_n=O(1/n^\alpha)$, $\epsilon_n=O(n^{\beta-\alpha})$. This theorem generalizes some results of Bosanquet and Kogbetliantz. (Received March 9, 1954.)

421. P. C. Rosenbloom: *Singularities of solutions of nonlinear hyperbolic equations.* Preliminary report.

We show that if $u_{tt}-u_{xx}=u^2$ for $|x|\leqq T-t$, $0\leqq t\leqq t_0\leqq T$, $u(x, 0)=u_0(x)$, $u_t(x, 0)=0$ for $|x|\leqq T$, and if $u_0(x)\geqq a>(K/T)^2$, where K is a certain numerical constant, then $t_0<T$, i.e., $u(0, t)$ has a singularity in the interval $[0, T]$. A similar conclusion holds for a certain class of nonlinear hyperbolic equations. Some remarks are also made concerning the singular solutions of $\Box u=\lambda u^3$, which arises in the theory of self-coupled photon fields, especially with respect to the nonanalyticity of their dependence on λ at $\lambda=0$. (Received March 12, 1954.)

422. Walter Rudin: *Multiplicative groups of analytic functions.*

For a proper subdomain D of the Riemann sphere, $G(D)$ denotes the multiplicative group of all nonvanishing regular single-valued analytic functions f on D, normalized by $f(z_0)=1$ for some fixed $z_0\in D$. $H(D)$ is the subgroup of $G(D)$ which consists of those f for which the equation $g^n=f$ has a solution $g\in G(D)$, for every integer $n\neq 0$; $f\in H(D)$ if and only if f has a single-valued logarithm in D. Results: (1) For any two domains D_1 and D_2, $H(D_1)$ and $H(D_2)$ are isomorphic. (2) $G(D)$ is the direct product of H and $G(D)/H$. (3) If the complement of D has $k+1$ components, $G(D)/H$ is the direct product of k infinite cyclic groups. (4) If the complement of D has infinitely many (countable or not) components, $G(D)/H$ is isomorphic to the additive group of all integer-valued functions on a countable space. It follows that the algebraic structure of $G(D)$ determines, and is determined by, the connectivity of D. (Received March 9, 1954.)

423. J. B. Serrin: *A uniqueness theorem for the parabolic equation $u_t=a(x)u_{xx}+b(x)u_x+c(x)u$.* Preliminary report.

Suppose that the coefficients of the equation $(*)$ $u_t=a(x)u_{xx}+b(x)u_x+c(x)u$ are bounded and Holder continuous for $-\infty<x<+\infty$, and that $a(x)\geqq k$ where k is a positive constant. Let S denote the strip $-\infty<x<+\infty$, $0<t<T$ in the (xt)-plane and $\overline{S}$ its closure (i.e. the strip $0\leqq t\leqq T$). Then if $u(x, t)$ is a non-negative solution of $(*)$ in S which is continuous in $\overline{S}$, it can be represented in terms of its initial values $u(x, 0)=f(x)$ by a functional operator $P(f)$. It follows that such a solution $u(x, t)$ is uniquely prescribed by its initial values. This represents a partial extension of a theorem of Widder concerning non-negative solutions of the equation of heat conduction (Trans. Amer. Math. Soc. vol. 55 (1944)). (Received March 10, 1954.)

424t. V. L. Shapiro: *Localization of conjugate multiple trigonometric series.*

Let $T=\sum_m a_m e^{i(m,x)}$ be an n-dimensional trigonometric series where $(m, x)=m_1x_1+\cdots+m_nx_n$, and let $K(x)=P_g(x)/|x|^{-(n+g)}$ where $P_g(x)$ is a homogeneous polynomial of degree g which satisfies Laplace's equation $\Delta P=0$. Then Calderón and Zygmund have shown that to the trigonometric series T one can associate a series $\overline{T}$, called the conjugate series of T with respect to the spherical harmonic kernel $K(x)$, which is an extension of the classical one-dimensional case. $\overline{T}$ is defined by $\overline{T}=\sum_m a_m\widehat{K}(m)e^{i(m,x)}$ where $\widehat{K}(u)$ is the principal-valued Fourier transform of $K(x)$, recently evaluated by Bochner. Using the technique of formal multiplication of series developed by Rajchman and Zygmund, and also Berkovitz, this paper studies the localization properties of the above conjugate series. In particular if a_m, the coefficients of the T, are $o(|m|^p)$, p an integer not less than $-(n-1)$, a function $F(x)=a_0|x|^{2q}/K_q+(-1)^q\sum_{m\neq 0}a_m|m|^{-2q}e^{i(m,x)}$ can be associated to T where q is the integer $[(p+n)/2]+1$ and $\Delta^q|x|^{2q}=K_q$. It is then shown in this paper that if $F(x)$ is in class $C^{(2q+n+1)}$ on a domain D contained in the fundamental cube R, then $\overline{T}$ is uniformly spherically summable $(C, p+n-1)$ in every interior closed subdomain of D. (Received March 9, 1954.)

425. F. M. Stewart: *Periodic solutions of a nonlinear wave equation.*

The differential equation (A) $u_{tt}-u_{xx}=\mu(u^3+f(x)\sin t)$ is replaced by an ordinary differential equation, (B) $Z'=\mu F(Z, t)$, in a space of sequences. Following Fatou [Bull. Soc. Math. France vol. 56 (1928) pp. 98–139] one can find a Z_0 such that $\overline{F}(Z_0)=(1/2\pi)\int_0^{2\pi}F(Z_0, t)dt=0$. It is proved that if μ is sufficiently small, then (B) has a periodic solution near Z_0. Difficulties, caused by the fact that the linear part of $\overline{F}$ is singular, are overcome by using several norms in the space of sequences. Using the periodic solution of (B) it is easy to construct a periodic solution of (A). (Received March 9, 1954.)

426. C. T. Taam: *Criteria of boundedness of the solutions of nonlinear differential equations.*

Let r and p_i be real-valued continuous functions of x on I: $a\leqq x<\infty$, r being positive. Let also rp_i be absolutely continuous on I. Consider only the real-valued solutions of (A): $(ry')'+\sum_{i=1}^{n}p_iy^{2i-1}=0$. Denote by f'_+ the max $(f', 0)$ and by f'_- the min $(f', 0)$. It is proved that if (1) $rp_i>0$ on I, $i=1, 2, \cdots, n$, (2) $(rp_i)'_-/rp_i$ belongs to $L(a, \infty)$ for some $i=k$ and $(rp_i)'_+/rp_i$ belongs to $L(a, \infty)$ for all $i\neq k$, then every solution of (A) is bounded on I. Other results derived from this theorem are also given. Let (B) denote what results in (A) if the power of y in (A) is i and n is a positive odd integer. The following theorem is also obtained: Every solution y of (B) and ry' are bounded on I if $(rp_i)'$ belongs to $L(a, \infty)$ for $i=1, 2, \cdots, n$ and rp_n has a positive lower bound on I. (Received March 10, 1954.)

427. J. L. Walsh and D. M. Young (p): *Lipschitz conditions for harmonic and discrete harmonic functions.*

Let R denote a plane region whose boundary S is a Jordan curve, let $u(z)$ be harmonic in R, continuous in $\Omega=R+S$, and let $u(z)\in\text{Lip}\ \alpha$ on S, $0<\alpha<1$, in the sense $|u(z_1)-u(z_2)|\leq L|z_1-z_2|^\alpha$ (L constant), for $z_1, z_2\in S$. If R is a disk or half plane, then $u(z)\in\text{Lip}\ \alpha$ in Ω. This conclusion is shown to hold when R is a semi-infinite strip or a rectangle. An "accordion folding process" is used, which studies, for instance, the functions $u(x, y)-u(2-x, y)+u(2+x, y)-u(4-x, y)+u(4+x, y)-\cdots$ in the strip $0\leq x\leq 1$, $y\geq 0$, if $u(x, y)$ is defined for $x\geq 0$, $y\geq 0$. Next, let $U(z)$ be discrete harmonic in R_L and assume the same values as $u(z)$ on S_L. (Here R_L and S_L denote respectively the set of points (mh, nh) (m, n integers) belonging to R and S.) An inequality of Allen and Murdoch (Proc. Amer. Math. Soc. (1953)) is used to show $U(z)\in\text{Lip}\ \alpha$ uniformly with respect to h in $\Omega_L=R_L+S_L$ if R is $\text{Im}\ (z)>0$. The conclusion also holds when R is a semi-infinite strip or a rectangle. For the rectangle the difference $|U(z)-u(z)|=O(h^{2\alpha/(\alpha+1)})$ uniformly in Ω_L as $h\rightarrow 0$. This improves a previous result of the authors. (Received March 9, 1954.)

428. František Wolf: *Simplicity of spectra in general operators.*

Let $E(M)$ be a spectral measure, i.e. a mapping of Borel sets in the plane into uniformly bounded idempotent operators in Hilbert space, completely additive with the usual other conditions. Then, if there exists a generator g such that $E(M)g$, considered for all Borel sets, generates all of Y, then $E(M)$ will be called simple. Then $A=\int\lambda DE(\lambda)$ is a scalar operator in Dunford's sense (cf. Pacific Journal of Mathematics vol. 2 (1952) p. 559) with a simple spectrum: Any bounded B commutative with A is of the form $\int b(\lambda)dE(\lambda)$, i.e. a function of A, $b(A)$. Any spectral operator (cf. Dunford loc. cit.) admitting $E(M)$ as its spectral measure is a scalar operator. By Wermer's result (to appear in the Pacific Journal of Mathematics (1954)) there is no loss of generality to suppose that $E(M)$ is a mapping into projections. Then the condition stated guarantees that there exists (cf. Akhiezer-Glazman, *Theory of lin. operators*, in Russian, p. 269) an isometric transformation of our space into a $L^2(d\sigma)$ and that A transforms into $x\cdot$, the multiplicative operator. Any operator commutative with A, by a well known theorem (cf., i.e., Akhiezer-Glazman, loc. cit. p. 300) must be a function of A, and so a scalar operator. (Received March 10, 1954.)

APPLIED MATHEMATICS

429. Milton Abramowitz and H. A. Antosiewicz (p): *Coulomb wave functions in the transition region.*

The regular and irregular solutions, $F_L(\eta, \rho)$ and $G_L(\eta, \rho)$, of the Coulomb wave equation $y''+(1-2\eta\rho^{-1}-L(L+1)\rho^{-2})y=0$ are obtained for $L=0$ and values of η and ρ in terms of the variable $t=(2\eta-\rho)(2\eta)^{-1/3}$ under the assumption that $|t|$ is small. These solutions are expressed in the form $\theta(t, \eta)U(t)+\psi(t, \eta)U'(t)$ where $U(t)$ is a solution of the equation $U''-tU=0$ and $\theta(t, \eta)$, $\psi(t, \eta)$ are power series in t so determined that these solutions yield for $t=0$ the values of F_0, G_0 at $\rho=2\eta$, respectively. By identifying $U(t)$ with the Airy integrals $Ai(t)$ for $F_0(\eta, \rho)$ and $Bi(t)$ for $G_0(\eta, \rho)$, the power series obtained for $\theta(t, \eta)$, $\psi(t, \eta)$ are identical in both cases and converge rapidly for sufficiently small $|t|$. The values of $F_L(\eta, \rho)$, $G_L(\eta, \rho)$ with $L>0$ may be obtained from those of $F_0(\eta, \rho)$, $G_0(\eta, \rho)$ by use of recurrence relations satisfied by $F_L(\eta, \rho)$, $G_L(\eta, \rho)$ without loss of accuracy for the pertinent range of values of η and ρ. This method may be generalized to equations $y''+\phi(t, \lambda)y=0$ where $\phi(t, \lambda)\to t^k$, $k\geq 0$, as $\lambda\to\infty$ and $\phi(t, \lambda)$ is analytic in t at $t=0$. (Received March 8, 1954.)

430t. F. H. Brownell: *Perturbation of the n-dimensional Schrödinger differential operator.*

Consider the Schrödinger operator $[Au](x)=-\nabla^2u(x)+V(x)u(x)$ for $u\in D$ dense in $L_2(R_n)$ over $x\in R_n$, n-dimensional euclidean space, where ∇^2 is the Laplacian $\sum_{j=1}^{n}\partial^2/\partial x_j^2$ and $V(x)$ is an arbitrary real-valued, measurable function over R_n. For $n=3$, if $\{\text{ess sup}_{x\in R_3}\int_{R_3}(|V(y)|/4\pi\|x-y\|)d\mu_3(y)\}<1$, then a natural self-adjoint extension of A has the same spectrum as that of the unperturbed operator A_0 defined by setting $V(x)=0$, namely the point spectrum is void and the continuous spectrum is precisely the positive axis $[0, +\infty)$. Similar results hold for $n>3$, but the conditions on $V(x)$ become more complicated. The question of unitary equivalence of the extensions of A and A_0 is also considered. (Received March 12, 1954.)

431. J. B. Diaz (p) and G. S. S. Ludford: *A transonic approximation.*

This approximation is based on obtaining that pressure-density relation which best fits that of a polytropic gas near the sonic point, whilst retaining the analytical simplicity of the Tricomi gas. Third order contact in the pressure-density relation can be obtained at the sonic point, and there still remains a parameter, which can be chosen in two ways so as to give good supersonic agreement. These two choices give practically identical results. For this approximation the general solution of the partial differential equation for the stream-function is expressible in terms of Bessel functions, as also are the product solutions. (Received March 10, 1954.)

432. L. C. Maximon: *Indefinite integrals involving the special functions.*

In the solution of one-dimensional linear homogeneous time dependent partial differential equations one may take the transform with respect to time and solve the resulting ordinary inhomogeneous differential equation, obtaining the transform in terms of indefinite integrals involving solutions of the homogeneous transformed equation. If the boundary and initial conditions are such that the solution to the original problem is actually time independent, its time-transform is then trivial and one may reverse the procedure just mentioned to solve for the indefinite integrals. Following this procedure a general method of evaluating in closed form a large class of integrals involving the special functions of mathematical physics has been developed and a number of typical examples are given. The method is also extended to evaluate integrals containing functions which are solutions of nth order linear differential equations. (Received March 10, 1954.)

433. Cathleen S. Morawetz: *A nonexistence theorem for transonic flows past a profile.* Preliminary report.

Let there be given a sufficiently smooth, two-dimensional, steady, irrotational compressible flow F_0, with potential ϕ_0 and complex velocity $qe^{-i\theta}$ about a convex profile P_0. The free stream Mach number of the flow is less than 1 but there is a finite supersonic region. The profile and flow are symmetric about the x and y axes. Let S be the segment of the profile cut out by the two Mach lines issuing from the intersection of the sonic curve with the y-axis. Let ϕ_1 be the velocity potential of an irrotational, steady, symmetric, continuous flow, with the same free stream Mach number as F_0, around a profile P_1 coinciding with P_0 except on S and very close to P_0. If products of the first two derivatives of $\omega=\phi_1-\phi_0$ are neglected, ω satisfies the equation (1) $K(\sigma)\omega_{\theta\theta}+\omega_{\sigma\sigma}=0$ with $K(\sigma)\gtrless 0$ for $q\gtrless c$, $\sigma=\sigma(q)$, and the boundary condition (2) $K(\sigma)\omega_\theta d\sigma-\omega_\sigma d\theta=0$ on the images of P_0-S and $y=0$ in the (θ, σ)-plane. We prove that there exists only one solution ω satisfying (1) and (2). Hence there exists at most a one-parameter family of profiles P_1 for which a continuous flow exists. This is a rigorous proof of the conjecture, frequently stated in the literature, that the perturbation problem belonging to a transonic flow has no solution in general. (Received March 23, 1954.)

434. Charles Saltzer: *Two-terminal relay and switching circuits*. Preliminary report.

A topological method is given for determining the Boolean hindrance function for a two-terminal relay and switching circuit from the incidence matrix of the graph of the circuit. (Received March 10, 1954.)

435. Raymond Sedney: *A maximum principle for the drag in linearized airfoil theory*. Preliminary report.

A function space is established for suitably restricted functions representing the pressure and angle of attack distributions on a lifting surface. Using linearized airfoil theory, the scalar product is defined as one-half the mutual interference drag of two distributions; the metric is then the square-root of the drag of a distribution. This idea was first introduced by E. W. Graham and P. A. Lagerstrom. Using simple geometrical reasoning, a maximum principle involving the drag is found. This can be used to obtain approximate solutions to the lifting surface integral equation, similar to the Rayleigh-Ritz procedure. A workable procedure has been found only for the case of subsonic flow. When used as an iteration method, the procedure gives an increasing sequence of lower bounds for the drag. However, singularities in the pressure distribution must be excluded so that the drag does not include leading edge suction. (Received March 10, 1954.)

436t. R. L. Sternberg: *Elementary methods in the numerical design of microwave dielectric lenses*. Preliminary report.

A number of apparently new techniques are presented which facilitate the design of zoned or unzoned axially symmetric dielectric lenses for microwave antenna applications in which it is required to produce an off axis beam from an off axis focus when the refractive index, the wave length, the beam angle, the focal distance and the lens diameter are preassigned. Among other results obtained in the meridional plane it is shown: (i) that the focus must be located on a particular hyperbola, (ii) that the locations of the zone extremities are unique and vary with the wave length, the beam angle, the focal distance, and the lens diameter, and (iii) that the required nonspherical surfaces may be fitted in a nontrivial manner to control several meridional rays perfectly including, in particular, the meridional rays through the zone and lens extremities. Elementary methods are indicated suitable for carrying out the required calculations. Applications to scanning antenna problems are briefly discussed. (Received March 5, 1954.)

437t. R. L. Sternberg, J. S. Shipman, and Hyman Kaufman: *Tables of Bennett functions for the two-frequency modulation product problem for the half-wave square law rectifier*. Preliminary report.

Eight decimal tables of the functions $A_{mn}^{(2)}(k) = (2/\pi^2)\iint_R(\cos u + k\cos v)^2 \cos mu\,du \cdot \cos nv\,dv$, R: $\cos u + k \cos v \geqq 0$, $0 \leqq u, v \leqq \pi$, for $k = 0.02\ (0.02)\ 1.0$ and $m+n \leqq 5$ are provided for application to the problem referred to in the title (see W. R. Bennett, *New results in the calculation of modulation products*, Bell System Technical Journal vol. 12 (1933) pp. 228–243). The evaluation of the functions $A_{mn}^{(2)}(k)$ is carried out by means of power series expansions and finite formulas in complete elliptic integrals while the actual arithmetic is performed on I.B.M. equipment. New recursion identities by means of which the higher order functions $A_{mn}^{(2)}(k)$ can be expressed in terms of those tabulated are given. Bennett functions $A_{mn}^{\nu}(k)$ of the νth kind are defined and recursion identities between the $A_{mn}^{(\nu+1)}(k)$ and $A_{m'n'}^{(\nu)}(k)$ are derived by means of which a number of the higher kind functions may be evaluated readily in terms of the second kind functions tabulated. (Received February 22, 1954.)

438t. R. L. Sternberg, J. S. Shipman, and W. B. Thurston: *Tables of Bennett functions for the two-frequency modulation product problem for the half-wave linear rectifier.*

Eight decimal tables of the functions $A_{mn}(k) = (2/\pi^2)\iint_R(\cos u + k \cos v) \cos mu\,du \cdot \cos nv\,dv$, R: $\cos u + k \cos v \geqq 0$, $0 \leqq u, v \leqq \pi$, for $k = 0.02\ (0.02)\ 1.0$ and $m+n \leqq 4$ are provided for application to the problem referred to in the title (see W. R. Bennett, *New results in the calculation of modulation products*, Bell System Technical Journal vol. 12 (1933) pp. 228–243). The evaluation of the functions $A_{mn}(k)$ is carried out by means of power series expansions and finite formulas in complete elliptic integrals while the actual arithmetic is performed on I.B.M. equipment. Known recursion identities by means of which the higher order functions $A_{mn}(k)$ can be expressed in terms of those tabulated are noted. (Received February 22, 1954.)

439. H. F. Weinberger: *A Rayleigh-Ritz procedure with arbitrary error estimate for differential operators with finite perturbations.*

Let B be a positive, symmetric operator with discrete spectrum, having known eigenvectors u_i and eigenvalues $\mu_i \to \infty$. Let A be an operator whose eigenvalues λ_i are to be approximated, and which satisfies $B \leqq A \leqq B + cI$, where c is a known constant and I the identity operator. Let $\lambda_i^{(N)}$ be the ith eigenvalue obtained by applying the Rayleigh-Ritz method to A, using the first N eigenvectors of B as "coordinate vectors." Then it is well known that $\lambda_i \leqq \lambda_i^{(N)}$. The following inequality then serves as an error estimate for the approximation of $\lambda_i^{(N)}$ to λ_i: $\lambda_i \geqq \lambda_i^{(N)} - c^2[4\mu_{N+1} - 4\lambda_i^{(N)} + 2c]^{-1}$. Since $\lambda_i^{(N)}$ decreases with N^{-1}, it may be replaced in the error term by an earlier Rayleigh-Ritz bound, the simplest being $\lambda_i^{(i)}$. Then the error term may be made arbitrarily small by choosing N sufficiently large, since $\mu_{N+1} \to \infty$. The above inequality is obtained by first proving a stricter but more complicated inequality for the special case where A and B are 2×2 matrices. This inequality is then extended to operators in Hilbert space by decomposing vectors into their projections into the subspaces $[u_1, \cdots, u_N]$ and its complement. (For a similar procedure, see the proof of Aronszajn's Inequality in Hamburger and Grimshaw, *Linear transformations in n-dimensional vector space*, Cambridge, 1951, p. 76.) The inequality so obtained is then simplified at the price of some precision. The method here presented is applicable to a Schroedinger operator differing by a bounded potential function from a known Schroedinger operator with discrete spectrum. (Received March 11, 1954.)

GEOMETRY

440. H. S. M. Coxeter: *Arrangements of equal spheres in non-Euclidean spaces.*

Fejes Tóth (*Lagerungen in der Ebene, auf der Kugel und im Raum*, Berlin, 1953, p. 114), in his work on arrangements of equal circles, proved that the elliptic plane cannot be packed as densely, nor covered as thinly, as the Euclidean plane. Accordingly, it is interesting to find a different state of affairs among the arrangements of equal spheres in space. It is found that, in elliptic space, sixty spheres of radius $\pi/10$, each touching twelve others, form a packing of density $0.774 \cdots$, and that somewhat larger spheres having the same centers form a covering of density $1.439 \cdots$, whereas in Euclidean space the maximum packing-density and the minimum covering-density are almost certainly $0.740 \cdots$ and $1.464 \cdots$ (Fejes Tóth, op. cit. pp. 171, 174). On the other hand, greater packing-densities and smaller covering-densities occur in the hyperbolic plane (Fejes Tóth, op. cit. p. 157) and in hyperbolic space. In fact, one can find a packing of horospheres (spheres of infinite radius) and a covering by horospheres, of respective densities $(1+2^{-2}-4^{-2}-5^{-2}+7^{-2}+8^{-2}-\cdots)^{-1}=0.855\cdots$, $(1-2^{-2}+4^{-2}-5^{-2}+7^{-2}-8^{-2}+\cdots)^{-1}=1.282\cdots$. These values were obtained with the aid of the following lemma: The volume of any solid "sector" of a horosphere (in natural measure) is equal to one-half the area of its horospherical boundary. (Received March 2, 1954.)

441. C. C. Hsiung: *Some integral formulas for closed hypersurfaces.*

Let V^n be a hypersurface with a closed boundary V^{n-1} twice differentiably imbedded in a Euclidean space E^{n+1} of $n+1$ $(n\geqq 2)$ dimensions, and let M_α be the αth mean curvature of V^n at a point P defined by the elementary symmetric function $M_\alpha=\sum\kappa_1\kappa_2\cdots\kappa_\alpha$ $(\alpha=1,\cdots,n)$, where $\kappa_1,\kappa_2,\cdots,\kappa_n$ are the n principal curvatures and M_n the Gaussian curvature K of V^n at P. The purpose of this paper is to derive for V^n two integral formulas from which follow immediately for a closed orientable V^n the following two formulas, which were obtained by H. Minkowski for a closed convex hypersurface V^n with $n=2$: $nA+\int_{V^n}M_1p\,dA=0$, $n\int_{V^n}Kp\,dA+\int_{V^n}M_{n-1}dA=0$, where dA is the area element of V^n at P and p an oriented distance from a fixed point 0 to the tangent hyperplane of V^n at P. From the above two formulas some simple conditions for a closed orientable hypersurface V^n to be a hypersphere are deduced. (Received March 8, 1954.)

442. C. C. Hsiung and C. B. Sensenig (p): *On convex surfaces.*

Let S_1 (S_2) be a convex surface of class $\geqq 3$ with a plane boundary C_1 (C_2) in an ordinary Euclidean space. If S_1 and S_2 satisfy a certain additional boundary condition, a one-to-one correspondence ϕ between the points of S_1 and S_2 is defined under which any two corresponding points P_1, P_2 of S_1, S_2 have the same parameters u and v. Let B_1, B_2 be the plane areas bounded by C_1, C_2 respectively, and let $K_1(u,v)$, $K_2(u,v)$ and dA_1, dA_2 be respectively the Gaussian curvatures and the area elements of S_1, S_2 at P_1, P_2. It is shown that if S_1, S_2 are in a correspondence ϕ and $dA_1=dA_2$, $K_1\geqq K_2$

for all corresponding points of S_1, S_2, then $B_1 \leq B_2$. Moreover, $B_1 = B_2$ implies that S_1 and S_2 are congruent if S_1 and S_2 are closed and analytic or are surfaces of revolution with axes of revolution perpendicular to the planes of C_1 and C_2 respectively. The first part of these results is an analogue of a theorem of A. Schur on two curves. (Received March 8, 1954.)

443. Valdemars Punga: *Theorem on the construction of affine connection in Cartan space of line elements.*

Cartan space of line elements (E. Cartan, *Les espaces de Finsler* II) is a set of objects in (1-1) correspondence with $2n$ ordered real numbers $(x^1, x^2, \cdots, x^n; p^1, \cdots, p^n)$ or simply (x^α, p^α) or (x, p), where line elements (x, p) and (x, cp), $c>0$, are considered equal. p^α transforms as contravariant vector. The covariant differential (δ) of a vector $V^\alpha(x, p)$ is given by $\delta V^\alpha = (\nabla^1_\gamma V^\alpha)dx^\gamma + (\nabla^2_\gamma V^\alpha)\delta p^\gamma = dv^\alpha + \Gamma^\alpha_{\beta\gamma} v^\gamma dx^\beta + C^\alpha_{\beta\gamma} v^\gamma \delta p^\beta$, where $\Gamma^\gamma_{\beta\gamma}(x, p)$ is homogeneous of degree 0 in p and tensor $C^\alpha_{\beta\gamma}(x, p)$ is homogeneous of degree (-1) in p. *Theorem*: Any five tensors $S^\alpha_{\beta\gamma}(x, p)$, $\bar{S}^\alpha_{\beta\gamma}(x, p)$, $Q_{\gamma\alpha\beta}(x, p)$, $\bar{Q}_{\gamma\alpha\beta}(x, p)$ and $f_{\alpha\beta}(x, p)$, where (a) all five tensors are homogeneous functions of degree 0 in p, (b) $S^\alpha_{\beta\gamma}$ and $\bar{S}^\alpha_{\beta\gamma}$ are skew-symmetric in β and γ indices, (c) $Q_{\gamma\alpha\beta}$ and $\bar{Q}_{\gamma\alpha\beta}$ are symmetric in α and β indices, (d) $f_{\alpha\beta} = \partial^2 f/\partial p^\alpha \partial p^\beta$ with determinant $|f_{\alpha\beta}| \neq 0$, determine uniquely an affine connection $(\Gamma^\alpha_{\beta\gamma}, C^\alpha_{\beta\gamma})$ which is such that $\nabla^1_\gamma f_{\alpha\beta} = Q_{\gamma\alpha\beta}$, $\nabla^2_\gamma = f_{\alpha\beta} = \bar{Q}_{\gamma\alpha\beta}$, $(\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta})/2 = S^\alpha_{\beta\gamma}$, $(C^\alpha_{\beta\gamma} - C^\alpha_{\gamma\beta})/2 = \bar{S}^\alpha_{\beta\gamma}$. The author proved the theorem by expressing $\Gamma^\alpha_{\beta\gamma}$ and $C^\alpha_{\beta\gamma}$ in terms of given tensors. (Received February 24, 1954.)

LOGIC AND FOUNDATIONS

444t. K. J. J. Hintikka: *A reduction in the simple theory of types.*

To each closed formula K of the simple theory of (finite) types one can associate, by an effective rule, a formula $\phi(K)$ of the first-order functional calculus in such a way that K is valid (in the sense of a standard truth-definition) if and only if the formula $(D \supset \phi(K))$ is valid, and K is logically false if and only if $(D \supset \sim\phi(K))$ is valid. Here, D is a constant formula of the second-order functional calculus containing only one quantified monadic predicate (functional) variable (in addition to a few free predicate variables and to a number of bound individual variables). If one allows for nonstandard models, then D may be reduced further so as to contain no bound predicate variables. (Received March 19, 1954.)

445t. J. R. Shoenfield: *Models of formal systems.*

Let P be a system of axioms for set theory, and let P' be the system formed from P by adding an additional axiom A. Suppose that the statement "P in consistent" is provable in P'. By a theorem of Gödel, if P is consistent, A is not provable in P. Moreover, if P is ω-consistent, then the statement "If P is consistent, then P' is consistent" is not provable in P. This shows that A is independent of P, and that the consistency of A with P is not provable by methods available in P. The above may be applied to give new proofs of certain independence results, and shows at the same time why corresponding consistency results are unavailable. For example, P may be taken as Zermelo-Fraenkel set theory, and A may be taken as the axiom asserting the existence of a strongly inaccessible cardinal. The main step, in each case, consists in developing the theory of models in P' so that "P is consistent" may be proved in P'. (Received March 12, 1954.)

STATISTICS AND PROBABILITY

446. M. D. Kruskal: *The expected number of components under a random mapping function.*

Let f be a mapping of a finite set S of N elements into itself. A subset T is ***invariant*** if $f(T)\subset T$ and $f^{-1}(T)\subset T$. A ***component*** is a minimal non-null invariant subset. (The components form a partitioning of S.) Metropolis and Ulam (Amer. Math. Monthly vol. 60 (1953) p. 252) have raised the problem of finding the expected number E of components if f is a uniformly distributed random function (i.e. if f has probability N^{-N} of being any specific mapping function). The author proves that $E=\sum_{n=1}^{N} N!/(N-n)!nN^{n}$ and that $E-(\log 2N+C)/2\to 0$ as $N\to\infty$, where C is Euler's constant. (Received March 8, 1954.)

TOPOLOGY

447t. A. F. Batholomay: *The Serre group $E_2^{p,q}$.*

The terms E_r of the "spectral sequence" [*Homologie singulière des espaces fibrés*, J.-P. Serre, Ann. of Math. (1951)] associated with the filtration $\cdots\subset A^p\subset A^{p+1}\subset\cdots$ of a differential group (A, d) are such that $E_{r+1}=H(E_r)$, where H denotes the homology group with respect to the operators (d_r) defined by d. If group A is graded, then each E_r is bi-graded; i.e., $E_r=\sum_{p,q}E_r^{p,q}$. The important term $E_2^{p,q}$ is expressible as $H(H_{p+q}(A^p/A^{p-1}))$. Another form for $E_2^{p,q}$ is known to be $E_2^{p,q}=\text{image } f$, where f: $H_{p+q}(A^p/A^{p-2})\to H_{p+q}(A^{p+1}/A^{p-1})$. An explicit form is obtained for such a homomorphism, at the same time that a proof by construction of the equivalence of the two forms is accomplished. First, the mapping $f_2=(\bar{i}\circ j')$ is defined: $A^p/A^{p-2}\xrightarrow{j'}A^p/A^{p-1}\xrightarrow{\bar{i}}A^{p+1}/A^{p-1}$, where the induced homomorphisms $\bar{i}_*$, j'_* are identified with the homology sequences of (A^{p+1}, A^p, A^{p-1}) and (A^p, A^{p-1}, A^{p-2}), respectively. After identifying d_1^p and d_1^{p+1} with corresponding homomorphisms in the homology sequences, the equivalence of the two forms then follows by observing that $E^{p,q}=(\text{kernel } d_1^p/\text{image } d_1^{p+1})=(\text{image } j'_*/\text{kernel } \bar{i}^*)$. (Received March 3, 1954.)

448t. Morton Brown: *N homogeneity implies $N-1$ homogeneity.*

A topological space S is N homogeneous if for any N distinct points $p_1, p_2, \cdots, p_N$ and any N distinct points $q_1, q_2, \cdots, q_N$ there exists a homeomorphism of S onto itself taking $p_1, p_2, \cdots, p_N$ onto some permutation of $q_1, q_2, \cdots, q_N$. It is a natural question to ask whether N homogeneity implies $N-1$ homogeneity. The author proves the more general theorem that if T is a group of 1-1 transformations of an infinite set onto itself such that for any two subsets X, Y, each containing N distinct points, there exists a $t\in T$ such that $t(X)=Y$, then for any two sets X', Y', each containing $N-1$ distinct points, there exists a $t'\in T$ such that $t'(X')=Y'$. (Received March 10, 1954.)

449. Eldon Dyer: *Continuous collections of decomposable continua on a spherical surface.*

Let S denote a spherical surface and G denote a continuous collection of mutually exclusive decomposable continua on S which is a continuum with respect to its elements. Then G with respect to its elements is a continuum such that each of its nondegenerate subcontinua contains uncountably many of its locally separating points. (O. G. Harrold, Jr. presented a list of characterizations of such continua in *Lectures in* *topology*, University of Michigan Conference of 1940, p. 248). If G fills up S, then with respect to its elements it is a dendron. For each dendron D there is a continuous collection of mutually exclusive decomposable continua filling up S which is with respect to its elements topologically equivalent to D. If G fills up S and none of its elements is the boundary of three of its complementary domains, G is an arc with respect to its elements. R. D. Anderson has obtained results of a similar nature for continuous collections of nondegenerate continuous curves in the plane (Proc. Amer. Math. Soc. vol. 3 (1952) pp. 647–657). (Received March 10, 1954.)

450. E. R. Fadell: *A property of compact absolute neighborhood retracts.*

A continuous mapping $f: X\rightarrow X$ is said to be a ϕ-mapping if for every $x\in X$, there exists an $x'\in X$ different from x such that $f(x')\neq x'$ and x' does not separate x and $f(x')$. For example, every fixed point free (fpf) mapping $f: X\rightarrow X$, where X is a continuum (metric), is a ϕ-mapping and *any* mapping $g: X\rightarrow X$, $g\neq 1$, where X is a cyclic (no cut points) continuum, is also a ϕ-mapping. The main result of this paper is the following theorem. Let $f: X\rightarrow X$ denote a ϕ-mapping, where X is a compact ANR (metric, not necessarily connected). Then X admits of a mapping $g: X\rightarrow X$ such that $g(X)=X$ and $g\sim f$ with homotopy H such that the set of fixed points under H_t, $0\leq t\leq 1$, is the set of fixed points of f. In particular, therefore, if X is a compact, connected ANR and f is fpf, then f is continuously deformable into a fpf mapping g which is onto. Also, if $f: X\rightarrow X$ is *any* mapping from a cyclic, connected, compact ANR, then f is continuously deformable into a mapping g which is onto and possesses the same set of fixed points as f. (Received February 26, 1954.)

451*t*. Gertrude I. Heller: *Interrelations between directed sets and filters.*

Moore and Smith in their general theory of limits (both authors, Amer. J. Math. (1952); the latter, National Mathematics Magazine (1938)) used the notion of directed sets. Frechet based his limit theory on the notion of filter, later elaborated by Bourbaki (*Topologie générale*, Paris, 1940). Since Marlow (Bull. Amer. Math. Soc. vol. 58) used directed sets to define upper and lower limits, the two theories became so parallel that the question of their interrelationships imposed itself. The paper compares the two theories and the structures on which they are based. The main result is: While any directed set obviously generates a filter, namely the collection of residuals (and therefore any theory of limits based on filters will necessarily contain the corresponding theory based on directed sets), the converse is not true. A necessary and sufficient condition for a filter F on a set E to be obtainable as the collection of residuals of some directing ordering of E is that F have a basis B with the property that the intersection of any collection of sets of B is either empty or belongs to F. An example of a filter which does not fulfill this condition was furnished by W. L. Chow. (Received March 10, 1954.)

452*t*. L. F. McAuley: *A relation between perfect separability, completeness, and normality in semi-metric spaces.*

Let S denote a regular semi-metric topological space. C. W. Vickery has stated (Bull. Amer. Math. Soc. vol. 46 (1940) p. 433) that S is a Moore space. In this paper, it is proved that S is not necessarily a Moore space. Indeed, S may have many properties such as complete separability (very subset of S is separable), complete normality, and weak completeness but S fails to be either a Moore space or a metric space. While it is shown that a complete Moore space is not necessarily a strongly complete space S, it is proved that if S is strongly complete and completely separable, then S is a perfectly separable complete Moore space. Other theorems are proved which include affirmative answers to questions raised by L. W. Cohen (Duke Math. J. vol. 5 (1939) p. 183) and W. A. Wilson (Amer. J. Math. vol. 53 (1931) p. 366). (Received March 8, 1954.)

453. Mary E. Rudin: *A connected set such that the complement of every connected subset is countable.*

Paul Erdős (*Some remarks on connected sets*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 442–446), raised the question of the existence of a nondegenerate connected set such that the complement of every nondegenerate connected subset is countable. This paper shows that, if the hypothesis of the continuum is true, there exists such a set in the plane. (Received March 9, 1954.)

454*t*. Hans Samelson: *Groups and spaces of loops.*

A construction is given which shows that every topological group with a universal bundle (contractible principal bundle) is homotopically equivalent to the space of loops of the corresponding classifying space (base space of the bundle); the equivalence includes the operations of multiplication and inversion. From this and from known properties of homotopy groups of spheres it is shown that the commutator map of the group of quaternions of norm 1 ($(x, y) \rightarrow xyx^{-1}y^{-1}$) is not null-homotopic, answering a question of Eilenberg's. (Another proof was found independently by G. W. Whitehead.) Another application: The generator s of the nth homology group of the Eilenberg-MacLane space $K(Z, n)$ satisfies, for odd n, $s * s = 0$, where $*$ is the Pontryagin product. This work was sponsored by the National Science Foundation. (Received January 21, 1954.)

L. W. COHEN,
Associate Secretary

THE APRIL MEETING IN CHICAGO

The five hundred second meeting of the American Mathematical Society included a Symposium on Applied Mathematics (sponsored by the Society and the Office of Ordnance Research) and was held at the University of Chicago on Thursday, Friday and Saturday, April 29, 30 and May 1.

There were a total of 421 registrations, including 308 members of the Society, representing almost twice the attendance at the 1953 meeting. (By order of the Council the detailed list of the names of members attending this and future meetings is deleted from the report.)

The Symposium was divided into three sessions which met respectively at 9:00 A.M. and 2:00 P.M. on Thursday, and at 9:00 A.M. on Friday. Major General L. E. Simon, Department of the Army, served as chairman of the first meeting at which the following papers were presented: *Operations research* by Professor P. M. Morse of Massachusetts Institute of Technology, *Problem of inductive inference*, by Professor Jerzy Neyman of the University of California, *Recent developments in analysis of variance*, by Professor H. O. Hartley of the University of London and Iowa State College. Dr. F. E. Grubbs of the Ballistic Research Laboratories served as the discussion leader.

The second session was presided over by Dr. T. J. Killian of the Office of Ordnance Research. He introduced Professor M. R. Hestenes of the University of California, Los Angeles, Mr. John Todd of the National Bureau of Standards, Professor A. A. Bennett of Brown University, and Professor J. E. Mayer of the University of Chicago, who spoke respectively on *Computational methods*, *Motivations for working in numerical analysis*, *Some numerical computations in ordnance problems*, and *Unsolved problems in statistical mechanics*. Mr. M. W. Leutert of the Ballistic Research Laboratories led the discussion.

The topics of the final session were *The simplest rate theory of pure elasticity*, *On the stability of mechanical systems*, *Problems associated with hyperbolic partial differential equations*, and *On the nature of differential operators and boundary value problems*, presented respectively by Professor C. A. Truesdell of Indiana University, Professor J. J. Stoker of New York University, Professor Florent Bureau of the University of Liege and the University of Chicago, and Professor William Feller of Princeton University. Mr. W. W. Leutert of the Ballistic Research Laboratories presided, and Professor Paul Germain of the Office National de l'Etude et des Recherches Aéronautiques and Brown University acted as the discussion leader.

By invitation of the Committee to Select Hour Speakers for Western Sectional Meetings, Professor S. C. Kleene of the University of Wisconsin addressed the Society on the topic *Hierarchies of number-theoretic predicates* at 2:00 P.M. on Friday. Presiding officer at this session was Professor Saunders MacLane.

Sessions for the presentation of contributed papers were held at 3:15 P.M. on Friday and at 9:00 A.M., 10:30 A.M. and 2:00 P.M. on Saturday. Presiding officers at these sessions were Professors Zeev Nehari, J. J. Gergen, R. M. Thrall, G. de B. Robinson, Michael Golomb, L. M. Graves, Dr. Serge Lang, and Mr. John Todd.

The society is indebted to the ladies of the Department of Mathematics, who entertained the assembled mathematicians and their guests at tea on Thursday and Friday afternoons.

The Council met on Friday evening, April 30, 1954.

The Secretary announced the election of the following thirty-six persons to ordinary membership in the Society:

Mr. Louis Richard Bragg, University of Wisconsin;
Professor Martin Dudley Burrow, McGill University;
Mr. Angelo Anthony Caparaso, New York University;
Mr. Joseph Biggi Chiccarelli, Fordham University;
Mr. Bernard H. Chovitz, Army Map Service, Washington, D. C.;
Mr. John Francis Daly, St. Louis University;
Mother Charlotte Anne Dames, Barat College;
Mr. Jesus Gil de Lamadrid, University of Michigan;
Dr. Eduardo Hernán del Busto, Universidad Nacional de La Plata, Eva Perón, Argentina;
Mr. George Welva Fairchild, Bendix Computer Division, Los Angeles, California;
Mr. Herman Paul Friedman, Reeves Instrument Corporation, New York, New York;
Mr. James Scott Hanna, Jr., Southwestern Engineering and Equipment Company, Dallas, Texas;
Dr. Heinz G. Helfenstein, University of Alberta;
Mr. Manfred Kochen, Columbia University;
Mr. Arthur John Leino, University of California, Berkeley;
Mrs. Helen White Lindley, Tulane University;
Dr. Viktors Linis, University of Saskatchewan;
Mrs. Ruth Johnson MacKichan, University of North Dakota;
Mr. Henrik Herman Martens, Consolidated Edison Company of New York, Inc.;
Mr. Robert M. Meisel, New York University;
Professor Earle Frederick Myers, University of Pittsburgh;
Sister Miriam Patrick, St. Mary's College, Notre Dame, Indiana;
Dr. Aubrey Hampton Payne, Aberdeen Proving Ground, Aberdeen, Maryland;
Mr. George Owen Peters, Aircraft Armaments Incorporated, Baltimore, Maryland;
Mr. Samuel I. Plotnick, Mathematics Research Incorporated, State College, Pennsylvania;

Mr. Donald Clayton Rose, University of Kentucky;
Dr. Joseph William Siry, Naval Research Laboratory, Washington, D. C.;
Mr. Elmo J. Stewart, Bendix Computer, Bendix Aviation Corp., Los Angeles, California;
Professor Tsuneo Suguri, Kyusyu University, Fukuoka, Japan;
Professor Robert F. Tidd, Canisius College;
Professor Yasuro Tomonaga, Utsunomiya University, Utsunomiya, Japan;
Mr. Henry Snowden Valk, George Washington University;
Dr. Guido Leopold Weiss, University of Chicago;
Mr. John Sheldon Youtcheff, General Electric Advanced Electronic Center, Ithaca, New York;
Mr. Wilson Miles Zaring, University of Kentucky;
Mr. Jack Ira Zektzer, University of Wisconsin;

It was reported that the following forty-seven persons had been elected to membership on nomination of institutional members as indicated:

University of Florida: Mr. Thomas Roscoe Horton.

Harvard University: Mr. Marshall Leonard Freimer, Dr. Dieter Gaier, Professor Albert Haertlein, Mr. James Brown Herreshoff, Mr. Frank Albert Raymond, Mr. Eugene R. Rodemich, Mr. Richard Steven Varga, and Mr. Tai Tsun Wu.

State University of Iowa: Mr. Frank Wylie Anderson.

Kenyon College: Mr. David Ryeburn.

University of Maryland: Mr. James Hill, Mr. Robert Hanson Moore, and Mr. George Norman Trytten.

University of Michigan: Mr. William Sherwood Bicknell.

University of Missouri: Mr. Charles David Gorman.

Oklahoma Agricultural and Mechanical College: Miss Margaret Ann Reiff, Mr. Tetsundo Sekiguchi, and Mr. Albert William Wortham.

Oregon State College: Miss Elvy Lennea Fredrickson.

Princeton University: Mr. Harvey James Arnold, Mr. Julian Brody, Mr. Bradley Dean Bucher, Mr. I. Thomas Cundiff, Jr., Mr. Eric Yngve Domar, Mr. Nathaniel Roy Goodman, Mr. David Kent Harrison, Mr. Robert Hermann, Mr. Joseph John Kohn, Mr. Henry Pratt McKean, Jr., Mr. Pinchas Mendelson, Mr. Peter Gerald Moore, Mr. Lionel MacLean Noel, Mr. Paul Emery Thomas, Mr. Hale Freeman Trotter, Mr. Alan John Weir, and Mr. John William Woll, Jr.

University of Texas: Mr. Theodore Parker Higgins, Mr. William Andrew Holley, Mr. Charles Albert Nicol, Mr. G. P. Owen, Jr., Mr. Leon Bruce Treybig, and Mr. Don Harrell Tucker.

Tulane University: Mr. James H. Case and Mr. Erwin Stuart Krule.

Vanderbilt University: Mr. Howard Leroy Rolf.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: Deutsche Mathematiker Vereinigung: Mr. Erik Sparre Andersen, Institute of Copenhagen, Dr. Alexander Peyerimhoff, University of Cincinnati, and Professor Wolfgang B. Jurkat, University of Cincinnati; Finnish Mathematical Society: Dr. Olli Erkki Lehto, University of Helsinki; London Mathematical

Society: Dr. James Edmund Gibbs, Carleton College, Ottawa 1, Ontario, Canada, and Mr. Kenneth Povey, University of Liverpool; Société Mathématique de France: Professor Jacques Deny, University of Strasbourg, and Dr. Mark Wilmet, 1 Place Street, Thomas d'Aquin, Paris 7, France; Svenska Matematikersamfundet: Mr. Hans Ivar Riesel, Matematiska Institutet, Stockholm, Sweden, and Mr. Lars Erik Zachrisson, Research Institute of National Defense, Stockholm 5, Sweden.

Goucher College, Baltimore, Maryland, was elected to institutional membership.

The following appointments by the President were reported: as a Committee to Nominate Officers and Members of the Council for 1955: C. B. Morrey, Chairman, R. H. Bing, Richard Brauer, A. W. Tucker, and A. D. Wallace; as a Committee to Nominate an Executive Director: E. G. Begle, J. R. Kline, A. E. Meder, Jr., and G. T. Whyburn; as a Committee to Study the Programs of Society Meetings: W. M. Whyburn, Chairman, E. G. Begle, R. P. Dilworth, P. R. Halmos, and J. L. Kelley; as a Committee to Recommend Policy on Reprinting of Books: W. T. Martin, Chairman, Einar Hille, and R. J. Walker; as a Committee on Subsidies to Journals: Professor A. E. Meder, Jr., Chairman, E. G. Begle, Saunders MacLane, and A. W. Tucker; as a Committee on the Columbia University Bicentennial: E. G. Begle, L. W. Cohen, B. P. Gill, and H. M. MacNeille; as a Committee on Arrangements for the Meeting to be held at the University of Alabama, November 26–27, 1954: Professors F. A. Lewis, Chairman, J. H. Hornback, F. W. Kokomoor, J. H. Roberts, C. L. Seebeck, Jr., and H. S. Thurston; as a Committee on Arrangements for the Annual Meeting to be held at the University of Pittsburgh, December 27–29, 1954: J. C. Knipp, Chairman, I. Barsotti, J. L. Blumberg, A. M. Bryson, L. W. Cohen, H. M. Gehman, George Laush, Norman Levine, E. F. Myers, J. S. Taylor, and Jean Teats; as a Program Committee for the Summer Institute of 1954: Salomon Bochner, Chairman, D. C. Spencer, Alternate Chairman, Lipman Bers, N. S. Hawley, and Oscar Zariski.

The Secretary reported that the following persons have accepted invitations to deliver hour addresses during 1954: F. I. Mautner, New York City, April 23–24, 1954; Harry Pollard, New York City, April 23–24, 1954; S. C. Kleene, Chicago, Illinois, April 30–May 1, 1954; V. L. Klee, Portland, Oregon, June 19, 1954; Richard Bellman, Laramie, Wyoming, Summer Meeting 1954; Edwin Hewitt, Laramie, Wyoming, Summer Meeting 1954; R. D. James, Laramie, Wyoming, Summer Meeting 1954; Ralph Phillips, Laramie, Wyoming, Summer

Meeting 1954; Vaclav Hlavatý, Iowa City, Iowa, November 26–27, 1954; and Edmund Pinney, Los Angeles, November 27, 1954.

The following items were reported for the information of the Council: selection of W. T. Martin as Managing Editor of the Bulletin Editorial Committee for 1954; A. C. Schaeffer as Managing Editor of the Proceedings Editorial Committee for 1954; J. L. Doob as Managing Editor of the Transactions and Memoirs Editorial Committee for 1954; R. J. Walker as Chairman of the Mathematical Surveys Editorial Committee for 1954; Einar Hille as Chairman of the Mathematical Reviews Editorial Committee for 1954; Deane Montgomery as Chairman of the Colloquium Editorial Committee for 1954; C. J. Rees as Chairman of the Committee on Printing and Publishing for 1954.

It was reported to the Council that the Trustees had approved a recommendation of the Council that reduced dues be authorized for members of the Society serving as enlisted men in the armed forces of the United States or Canada. Dues have been fixed at one dollar per year for such individuals. Members taking advantage of these reduced dues will continue to receive all the privileges of membership in the Society, including the Bulletin, Proceedings, and Notices. In order to obtain this privilege of reduced dues, a member must be in good standing at the time he requests the privilege and, in particular, must have discharged any past dues and other financial obligations to the Society. This privilege is normally available for a period of two years, but extensions may be made at the discretion of the Secretary.

It was reported that Dr. Paul Erdős has accepted an invitation to be the Society's Visiting Lecturer during the academic year 1955–1956.

It was reported that the Society for Industrial and Applied Mathematics will meet in conjunction with the Mathematical Association of America, the Association for Symbolic Logic, and the Society at the time of the Annual Meeting of 1954 in Pittsburgh.

The following actions taken by mail vote of the Council were reported: The election of Professors J. L. Doob and G. B. Price to serve as members of the Executive Committee of the Council for a period of two years beginning January 1, 1954; and the election of Professor Leo Zippin to the Policy Committee for Mathematics for a period of four years beginning January 1, 1954.

The Council set meetings at the University of California, Los Angeles, November 27, 1954, and at Brooklyn Polytechnic Institute on April 15–16, 1955.

The Council voted to co-sponsor with the Office of Ordnance Research a Symposium in Applied Mathematics in conjunction with the April 1955 meeting in New York. The topic of this Symposium will be ***Mathematical probability and its applications.***

The Council voted to approve the following substitutions on the Editorial Committee for the Transactions and Memoirs: Leo Zippin for Herbert Busemann for the academic year 1954–1955; M. M. Day for J. L. Doob for the period from June through November 1954; O. F. G. Schilling for Saunders MacLane for the summer of 1954.

The Council approved amendments to the by-laws to create a new class of members, namely corporate members.

The Council voted that lists of names of members attending Society meetings be omitted from the reports of meetings published in the Bulletin.

Abstracts of the papers presented follow below. Abstracts whose numbers are followed by the letter "*t*" were presented by title. In the case of joint papers, (p) following one of the authors' names indicates the one who actually read the paper. Mr. Marsh was introduced by Professor B. W. Jones and Dr. Livesay by Professor A. J. Lohwater.

ALGEBRA AND THEORY OF NUMBERS

455. J. L. Brenner: *A bound for a determinant with dominant principal diagonal.*

Let $A=(a_{ij})$ be an $n\times n$ matrix in which the relations $|a_{kk}|>\sum'|a_{kj}|$ hold $(k=1, 2, \cdots, n)$. The following bound is obtained by an elementary inductive proof: $|\det A|\geq|a_{11}|\prod_{k=2}^{n}(|a_{kk}|-m_k+L_k)$. Here, m_k is $\sum_{l<k}|a_{kl}|$ and L_k is $R_k+|a_{k1}/a_{11}|\cdot\sum_{j>k}|a_{1j}|$, where R_k is a certain sum of absolute values of minor determinants of A; a corollary bound is obtained by replacing R_k by 0. The bound of Price (Proc. Amer. Math. Soc. vol. 2, p. 492) is the corollary bound obtained by replacing all of L_k by 0. (Received December 17, 1953.)

456*t*. J. L. Brenner: *Linear recurrence relations.*

Corresponding to the arbitrary linear recurrence relation (over rational integers) $u_{n+1}=a+\sum_0^{k-1}a_t u_{n-t}$, homogeneous or not, a matrix B is defined such that B^{n+1} has $(0, 1, u_{n-1}, \cdots, u_{n+k-2})$ for a first row. A theorem is deduced concerning the order of the recurrence modulo a prime. (Received January 18, 1954.)

457*t*. J. L. Brenner: *The factorization of orthogonal matrices.*

A matrix A is orthogonal if the product AA^T is the identity. If the field of coefficients has more than two elements, such a matrix is a product of plane rotations T diag $(R, I)T^{-1}$, where T is a permutaton matrix and R has dimension 2, $RR^T=I$. (Received January 18, 1954.)

458. Sarvadaman Chowla and W. E. Briggs (p): *On discriminants of binary quadratic forms with a single class in each genus.*

Consider the classes of positive, primitive binary quadratic forms $ax^2+bxy+cy^2$ of discriminant $d=b^2-4ac<0$ and let $\Delta=-d$. There are 101 known values of Δ such that $d=-\Delta$ is a discriminant which has a single class in each genus. The largest of these values is 7392 and Swift (Bull. Amer. Math. Soc. vol. 54 (1948) pp. 560–561) has shown that there are no more up to 10^7. It is also known that there is only a finite number of discriminants of this type. Using the L-series $L_k(s)=\sum_1^\infty \chi(n)n^{-s}$, where χ is a real nonprincipal character modulo k and $R(s)>0$, and some facts about the number of genera into which the classes are divided, it is proved that: (1) There is at most one fundamental discriminant with a single class in each genus with $\Delta>10^{[illegible]}$. (2) If $L_k(53/54)\geqq 0$, $k>10^{14}$, then there are no such discriminants with $\Delta>10^{14}$. The first result is the counterpart of the result of Heilbronn and Linfoot (Quart. J. Math. vol. 5 (1934) pp. 293–301) which says that there is at most one fundamental discriminant with class-number one for $\Delta>5\ 10^9$. (Received February 22, 1954.)

459*t*. W. F. Darsow: *On certain division algebras.*

Let G_n be the group under symmetric difference of all subsets of the set $(1, \cdots, n)$ of the first n positive integers; and let A_n be the linear space of all functions on G_n to the field R of reals. For a function w on the cartesian product $G_n\times G_n$ to R a weighted convolution $f*g$ is defined for all f, g in A_n by $(f*g)(X)=\sum_Y f(Y)g(Y+X)w(Y, Y+X)$ with respect to which A_n is a linear algebra (not necessarily associative) over R of order 2^n. Two proper choices of w yield the algebra of Clifford numbers and Grassmann's algebra for each n. For $n=1, 2, 3$, appropriate choices of w yield the complex numbers, the quaternions, and the Cayley numbers; and a variety of other division algebras exist. However, as is not surprising, for $n>3$ no choice of w yields a division algebra A_n over any ordered field R. (Received February 17, 1954.)

460*t*. W. E. Deskins: *A radical for near-rings.*

In a recent paper (*Simple and semisimple near-rings*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 772–785) D. W. Blackett developed a structure theory for semisimple near-rings, analogous to the structure theory for semisimple rings (with DCC). Let N be a near-ring and let the right modules of N satisfy the DCC. If R is a minimal right module of N, define $r(R)$ to be the (two-sided) ideal of N consisting of all the elements of N which annihilate R from the right. Then K, the intersection of all $r(R)$, is an ideal of N such that $N-K$ is semisimple. If N is a ring, then K coincides with the usual nilpotent radical. (Received January 25, 1954.)

461. W. E. Deskins: *On the homomorphisms of an algebra onto a Frobenius algebra.*

A linear associative algebra A which possesses a nonsingular parastrophic matrix is called a Frobenius algebra. Such algebras have been studied by a number of mathematicians, notably R. Brauer, C. C. MacDuffee, T. Nakayama, and C. Nesbitt. Simple examples may be given to show that in general a singular parastrophic matrix of rank m does not determine a homomorphism of A onto a Frobenius algebra of order m. However, the following result is proved in this paper. Theorem. If Q is a parastrophic matrix of rank m of the algebra A of order n, if Q is congruent to a corner matrix, and if Q is associated with an element of a subalgebra of A which has either a left or right identity element, then Q determines a homomorphism of A onto a Frobenius algebra of order m. Furthermore, every homomorphism of A onto a Frobenius algebra is determined by such a parastrophic matrix. (Received January 25, 1954.)

462. Leonard Gillman (p) and Melvin Henriksen: *On rings of continuous functions in which every finitely generated ideal is principal.*

Let X be a completely regular Hausdorff space, $C=C(X)$ the ring of all real-valued continuous functions on X. X is (i) P, (ii) P', (iii) P'', if for every $p\in X$ and $f\in C, f(p)=0$ implies that there is a deleted neighborhood U of p such that (i) $f(U)=0$ (see Bull. Amer. Math. Soc. Abstract 59-4-445), (ii) $f(U)=0$ *or* $f(U)>0$ *or* $f(U)<0$, (iii) $f(U)\geqq 0$ *or* $f(U)\leqq 0$ (respectively). Clearly, P implies P' implies P''. Every extremally disconnected (ED) space (i.e., the closure of every open set is open) is P''. *Results.* (1) If (a) for all $f\in C$, f is a multiple of $|f|$, *or* (b) for all $f\in C$, the ideal $(f, |f|)$ is principal, *or* (c) every finitely generated ideal of C is principal, then X is P''. (2) If X is P' *or* ED *or* normal P'', then (a), (b), *and* (c) hold. (3) If X is P' *or* ED, then (d) for every $f\in C$, f is a *unit* multiple of $|f|$, *and* (e) for every $p\in C$, the ideal of all $f\in C$ that vanish on a neighborhood U_f of p is prime; if (e), then X is P''. (4) Examples are given of spaces that are P' but not P, ED but not P', P' but not ED. (Received March 10, 1954.)

463. Leonard Gillman and Melvin Henriksen (p): *Concerning adequate rings and elementary divisor rings.*

A commutative ring S with unit is *adequate* if (A_1) every finitely generated ideal is principal, *and* (A_2) for all a, b, with $a\neq 0$, there exist r, s, with $a=rs$, $(r, b)=1$, and $(s', b)\neq 1$ for every non-unit divisor s' of s (Helmer, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 225–236; Kaplansky, Trans. Amer. Math. Soc. vol. 66 (1949) pp. 464–491). The problem has been open as to whether there exist S satisfying A_1 but not A_2. *Results* (see preceding abstract). (1) $C(X)$ satisfies A_2 *iff* $C(X)$ is adequate *iff* X is P (hence *iff* $C(X)$ is regular—see Bull. Amer. Math. Soc. Abstract 59-4-446). (2) If X is P' *or* ED *or* normal P'', but not P, then $C(X)$ satisfies A_1 (but not A_2). S is an *elementary divisor* ring if for every (finite) matrix M, there exist unimodular matrices K, L such that KML is diagonal. Generalizing a result of Helmer, Kaplansky has shown (loc. cit. Theorem 5.3) that if S is adequate and its zero-divisors are in the radical, then S is an elementary divisor ring. Clearly, for any X, the zero-divisors of $C(X)$ are not in the radical. (3) If X is P' *or* ED, but not P, then $C(X)$ is an elementary divisor ring (satisfying A_1 but not A_2). (Received March 10, 1954.)

464. H. E. Goheen: *The Wedderburn theorem.*

Using the lemma that any invariant division subring of a skew field must be in its center (Hua, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) p. 533) and the lemma that any finite group must be Abelian if the normalizers of all its Abelian subgroups are their centralisers (Zassenhaus, Proc. Glasgow Math. Assoc. vol. 1 (1952) p. 53), the author provides a simplified version of Zassenhaus' proof of the theorem of Wedderburn on the non-existence of finite skew fields. (Received March 8, 1954.)

465. Franklin Haimo: *Power-type endomorphisms of some class 2 groups.*

An endomorphism ϕ of a group G is said to be of power-type if $\phi(x)\equiv x^{n(\phi)}$ mod Q for every $x\in G$, where Q is the derivative of G. If G is non-abelian of class 2, and if G/Q is an elementary p-group, then N, the set of endomorphisms of G into Q, is a prime ideal in the ring R of all power-type endomorphisms of G for which $p \mid n(\phi)$. If p is odd, then such a group G has precisely p endomorphisms of the form $\phi(x)=x^n\in Z$, where Z is the center of G. (Received March 10, 1954.)

466. I. N. Herstein: *On the Lie ring of a simple ring.*

If A is any associative ring one can transform into a Lie ring by defining a new product, the Lie product, by means of $[a, b] = ab - ba$ for all $a, b \in A$. It has been conjectured by Kaplansky that if A is a simple associative ring of characteristic not 2, then the only Lie ideals of $[A, A]$ are in the center of A. This conjecture is proved in this paper as long as A is also of characteristic different from 3. (Received March 13, 1954.)

467t. W. G. Leavitt: *Finite-dimensional modules.*

This paper considers the problem of dimensionality for a finitely based module M over a ring K with unit. Such a module is said to be *finite-dimensional* if, for some integer n, (a) every basis has length n, and (b) M contains no independent set of length more than n. Examples are known of finitely based modules satisfying both conditions, condition (a) alone, or neither condition. The main results proved are: (1) a finitely based module containing no infinite independent set is finite-dimensional; (2) if a module contains two bases of different lengths, then no upper bound exists for the length of a basis; (3) a sufficient condition that every finitely based module over K be finite-dimensional is that there exist an integer m such that every set $\{a_1 \cdots a_m\} \subset K$ is related by $\sum_1^m a_i x_i = 0$, with either (i) at least one x_i is not a left zero divisor or (ii) at least one $a_i x_i \neq 0$; (4) If K is imbeddable in a ring admitting the descending chain condition, then any finitely based module over K satisfies condition (a). (Received March 1, 1954.)

468. D. C. B. Marsh: *An investigation of the number of classes in the genus of certain indefinite ternary quadratic forms.* Preliminary report.

The greatest common divisors of the two-rowed minors of a symmetric matrix and its reciprocal matrix are invariant under unimodular transformations; necessary and sufficient conditions that the removal of a factor from such an invariant leave the class number of the matrix's form unaltered are found in terms of automorphs of the matrix. Application of a simplified expression for these automorphs (Duke Math. J., forthcoming) and a study of sufficient conditions that integral automorphs exist with elements of specified congruential nature enabled one to replace the form by another of reduced invariants for consideration in determining the class number. Employing an early theorem by Adolf Meyer (J. Reine Angew. Math. vol. 108 (1891) p. 139) and restricting consideration to forms where neither invariant is divisible by four, one developed an algorithm whereby the invariants might be further reduced without altering the class number. Finally, group theoretic considerations showed precisely how the class number was altered by a final transition to a form in a genus of one class. (Received February 23, 1954.)

469t. H. T. Muhly: *A remark on a paper of P. Samuel.*

It is pointed out that in the case of the ideals of an integral domain, the notion of asymptotic equivalence introduced by P. Samuel (Ann. of Math. vol. 56 (1952) pp. 11–21) coincides with the notion of valuation equivalence (b-equivalence) introduced by W. Krull (Math. Zeit. vol. 41 (1936) pp. 545–577). (Received February 26, 1954.)

470*t*. R. J. Nunke: *A relation between the functors* Ext *and* Tor. Preliminary report.

Definitions of the functors Ext and Tor may be found in the *Foundations of algebraic topology*, Princeton University Press, 1952, by S. Eilenberg and N. Steenrod. For abelian groups A, B, C a natural isomorphism is given between Ext (Tor(A, B), C) and Ext (A, Ext (B, C)). The following theorems are consequences: (1) For any prime p, Ext (A, C) is divisible by p if and only if either A has no elements of order p or C is divisible by p; (2) if all elements of A are of finite order, then Ext (A, C) has no nontrivial divisible subgroups; (3) Ext (A, C) is the direct sum of Ext ($T(A)$, C) and a homomorphic image of Ext ($A/T(A)$, C), where $T(A)$ is the torsion subgroup of A; (4) Ext (A, C) is never free. (Received March 10, 1954.)

471*t*. Alex Rosenberg: *Finite-dimensional simple subalgebras of the ring of all continuous linear transformations.*

Let M, N be dual spaces over a division ring D with center Z. Let $A = L(M, N)$, the ring of all continuous linear transformations, and let $S = F(M, N)$ be its socle. Let B be a simple finite-dimensional subalgebra of A over Z containing the unit of A, and let C' denote the A-centralizer of a subring C of A. Then B' is again a ring of the form $L(R, T)$, R, T dual spaces over some division ring, and the socle, U, of $B' \subset S$ and $MU = M$, $U^*N = N$. In case $D = Z$, an algebraically closed field, it is further shown that $M = \sum_1^* M_i$, $N = \sum_1^* N_i$, $(M_i, N_j) = 0$, M_i, N_i dual spaces over Z and $B' \cong L(M_i, N_i)$. In this case $B'' = B$ and one gets the usual one-to-one correspondence between finite-dimensional simple subalgebras of A containing the unit and those subrings of A which are of the form $L(R, T)$ with socle $U \subset S$ and $MU = M$, $U^*N = N$. Furthermore if σ is an isomorphism of B into A, σ can be extended to an inner automorphism of A if and only if $B' \cong (B^\sigma)'$. These results generalize those obtained in the case $\dim M = \dim N \aleph_0$, Bull. Amer. Math. Soc. Abstract 59-5-532. (Received March 11, 1954.)

472*t*. A. R. Schweitzer: *Grassmann's extensive algebra and modern number theory.*

In this paper the quadratic form is interpreted as a link connecting Grassmann's algebra with Minkowski's *Geometrie der Zahlen* (Leipzig and Berlin, 1910). In Grassmann's theory a quadratic form is obtained as the inner square of a linear vector. Minkowski's theory is viewed primarily as an arithmetic of quadratic forms generalizing certain researches of Hermite. Minkowski succeeds in arriving at interesting results in number theory including algebraic numbers (loc. cit., p. 123) and references to Gauss (p. 166). In the course of his investigation Minkowski uses concepts pertaining to geometry (pp. v, vi), analysis (pp. 1, 37), and point set theory (pp. iv, 5, 18, 36, 201). (Received March 8, 1954.)

473. Leonard Tornheim: *Approximation to irrationals by classes of rational numbers.*

Let $(r, s, m) = 1$. Then for every irrational number x there exist infinitely many rational numbers a/b in lowest terms with $a \equiv rt$, $b \equiv st$ (mod m) such that $|x - a/b| < km/b^2$ if and only if when $m = p^e$ (p an odd prime) then $k \geqq 1/5^{1/2}$, or when $m = 2^e$ ($e \geqq 1$) then $k \geqq 1/2$, or when $m = p^eq^f$, where p, q are distinct primes and $e, f \geqq 1$, then

$k \geqq 1$. These results for the case $m=2$ have been proved by W. T. Scott, R. M. Robinson, A. Oppenheim, and L. Kuipers and B. Meulenbeld. (Received March 9, 1954.)

474. J. H. Walter: *Automorphisms of the projective unitary groups.*

The projective unitary group $PU_n(K, f)$ can be defined as the factor group $U_n(K, f)/C_n$, where $U_n(K, f)$ is the unitary group acting on a right vector space E of dimension n over a sfield K which possesses an involutive anti-automorphism J, C_n is its center, and f is a nondegenerate hermitian sesquilinear form defined on E with respect to J. If $n \geqq 7$ and $n \neq 8$ and K is not of characteristic 2 and is not the finite field $GF(3)$, it is shown that every automorphism of $PU_n(K, f)$ is induced by an automorphism of $U_n(K, f)$. The crucial step is to characterize the extremal involutions of $PU_n(K, f)$, which is done by using the commuting sets of involutions introduced by Rickart [Amer. J. Math. vol. 72 (1950) pp. 451–464 and vol. 73 (1951) pp. 697–716]. This work was sponsored in part by the OOR, Contract DA-20-018-ORD-13281. (Received March 10, 1954.)

475. L. M. Weiner: *J-simple algebras.*

An algebra A is called J-simple if the attached algebra $A^{(+)}$ is a simple Jordan algebra. This paper considers power associative algebras A which are both J-simple and Lie admissible. Albert has shown (Summa Braziliensis Mathematicae vol. 5 (1951) pp. 183–194) that when $A^{(+)}$ is a central simple Jordan algebra of matrices over an algebraically closed field, there exists a linear transformation T such that multiplication x, y in A is given in terms of the matric multiplication by $x, y = (xy+yx)/2+(xy-yx)T$. The present paper shows that when $A^{(+)}$ is the algebra of all matrices of order $t \geqq 3$ with multiplication given by $(x, y) = (xy+yx)/2$, and T is nonsingular, there exists a scalar α and a linear function $\lambda(B)$ such that $BT = \alpha B + \lambda(B)I$ for each matrix B of trace zero. If there exists a scalar extension K of the base field such that $A_K^{(+)}$ is the set of all t by t Hermitian matrices with elements in a central simple alternative algebra C of order $r=1$, 4, or 8, and $t \geqq 3$ for $r=1$, and $t \geqq 2$ for $r=4$ or 8, then T must be singular. (Received March 3, 1954.)

476. L. R. Wilcox (p) and R. J. Mihalek: *A generalisation of independence in semi-modular lattices.*

Let L be a lattice in which modularity is symmetric, i.e., if $(b, c)M$ means $a \leqq c$ implies $(a \cup b) \cap c = a \cup (b \cap c)$, then $(b, c)M$ implies $(c, b)M$. If $a, b, c \in L$, define a relation θ so that $(a, b, c)\theta$ means $(a \cup b) \cap (b \cup c) = b$, $(a \cup b, b \cup c)M$, together with the two other conditions obtained by cyclic permutation of a, b, c. This definition may be extended in an obvious way to n terms. If $(a, b)M$, then $(a, b, c)\theta$ reduces to $c \cap (a \cup b) = a \cap b$, $(c, a \cup b)M$, whence $(a, b, c)\perp$ in the sublattice of all $x \geqq a \cap b$ (see L. R. Wilcox, *Modularity in the theory of lattices*, Ann. of Math. vol. 40, pp. 491–496, where the properties of $\perp$ are developed). Thus θ is a generalization of ordinary independence $\perp$. In the present paper, formal properties of θ are obtained, most of which are identical to or modifications of properties of $\perp$. The present generalization of the theory of independence has been developed by the authors as a tool in the imbedding of certain Birkhoff lattices into modular lattices. (Received March 11, 1954.)

477. Ti Yen: *Trace on AW^*-algebras.*

An AW^*-algebra is [Kaplansky, Ann. of Math. vol. 53 (1951) pp. 235–249] a C^*-algebra satisfying the following two conditions: (a) In the set of projections any collection of orthogonal projections has a least upper bound. (b) Any maximal commutative self-adjoint subalgebra is generated by its projections. Let A be an AW^*-algebra of type II_1, Z its center. Denote by (P) and (P″) respectively the following conditions: (P): There exists a total set of completely additive positive functionals on A. (P″): There exists a completely additive positive linear transformation from A to Z which is identity on Z. (1) A has a trace if it satisfies (P) or (P″). (2) Any finite AW^*-algebra A satisfying (P) has a representation as an AW^*-subalgebra of some W^*-algebra. Its weak closure is also finite. A is already weakly closed if it is commutative. The underlying Hilbert space is one induced by the functionals in (P) and the trace. As a corollary of (2) and a theorem of Kaplansky [Ann. of Math. vol. 56 (1952) p. 470], an AW^*-subalgebra of type I of a W^*-algebra is weakly closed. (Received March 8, 1954.)

478*t*. Daniel Zelinsky: *Every linear transformation is a sum of nonsingular ones.*

Let α be a linear transformation on a vector space over a division ring. Then $\alpha=\beta+\gamma$ where β and γ are nonsingular linear transformations (automorphisms) of the same space—except when the vector space contains only two elements and $\alpha\neq 0$. This theorem is well known if the vector space is finite-dimensional; one possible proof constructs β as any isomorphism of the kernel of α onto a complement of the range of α, coupled with a suitable isomorphism of a complement of the kernel onto the range. The present proof provides the modifications necessitated in the infinite-dimensional case by the fact that a complement of the range need no longer be isomorphic to the kernel. (Received February 26, 1954.)

479. J. L. Zemmer: *Some remarks on p-rings and their Boolean geometry.*

The idempotents in a commutative ring R form a Boolean ring with multiplication that of R and addition defined by $x\oplus y=x+y-2xy$. The following results are obtained: (i) if B is a Boolean ring with identity, p a fixed prime, then there exists a unique p-ring R with identity, with B as its Boolean ring of idempotents; (ii) the automorphism group of a p-ring with identity is isomorphic to the automorphism group of its Boolean ring of idempotents. The remaining results in this paper extend some of the theorems of Ellis (*Autometrized Boolean algebras*, I, Canadian Journal of Mathematics vol. 3 (1951) pp. 87–93; and *Autometrized Boolean algebras*, II, ibid. pp. 145–147). Denote by $\mathfrak{B}$ the Boolean algebra associated with the Boolean ring of idempotents of the p-ring R with identity. It is shown that R is a "distance" space with respect to the function $d(x, y)=(x-y)^{p-1}$, defined on R to $\mathfrak{B}$. The group of isometries of this "distance" space is determined, and finally it is shown that the space has the property of free mobility if and only if $\mathfrak{B}$ is a complete Boolean algebra. (Received March 10, 1954.)

ANALYSIS

480*t*. H. A. Antosiewicz: *On nonlinear differential equations of the second order with integrable forcing term.*

Differential equations of the form $\ddot{x}+\phi(x, \dot{x})\dot{x}+h(x)=e(t)$ are considered where $\int^{\infty}|e(t)|dt<\infty$ and $\phi(x, \dot{x})$, $h(x)$ are sufficiently smooth so as to guarantee the existence of a unique solution for all $t\geqq 0$. It is proved that if $\phi(x, \dot{x})\geqq 0$ for all x, $\dot{x}$, $H(x)=\int_0^x h(u)du>0$ for $x\neq 0$, $H(x)\rightarrow\infty$ with $|x|$, then every solution satisfies $|x(t)|<C_1$, $|\dot{x}(t)|<C_2$ as $t\rightarrow\infty$. If $\phi(x, \dot{x})=f(x)+g(x)\dot{x}$, the same boundedness result holds pro-

vided $a(x) = \exp \left(\int_0^x g(u)du\right) \leq \alpha$ $(\alpha \geq 1)$ for all x, $h(x)\int_0^x a(u)f(u)du \geq 0$ and $H^*(x) = \int_0^x a^2(u)h(u)du > 0$ for $x \neq 0$, $H^*(x) \to \infty$ with $|x|$. These results are a considerable generalization of a result by G. Sestini (Atti 4° Congresso Un. Mat. It., Messina, 1951). (Received March 4, 1954.)

481*t*. W. B. Caton: *On the zeros of certain Laplace integrals*. I.

Let $F(t)$ defined on $(0, \infty)$ be non-negative, non-increasing, and positive in a set of positive measure. Let $F(t) \in L(0, R)$, for every $R > 0$. The Laplace transform $f = \mathcal{L}\{F\}$ exists and has no zeros in $R(s) > 0$. If $f_i = \mathcal{L}\{F_i\}$, $i = 1, 2, 3, \cdots$, where each F_i has the above properties and $c_i \geq 0$ are real numbers, then $\sum_{i=1}^{n} c_i f_i$ has no zeros in $R(S) > 0$. Finally, $\sum_{i=1}^{\infty} c_i f_i$ will have no zeros in $R(s) > 0$ if the sequence $\{c_i\}$ of non-negative real numbers is chosen so that the series is uniformly convergent in any bounded closed domain in $R(s) > 0$. Let F_i $(i = 1, 2)$ have the properties mentioned above and assume that $F_i \in L^2(0, R)$ for every $R > 0$. Then $\mathcal{L}\{F_1F_2\}$ exists and has no zeros in $R(s) > 0$. In particular, $\mathcal{L}\{F_1F_2\} = f_1 * f_2$ in case F_i $(i = 1, 2)$ are also entire functions of order one and minimal type. (Received March 5, 1954.)

482. Lamberto Cesari: *Orbital stability*.

Given a system of ordinary differential equations $x_i' = f_i(x, t)$, $x = (x_1, \cdots, x_n)$, $i = 1, 2, \cdots, n$, and a periodic solution (cycle) $C: x = x(t)$, $-\infty < t < +\infty$, in E_n, the question of the orbital stability of C can be discussed by using a function $V(x)$ of Liapounoff's type defined in a neighborhood U of C (tube). Then V could be for instance a definite positive quadratic form in a system of orthogonal local coordinates $y_1, y_2, \cdots, y_n$ in the hyperspace π normal to C at the point $x = x(t)$. For systems $x_i = f_i(x, t, \epsilon)$, weakly nonlinear (autonomous, or not) families $[C_\epsilon]$ of cycles C_ϵ can be determined, by using known methods, around some periodic solutions C_0 of the linear system $x_i = f_i(x, t, 0)$, $i = 1, 2, \cdots, n$. Then functions V can be determined by means of various processes. Examples are discussed. (Received March 11, 1954.)

483*t*. W. L. Duren, Jr.: *Real measurement operators*.

In a set X which is linearly ordered, dense, and Dedekind complete, a second ordering, called a comparability relation, is introduced locally. For every point x in X it is assumed that there exists a neighborhood U_x of x and a comparability relation which is a total ordering of the pairs (y, z) in $U_x \times U_x$ and is related to the linear order of points x in a natural way. A complete measurement of a set $Y \subset X$ is a 1-1 mapping of Y onto itself which preserves comparability. It is proved that every complete measurement of an open interval in X consists of an element of a field of multiplication operators followed by an element in an abelian group of translations. Moreover the field of multiplication operators with its induced linear order is isomorphic to the real numbers. The comparability is a local property so that X does not necessarily possess the algebra of the reals. It is a linearly ordered real manifold. Applications in a theory of scientific measurement and particularly in econometrics are indicated. (Received March 11, 1954.)

484. W. F. Eberlein: *Theory of numerical integration*. I. Preliminary report.

Let $E(s) = (s+1)^{-1} \sum_1^n A_m e^{-\alpha_m s}$. It is shown that from the standpoint of operator theory the "best" asymmetric quadrature formulae for fixed n are obtained by choosing the $2n$ constants $(A_m)(\alpha_m)$ to minimize E in a suitable norm. For example, the "Laguerre procedure" for the interval (0, 1) amounts to setting $E^{(k)}(0)=0$ ($k=0, 1, \cdots, 2n-1$) and yields the "best one-sided" formula. (Received April 16, 1954.)

485. W. H. Fleming: *An example concerning the problem of least area with unrestricted topological type.*

The existence of a surface of least area with a given simple closed curve C in euclidean 3-space as boundary, and prescribed character of orientability and Euler characteristic, is well known (see R. Courant's book, *Dirichlet's principle*, and references cited there). The problem for unrestricted rather than fixed topological types is unsolved, however. An example is given of a simple closed rectifiable curve C in 3-space for which the solution of the problem with unrestricted types must be of infinite topological type. This paper was sponsored by the Office of Ordnance Research. (Received March 12, 1954.)

486. R. E. Fullerton: *An inequality for linear operators between L^p spaces.*

Let R, S be two sets with completely additive measures ϕ, γ defined over σ-rings $\mathfrak{R}(R)$, $\mathfrak{R}(S)$ respectively and assume that R and S are σ-finite. Let $L^p(R, \phi)$, $L^q(S, \gamma)$ be the L^p and L^q spaces defined over (R, ϕ), (S, γ), $1 \leqq p, q \leqq \infty$. A bounded linear operator T defined from L^p to L^q is known to be representable in the form $Tx = (d/de)\int_R K(e, t)x(t)d\phi$ where $K(e, t)$ is defined on $\mathfrak{R}(S)\times R$ and satisfies certain known conditions and where the symbol d/de $(F(e)=f(t))$ denotes the integrable function associated with the completely additive, absolutely continuous set function $F(e)$ by the Radon-Nikodym theorem. It is shown that the kernel $K(e, t)$ must satisfy the inequality $[q\text{-var}_S(\int_R |K(e, t)|^{p'}d\phi)]^{1/p'}/\sup_{\{e_j\}}(\sum_{j=1}^{n} \gamma(e_j)^{(p'-1)(q-1)})^{1/qp'} \leqq \|T\|$, $(1/p)+(1/p')=1$, where $\{e_j\}$ varies over all finite disjoint families in $\mathfrak{R}(S)$ of finite nonzero measure. In particular, if $p \leqq q$, $\gamma(S) \leqq 1$, $[q\text{-var}_S\ (\int_R |K(e, t)|^{p'}d\phi)]^{1/p'} \leqq \|T\|$.(Received March 10, 1954.)

487. R. A. Gambill and J. K. Hale (p): *Subharmonic and ultraharmonic solutions for weakly nonlinear systems.*

Consider the system of differential equations (1) $\dot{y}_j = \rho_j y_j + \epsilon q_j(y_1, \cdots, y_n, t; \epsilon)$, $j=1, 2, \cdots, n$, where $\rho_1, \cdots, \rho_n$ are complex numbers, ϵ is a small parameter, and each q_j is a holomorphic function of $y_1, \cdots, y_n$, ϵ for $|y_j| < A$, $|\epsilon| < \epsilon_0$, with coefficients periodic in t of period $2\pi/\omega$. This abstract refers to research toward the determination of subharmonics and ultraharmonics of system (1) by using a variant of the Poincaré method of successive approximations analogous to the variants considered for linear systems (1) [L. Cesari, Mem. Accad. Italia (6) vol. 11, pp. 634–692] and for autonomous nonlinear systems [J. K. Hale, Bull. Amer. Math. Soc. Abstract 60-1-114]. By using a convergent process which at the same time furnishes approximate solutions, the following conditions, among others, have been obtained. I. The numbers $\rho_j = ik_j\omega/m_j + O(\epsilon)$ can be determined so that there is a solution of (1) of the form (2) $y_j = y_{j0} + \epsilon w_j(t; \epsilon)$, $j=1, \cdots, n$, where $y_{j0} = a_j e^{ik_j\omega t/m_j}$, k_j, m_j positive integers, a_j arbitrary complex constants, $|a_j| < A$, and $w_j(t; \epsilon)$ holomorphic in ϵ with coefficients of period $T = 2\pi m_1 \cdots m_n/\omega$. II. If $\rho_j = k_j\omega/m_j$ and if $S_j(a_1, \cdots, a_n) = \int_0^T q_j(y_{10}, \cdots, y_{n0}, t; 0)e^{-\rho_j t}dt$, then there is a periodic solution of (1) of the form (2) if $S_j(a_1^0, \cdots, a_n^0)=0$, and the Jacobian $|\partial S_j(a_1^0, \cdots, a_n^0)/\partial a_k| \neq 0$. This condition on $S_j(a_1, \cdots, a_n)$ has been obtained [E. A. Coddington, N. Levinson, Annals of

Mathematics Studies, no. 29, pp. 19–35]. However, cases where the Jacobian is zero are discussed when system (1) arises from a real system of second order equations. (Received March 11, 1954.)

488t. R. P. Gosselin: *On the convergence behaviour of trigonometric interpolating polynomials.*

It is known that there exist continuous functions for which the sequence of trigonometric interpolating polynomials taken at various sets of fundamental points diverges almost everywhere or even everywhere (cf. e.g., J. Marcinkiewicz, *Sur la divergence des polynomes d'interpolation*, Acta Univ. Szeged. vol. 8 (1936–37) pp. 131–135). Fundamental points of the form $\alpha+2\pi i/(2n+1)$, $i=0, \pm 1, \pm 2, \cdots$, are considered. It is shown by refinements of the Marcinkiewicz example that for any number α_0, irrational with respect to π, there is a continuous function for which the sequence of interpolating polynomials corresponding to $\alpha=0$ diverges for $x\neq 0 \pmod{2\pi}$ while the sequence corresponding to $\alpha=\alpha_0$ converges uniformly. (Received March 8, 1954.)

489. L. M. Graves: *A generalization of the Riesz theory of completely continuous transformations.*

Let $T_c=E-cK$ be a family of linear transformations mapping a Banach space X into another such space Y, where E maps X onto the whole of Y (but *not* one-to-one), K is completely continuous, and c is a complex parameter. A *proper value* c is one for which $W_1=T_cX\neq Y$. The range W_1 of T_c is the orthogonal complement of the null space of the adjoint transformation T_c^*. As in the classical theory of Riesz, the set of proper values has no finite accumulation point. The spaces W_k defined inductively by $W_k=T_cE^{-1}W_{k-1}$ are closed linear subspaces of Y, and there exists a finite ν such that $W_{\nu+1}=W_\nu$. Each proper value c_0 has a positive integral order, defined in terms of the behavior of the adjoint transformation T_c^*, and this order is proved to equal the smallest ν for which $W_{\nu+1}=W_\nu$. A decomposition $K=K_1+K_2$ relative to a proper value c_0 is exhibited, where K_1 has only c_0 as a proper value, while K_2 has as proper values only the remaining proper values of K. Furthermore, $K_1=\sum K_{1j}$, where each K_{1j}^* has only one proper vector. (This work was done under a contract with the Office of Ordnance Research.) (Received March 9, 1954.)

490t. L. M. Graves: *Remarks on singular points of functional equations.*

Let $G(x, y)=0$ be an equation between two Banach space variables, having $(0, 0)$ as an initial solution at which the partial differential $L(dx)=d_xG(0, 0; dx)$ (mapping X into X) fails to have an inverse. There may then be several continuous solutions $x=x(y)$ defined near $y=0$ and vanishing with y. Bartle (Trans. Amer. Math. Soc. vol. 75 (1953) pp. 366–384) was able to count the exact number of real solutions in cases when the null space of L is one-dimensional, with the help of Newton's polygon method. By an extension of the notion of Newton's polygon to equations involving several variables it is possible to count the exact number of real solutions when the null space of L is two-dimensional. As an illustration, the method is applied to a nonlinear elliptic partial differential equation. The theory is applicable to cases when the null space of L has more than two dimensions, but the computations become unmanageable. (This work was done under a contract with the Office of Ordnance Research.) (Received March 9, 1954.)

491t. I. I. Hirschman, Jr.: *Weighted quadratic norms for Legendre polynomials.*

Let $\omega_n(x)=[(n+1)/2]^{1/2}P_n(x)$, $n=0, 1, \cdots$, be the normalized Legendre polynomials and let $f(x)=\sum_0^\infty a_n\omega_n(x)$. Set $N_\alpha[f]^2=\int_{-1}^{1}(1-x^2)^\alpha|f(x)|^2dx$. It is established that if $0<\alpha<1/2$, then $A'(\alpha)\leq N_\alpha[f]^{-2}\sum_{k>j}|a_j/(2j+1)^{1/2}-a_k/(2k+1)^{1/2}|^2\cdot p(j, k)(2j+1)(2k+1)/|j-k|^{1+2\alpha}|j+k|\leq A''(\alpha)$. Here $p(j, k)=1$ or 0 as j and k are of the same or different parity. Many properties of expansions in Legendre polynomials relative to the weighted quadratic norm $N_\alpha[f]$ can be deduced from this relation. (Received April 30, 1954.)

492. Meyer Jerison (p) and Gustave Rabson: *Induced homomorphisms of group algebras.*

Theorems announced earlier for compact zero-dimensional groups [Bull. Amer. Math. Soc. Abstract 59-4-467] are proved for a wider class of groups. Let G be a locally compact group which is the inverse limit of a sequence of groups $\{\Gamma_n\}$, and such that for each n, the kernel K_n of the homomorphism of G onto Γ_n is compact. If f is a locally integrable function on G, define $f_n(x)=\int_{K_n}f(xk)d\mu_n(k)$, where μ_n is the Haar measure in K_n, and $\mu_n(K_n)=1$. The sequence $\{f_n\}$ is a martingale, and consequently, $f_n(x)\rightarrow f(x)$ for almost every x, and if $f\in L^p$, $p\geq 1$, then $f_n\in L^p$ and $f_n\rightarrow f$ in L^p. Moreover, if f is continuous at x_0, then $f_n(x_0)\rightarrow f(x_0)$, and if f is uniformly continuous on $S\subset G$, then $f_n\rightarrow f$ uniformly on S. As in the special case considered previously, f_n can be interpreted as a partial sum of the Fourier series of f. If G is a countable product of circles, then application of the results announced here leads to generalizations of some theorems of Jessen [Acta Math. vol. 63 (1934) pp. 249–323]. (Received March 9, 1954.)

493. E. R. Johnston: *Two theorems concerning the function spaces $H(C, p)$ and $\mathcal{L}_D^2$.* Preliminary report.

Let $D=I(C)$ denote the interior of a closed rectifiable Jordan curve C. Let $z=k(w)$ map $|w|<1$ conformally one to one onto D, $\zeta=k(0)$, $\Gamma_r=\{z=k(re^{i\theta}), 0\leq\theta\leq 2\pi\}$, and $\Delta_r=I(\Gamma_r)$. In this paper the following two theorems are proved. (1) $\mathcal{L}_D^2\supset H(C, p)$ for $p\geq 1$. If $f(z)$ is analytic in D, then $[\iint_{\Delta_r}|f|^2dxdy]^{1/2}\leq(1/2\pi^{1/2})\int_{\Gamma_r}|f||dz|$ for $0<r\leq 1$ $(z=x+iy)$. (2) Let $\mathfrak{J}(\mathcal{L}_D^2)$ represent the class of all functions which are integrals of functions in $\mathcal{L}_D^2$. If $\int_0^{2\pi}|k(e^{i\theta})|^q d\theta<\infty$ for some $q>1$, then $H(C, p)\supset\mathfrak{J}(\mathcal{L}_D^2)$ for every $p>0$. If $F(z)=\int_\zeta^z f(t)dt$, $f(z)\in\mathcal{L}_D^2$, then $[\int_{\Gamma_r}|F|^p|dz|]^{1/p}\leq K(C, p)[\iint_{\Delta_r}|f|^2dxdy]^{1/2}$ for $0<r\leq 1$. In addition there exists $\omega>0$ (which depends on f) such that $\int_C\exp\omega|F||dz|<\infty$. (Received March 8, 1954.)

494. N. D. Kazarinoff and R. W. McKelvey (p): *Asymptotic solution of differential equations in a domain containing a regular singular point.*

This paper concerns the behavior of the solutions of the differential equation (A): $(z-z_0)^2d^2u/dz^2+\lambda(z-z_0)P_1(z, \lambda)du/dz+\lambda^2P_2(z, \lambda)u=0$, in a finite region R about z_0 in the complex z-plane and for large absolute values of the complex parameter λ. $P_1(z, \lambda)$ and $P_2(z, \lambda)$ are assumed to be regular in z throughout R and in λ at infinity. The theory developed is in some respects analogous to that known to apply to the equation (B): $d^2u/dz^2+\lambda P_1(z, \lambda)du/dz+\lambda^2P_2(z, \lambda)u=0$, under the assumption that $P_2(z, \infty)-P_1^2(z, \infty)/4$ be bounded from zero for z in R. With this hypothesis the theory for (A) develops along lines which are formally similar to those for (B) when considered under the same hypothesis. This analogy seems not to have been fully appreciated previously, although a paper by Cashwell [Pacific Journal of Mathematics vol. 1 (1951) pp. 337–352] in some degree anticipates the method used here. Cashwell dealt with an equation much less general than (A) and determined only one term of the asymptotic series for its solutions. Here, complete asymptotic series for solutions of (A) are found. Following the development of the general theory, an application is made to determine the structure of the Whittaker functions $M_{-m+k,m}(x)$ for $|m|$ large, x bounded, and $R(m)\geqq 0$. (Received March 4, 1954.)

495. Jacob Korevaar: *Numerical Tauberian theorems for Dirichlet and Lambert series.*

In three papers published in Neder. Akad. Wetensch. vols. 56, 57, the author obtained various numerical Tauberian theorems for power series. The transformation $\sum a_n n^{-w} = \{\Gamma(w)\}^{-1}\int_0^\infty(\sum a_n e^{-nx})x^{w-1}dx$ and its inverse are used in this paper to derive the following result on Dirichlet series. Let the series $\sum a_n n^{-w}$ $(w = u+iv)$ have a half-plane of convergence. Let $F(w)$, the analytic continuation of the sum of the series, be regular for $u \geqq -a$ $(a>0)$, and let $F(u+iv) = O(e^{A|v|})$ $(u\geqq -a)$. Then under the Tauberian condition (1) $a_n \geqq -B/n$ one has the best possible estimate $|s_n - F(0)| \leqq C/\log(n+1)$ $(n=1, 2, \cdots)$. If $F(w)$ is known to be regular and $O(e^{A|v|})$ only in a region (2) $u \geqq -a/\{\log(2+|v|)\}^b$ $(a>0,\ b>0)$, then (1) implies the slightly weaker estimate (3) $|s_n - F(0)| \leqq C\{\log\log(n+e)\}^b/\log(n+1)$. It is known that $\zeta(w+1)$ is free of zeros in a region (2) $(b>3/4)$. Thus the transformation $(\sum a_n n^{-w})\zeta(w+1)\Gamma(w+1) = \int_0^\infty\{\sum a_n nx/(e^{nx}-1)\}x^{w-1}dx$ enables one to prove the following numerical Tauberian theorem for Lambert series. Let the Lambert series $\sum a_n nx/(e^{nx}-1)$ be convergent for $x>0$, and let its sum $f(x)$ for a certain s and positive D, α satisfy the inequality $|f(x)-s| < Dx^\alpha$ $(x>0)$. If $\zeta(w+1)$ is free of zeros in the region (2), then under the Tauberian condition (1) one has the estimate (3). This result improves an earlier result of the author on Lambert series (Bull. Amer. Math. Soc. Abstract 59-6-627). (Received February 26, 1954.)

496. M. Z. Krzywoblocki: *On the generalization of Iglisch's existence theorem to parametric differential equation.*

Iglisch considered the existence for a third order ordinary differential equation of the boundary layer problem, subject to two point boundary conditions. Superposition of additional conditions upon the boundary value problem in question furnished the existence proof of the flow in the boundary layer. The author in the past generalized some Whyburn's existence theorems of ordinary differential systems to differential systems with a parameter, thus furnishing a much wider class of possible solutions of the boundary layer problem. On this base, as well as using Rellich's criteria for continuation of a solution of a differential system, the author generalized Iglisch's proof of existence to a parametric third order ordinary differential equation subject to two point boundary conditions. (Received February 22, 1954.)

497t. W. S. Loud: *A nonexceptional element of Wiener space.*

Let C be the space of continuous functions on $0\leqq t\leqq 1$ which vanish at 0. Cameron and Martin (Bull. Amer. Math. Soc. vol. 53 (1947) pp. 130–137) define $\int_0^1|dx(t)|^2$, which they show exists and equals 1/2 for almost all points of C (Wiener measure). The quantity $\int_0^1|dx(t)|^2$ is equal to zero for any element of C which is of bounded variation, and for many totally nondifferentiable functions of C as well. An explicit example is given of an element of C for which $\int_0^1 |dx(t)|^2 = 1/2$. The construction is a modification of the author's (Proc. Amer. Math. Soc. vol. 2 (1951) pp. 358–360) with $A = 1/2$, $\alpha = 1/2$. (Received March 10, 1953.)

498. E. B. McLeod, Jr.: *The solution of a free-boundary problem by conformal mapping.*

By considering either a special case of the physical problem of finding the two-dimensional, irrotational flow about a bubble having a significant surface tension, or an equivalent variational problem in the complex domain, we obtain a condition on the free boundary which states that the square of the absolute value of the derivative of the velocity potential is proportional to the curvature of the boundary. Examination of the conformal mapping properties of the function whose derivative is equal to the square of the derivative of the velocity potential leads to a first order differential equation in $Z(\zeta)$, which maps a point ζ in the exterior of the unit circle into a point Z in the exterior of the free boundary. This differential equation has the explicit solution $Z = \zeta - 2/3\zeta - 1/27\zeta^3$. (Received March 3, 1954.)

499. E. J. McShane: *A dominated convergence theorem.*

In a previous publication Daniell's method of defining an integral has been generalized so as to apply to a wide class of elementary mappings (integrals) of a subset E of a lattice F into a partially ordered set G. This elementary map is extended to a larger subset of F so as to attain useful convergence properties. The program of defining the extended mapping by use of order properties alone was incomplete in two respects; G was assumed to be a group, and in its partial ordering was assumed "normal," a concept involving real functions on G and stronger than the Hausdorff property. Here it is shown that the hypothesis that G is a group can be replaced by a weaker hypothesis involving order alone, and that normality of G can be replaced by a hypothesis weaker than assuming that G is a Hausdorff space in its lattice topology. With these hypotheses, the integral can be defined and the generalization of Lebesgue's dominated-convergence theorem established. (Received March 10, 1954.)

500. Immanuel Marx: *On the structure of recurrence relations.* II.

Consider differential recurrence relations $p(x)A_n(x)dy_n/dx + B_n(x)y_n = y_{n-1}$, $p(x)F_n(x)dy_{n-1}/dx + G_n(x)y_{n-1} = y_n$ between contiguous solutions of a differential equation $(d/dx)(pdy/dx) + q(x, \lambda)y = 0$, With $y_m^{(1)}$, $y_m^{(2)}$ denoting, for each m, a fundamental system of solutions corresponding to the characteristic value λ_m, a pair of relations is called *general* if it is valid both for $y_n = y_n^{(1)}$, $y_{n-1} = y_{n-1}^{(1)}$ and for $y_n = y_n^{(2)}$, $y_{n-1} = y_{n-1}^{(2)}$. A pair of relations is called *particular* if it is valid for a pair y_n, y_{n-1} of particular solutions, but not valid for any pair of solutions linearly independent of y_n, y_{n-1}. It is proved that the solutions of every differential equation of the type considered have a pair of general recurrence relations with uniquely determined coefficients; that every set of contiguous particular solutions y_n, y_{n-1} has a family of pairs of particular recurrence relations depending on two arbitrary functions of x; and that all relations are included under these two cases. Explicit expressions for the coefficients are given. Similarly obtainable are differential recurrence relations between solutions corresponding to any two values λ', λ'' of the parameter, or indeed (suggestion of G. Y. Rainich) first-order differential relations between the solutions of any two second-order linear differential equations. Relations among three contiguous solutions, etc., are trivially obtainable from the differential relations. (Received February 11, 1954.)

501*t*. E. P. Merkes: *Binary complex functions.*

A binary function $w=f(z)=u(x, y)+jv(x, y)$, defined on a domain D of the binary complex $z=x+jy$ plane where x, y are real variables and $j^2=aj+b$, is binary polygenic on D if the real functions u, v are of class C' on D. In this paper the derivative of one binary polygenic function with respect to a second is obtained from which the generalized binary Cauchy-Riemann equations are derived. It is noted that when the discriminant of the quadratic in j is non-negative, binary monogenity does not imply binary analyticity. In the non-monogenic case analogues of the three characteristic properties of Kasner for the derivative of one polygenic function with respect to another are secured. Also studied are general linear fractional transformations in the binary variables z and $\bar{z}$ of which special sub-groups are analogues of the Moebius and the Laguerre transformations. Moreover, the analogues of the Cauchy integral theorem, of Cauchy's integral formulas, of Morera's theorem, and of the Pompieu areolar derivative are derived. (Received March 11, 1954.)

502. E. P. Merkes: *On a generalization of polygenic functions.*

A complex function $f(z)$ of a complex variable z which is continuous on an open set G is said to be quasi-polygenic if there exists a finite constant $M\geqq 0$ such that $|J(f, I)| \leqq 2M|I|$, where I is any interval in G and where $J(f, I)$ is the line integral of f along the boundary of I. The constant M is called a radius of f on G. The class of quasi-polygenic functions with radius M on G includes the class of polygenic functions with radii of the derivative circles for z on G not exceeding M. Theorem: A continuous function $f(z)$ on an open set G is quasi-polygenic with radius M if and only if, for all z in G except at most for a denumerable number, lim sup $|J(f, Q)|/|Q| < \infty$ as diam $Q\rightarrow 0$, and for almost all z of G, lim inf $|J(f, Q)|/|Q| \leqq 2M$ as diam $Q\rightarrow 0$, where Q is any square in G containing z. An analogue of the Looman-Menchoff theorem and other theorems are obtained. (Received February 15, 1954.)

503*t*. Josephine Mitchell: *Potential theory in the geometry of matrices.* Preliminary report.

Let D be the domain consisting of the set of points for which the quadratic form $I-Z\bar{Z}'$ is positive definite, where Z is an n by n matrix of complex numbers, $\bar{Z}'$ its conjugate transpose, and I the identity matrix; let B be that part of the boundary for which $Z\bar{Z}'=I$. The Laplace equation corresponding to the invariant metric $[\sigma((I-Z\bar{Z}')^{-1}dZ(I-\bar{Z}'Z)^{-1}d\bar{Z}')]^{1/2}$ ($\sigma(A)$ being the trace of the matrix A) was calculated and a fundamental solution of the equation obtained. The Green and Neumann's functions with respect to the set $D\cup B$ were constructed and their properties found. The Poisson formula for the real part of an analytic function was obtained easily from the Cauchy formula for analytic functions which is already known [Bochner, Ann. of Math. vol. 45 (1944)]. The relations among the Poisson, Bergman, and Szegö kernel functions and the Green and Neumann's functions were found to be exact generalizations of the case $n=1$. Finally a boundary value problem of Dirichlet type was solved for the set $D\cup B$. (Received March 9, 1954.)

504*t*. S. B. Myers: *Algebras of differentiable functions.*

Let M, N be compact differentiable manifolds of class C^r $(1\leqq r<\infty)$, let $C^r(M)$

and $C^r(N)$ be the algebras of differentiable functions of class C^r on M and N respectively. The following theorems are proved. (1) If $C^r(M)$ is isomorphic to $C^r(N)$, there is a nonsingular differentiable homeomorphism of class C^r of M onto N. (2) If M and N are provided with Riemannian metrics of class C^{r-1}, and if $C^r(M)$ is made into a normed algebra (a Banach algebra if $r=1$) by $\|f\| = \max_{x\in M} |f(x)| + \max_{\theta \in \overline{M}} |\partial f/\partial\theta|$, where $\partial f/\partial\theta$ is the directional derivative of f in the unit direction θ and $\overline{M}$ is the compact unit tangent bundle to M, and if $C^r(N)$ is similarly normed, then if $C^r(M)$ is equivalent to $C^r(N)$ there is an isometry of class C^r of M onto N. (Received February 17, 1954.)

505t. Leopoldo Nachbin: *Closed algebras of analytic functions of one variable with an isolated singularity.*

Let P be the vector field of all convergent power series $f(z) = \sum_{-\infty}^{+\infty} a_n z^n$ regular or with a pole in $z=0$. For each $r>0$ and integer N, let P_{rN} be the vector subspace of P of all $\sum_N^{+\infty} a_n z^n$ converging for $\{z;\ 0<|z|<r\}$, with the topology of uniform convergence on compact subsets of this set. Endow P with the inductive limit topology as $N\to -\infty$ and $r\to 0$. Consider also on P the natural valuation given by $v(f)=N$ if $a_N\neq 0$. Theorem 1. Every subalgebra of P closed in the inductive limit topology is also closed in the valuation topology and vice versa. The closed subalgebras satisfy the ascending chain condition. Let next E be the algebra of all $f(z) = \sum_{-\infty}^{+\infty} a_n z^n$ converging around 0, essential singularities being allowed. For each $r>0$, let E_r be the subalgebra of all $f\in E$ converging in $\{z;\ 0<|z|<r\}$, with the topology of uniform convergence on compact subsets of this set. Endow E with the inductive limit topology as $r\to 0$. The residue functional $r(f) = (1/2\pi i)\int f(z)dz$, $f\in E$ (integral around 0) is a trace on E in the sense that, if to every $g\in E$ one associates the linear continuous functional $\tilde{g}$ on E given by $\tilde{g}(f) = r(fg)$, $f\in E$, the mapping $g\to\tilde{g}$ is a topological linear isomorphism of E onto its dual E^* endowed with the strong topology. Call $\tilde{g}$ truncated if g is regular or has a pole at 0. Theorem 2: Every subalgebra of E closed in the inductive limit topology is also closed in the weak topology defined by the truncated functionals and vice versa. (Received March 10, 1954.)

506t. Zeev Nehari: *On the coefficients of R-univalent functions.*

An analytic function $w=f(z)$ is said to be R-univalent in a region B if it maps B onto a domain embedded in a given closed Riemann surface R. If S_R denotes the class of functions $f(z) = z + a_1 z^{-1} + a_2 z^{-2} + \cdots$ which are regular and R-univalent in $1<|z|<\infty$, it is shown that the region of variability of the coefficient a_1 within the class S_R is contained in a disk of radius 1. This result is used to obtain estimates for a_1 in a number of special cases. (Received April 30, 1954.)

507. Howard Osborn: *On conservation laws.* Preliminary report.

A system of n quasi-linear homogeneous partial differential equations in two independent variables and n unknowns, the coefficients depending only on the unknowns, can sometimes be represented as a system of *conservation laws*, i.e., n or more two-vectors depending only on the unknowns are solenoidal in the space of the independent variables. For $n=1$, 2 one can always represent the given system by ∞^n conservation laws, i.e , n functions of one variable may be arbitrarily prescribed. For $n>2$ the determination of those systems with such a representation leads to an *overdetermined* system of $C_{n,2}$ second order linear homogeneous partial differential equations in n variables and one unknown. With certain restrictions it is shown that when a set of integrability conditions is satisfied, one can solve the latter system with Cauchy data given on n characteristic curves. This leads to a representation of the original system by ∞^n conservation laws. As an application one can display many systems which *cannot* be derived from a variational principle. (Received March 11, 1954.)

508*t*. L. E. Payne and H. F. Weinberger: *Bounds for harmonic and biharmonic functions*. II.

In a previous abstract (Bull. Amer. Math. Soc. Abstract 60-3-352), the authors gave bounds for a harmonic function at a point with respect to which the boundary B is star-shaped. The authors here give bounds at any point assuming only that B is star-shaped with respect to a single point 0. More precisely, if r is the distance from 0, it is required that $h=r\partial r/\partial n$ is positive on B. If the Green's function in $N\geqq 3$ dimensions is written as $G(P, Q)=-(\omega_N r_{PQ}^{N-2})^{-1}+g(P, Q)$, it is proved that (1) $(N-2)g(P, P)+2r(P)g_r(P, P)=\int_B h(\partial G/\partial n)^2 d\sigma$. From this together with the Green's function representation of the harmonic function u and Schwarz's inequality follows the inequality (2) $u^2(P)\leqq[(N-2)g(P, P)+2r(P)g_r(P, P)]\int_B h^{-1}u^2 d\sigma$. Putting $u(Q)=(N-2)g(P, Q)+2r(P)g_r(P, Q)$ results in the inequality (3) $(N-2)g(P, P)+2r(P)g_r(P, P)\leqq\int_B h^{-1}[(N-2)\omega_N^{-1}(r(Q)^2-r(P)^2)r_{PQ}^{-N}]^2 d\sigma_Q$ which, together with (2), gives bounds for u in terms of a mean square integral of its boundary values and the geometry of B. In two dimensions the analogue of (1) is (4) $(2\pi)^{-1}+2r(P)g_r(P, P)=\oint_B h(\partial G/\partial n)^2 ds$ and inequalities corresponding to (2) and (3) are easily obtained from this. A similar result also holds for biharmonic functions. Let the clamped plate Green's function in two dimensions be $\Gamma(P, Q)=(8\pi)^{-1}r_{PQ}^2 \ln r_{PQ}+\gamma(P, Q)$. Then the analogue of (1) is $\gamma(P, P)-r(P)\gamma_r(P, P)=2^{-1}\oint h(\Delta\Gamma)^2 ds$. (Received March 4, 1954.)

509. Pasquale Porcelli: *Concerning integrals*. IIa.

The author establishes relationships between the Lebesgue-Stieltjes integral $L\{f, \phi\}_a^b$, f and ϕ functions on $[a, b]$, and the mean integrals $\oint_a^b f d\phi$ studied recently by R. E. Lane and the author (Bull. Amer. Math. Soc. Abstracts 59-5-538 and 59-6-638). The author starts with $L\{f, \phi\}_a^b=\sum f(x_i)[\phi(x_i+)-\phi(x_i-)]$, where f is bounded on $[a, b]$ and ϕ is a nondecreasing step function on $[a, b]$, and then by use of a theorem reported earlier (Abstract 59-6-638) arrives at an integral $L\{h, g\}_a^b$ which exists when (i) one of h and g is of bounded variation and the other has discontinuities of the first kind only, and (ii) g is of bounded variation and h is bounded and g-measurable, in this case $L\{h, g\}_a^b$ is the Lebesgue-Stieltjes integral. We show in case (i) that $L\{h, g\}_a^b=h(b)g(b)-h(a)g(a)-L\{g, h\}_a^b$ if, and only if, $L\{h, g\}_a^b=\oint_a^b h dg$ where the latter integral is a mean integral. (Received March 5, 1954.)

510*t*. Pasquale Porcelli: *Concerning integrals*. IIb.

In considering the integration by parts formula for the Lebesgue-Stieltjes integral, the author uses the solution to the following operator problem: Let $Q(a, b)$ denote the collection of all real-valued functions on $[a, b]$ such that $f\in Q(a, b)$ if, and only if, f is the uniform limit of a sequence $\{u_n\}_{n=1}^{\infty}$ of normalized step functions on $[a, b]$ and u_n is continuous at each of a and b for $n=1, 2, \cdots$; let L be a linear, homogeneous, and bounded operator on $Q(a, b)$, then there exists one, and only one, function γ of bounded variation on $[a, b]$ such that $\gamma(a)=0$ and $L(f)=\oint_a^b f d\gamma$ for each f in $Q(a, b)$. Theorem: If ϕ is of bounded variation on $[a, b]$, then there exists one, and only one, function γ of bounded variation on $[a, b]$ such that $\phi(a)=\gamma(a)$ and $L\{f, \phi\}_a^b=\oint_a^b f d\gamma$

for each f in $Q(a, b)$; moreover, $\gamma(x) = (1/2)\{\phi(x+)+\phi(x-)\}$ if $a<x<b$. Corollary: If $f \in Q(a, b)$, ϕ a normalized function of bounded variation on $[a, b]$, then $L\{f, \phi\}_a^b = f(b)\phi(b) - f(a)\phi(a) - L\{\phi, f\}_a^b$. (Received March 5, 1954.)

511. W. T. Reid: *A note on the Hamburger and Stieltjes moment problems.*

For $\{\mu_n\}$ $(n=0, 1, \cdots)$, a sequence of real numbers satisfying with a monotone decreasing sequence of positive numbers $\{a_n\}$ the condition $(*)$ $\mu_0 \geqq a_0$, $\mu_{2n} \geqq a_n + (a_{n-1}-a_n)^{-1}\sum_{j=0}^{n-1} \mu_{n+j}^2$ $(n=1, 2, \cdots)$, it is shown that the Hamburger moment problem $\mu_n = \int_{-\infty}^{\infty} t^n d\alpha(t)$ $(n=0, 1, \cdots)$ has a nondecreasing solution $\alpha(t)$ with infinitely many points of increase; moreover, if $0<\lim a_n$ then the problem is indeterminate. If in addition to condition $(*)$ there is a monotone decreasing sequence of positive numbers $\{b_n\}$ such that $\mu_1 \geqq b_0$, $\mu_{2n+1} \geqq b_n + (b_{n-1}-b_n)^{-1}\sum_{j=0}^{n-1} \mu_{n+j+1}^2$ $(n=1, 2, \cdots)$, then the Stieltjes moment problem $\mu_n = \int_0^{\infty} t^n d\alpha(t)$ $(n=0, 1, \cdots)$ has a nondecreasing solution with infinitely many points of increase, and the problem is indeterminate if $0<\lim a_n$, $0<\lim b_n$. In particular, this latter result contains as a very special instance a result proved by R. P. Boas in 1939 (see D. V. Widder, *The Laplace transform*, pp. 140–142). (Received March 10, 1954.)

512. L. V. Robinson: *Solutions of some irreducible linear partial differential equations of the second order.*

It is shown that, when but two independent variables are involved, all linear partial differential equations of the second order reduce to $[(\lambda+q)(\Lambda+Q)+N]u = V$, where $\lambda = \partial/\partial x + p\partial/\partial y$, $\Lambda = \partial/\partial x + P\partial/\partial y$, the p, P, q, Q, N, and V being functions of x and y. It is then shown how some of these equations can be solved. (Received March 9, 1954.)

513. R. J. Silverman: *Semigroups with the Hahn-Banach extension property.*

An abstract semigroup $\overline{G}$ has the *Hahn-Banach extension property* (HBEP), if and only if for every collection (Y, G, X, p, f, V) satisfying the following properties: (a) Y is a linear space, (b) G is a representation of $\overline{G}$ on Y, (c) X is a subspace of Y which is invariant under G, (d) V is a partially ordered linear space with the property that every subset of V with an upper bound has a least upper bound, (e) p is a positive homogeneous subadditive function from Y to V such that $p(gy) \leqq p(y)$ for all y in Y and g in G, (f) f is a linear function from X to V such that $f(gx) = f(x)$ and $f(x) \leqq p(x)$ for all x in X and g in G, there exists a linear function F defined on all of Y to V such that $F(gy) = F(y)$, $F(y) \leqq p(y)$, $F(x) = f(x)$ for all g in G, y in Y and x in X. The following results are obtained: $\overline{G}$ has the HBEP if $\overline{G}$ satisfies one of the listed conditions. 1. $\overline{G}$ is a commutative semigroup. 2. $\overline{G}$ is a solvable semigroup. 3. $\overline{G}$ is a finite group. 4. $\overline{G}$ is a semigroup which contains a collection of subsemigroups directed by set inclusion and whose union is all of $\overline{G}$ and such that every member of the collection has the HBEP. 5. $\overline{G}$ is a homomorphic image of a group with the HBEP. 6. $\overline{G}$ is an extension of a group with the HBEP by a group with the HBEP. 7. $\overline{G}$ is a subgroup of a group with the HBEP. (Received February 25, 1954.)

514t. C. J. Titus: *Linear vector spaces of elliptic mappings.* Preliminary report.

Let F be the family of mappings into the plane which are continuously differentiable on a plane domain D for which the Jacobian of any mapping in F is non-negative and zero only if rank zero. All real linear vector spaces V of mappings in F are characterized. These spaces lead to a concept of elliptic spaces of 1st and 2nd kind so-called since the Bers' theory of pseudo-analytic functions of 1st and 2nd kind is characterized therein. Various other theories of pseudo-analytic functions as related to systems of partial differential equations are also characterized therein. It is noted that all continuously differentiable mappings of bounded eccentricity, thus all quasiconformal mappings, are in F. The relationship between the elliptic spaces, quasiconformal mappings, and 1st order elliptic systems of partial differential equations is discussed. (Received March 11, 1954.)

515. J. L. Ullman: *On Tchebycheff polynomials.*

Let C be a bounded, closed set of positive capacity, with complement D, $T_n(z)$ be the Tchebycheff polynomial of degree n associated with C, $u_n(E)$ the total multiplicity of zeros of $T_n(z)$ in the set E, $U(z)$ the generalized conductor potential of C, and $m(E)$ the set function corresponding to the equilibrium distribution of charge on C. Theorem I. If E is a closed set bounded from C, $u_n(E)=o(n)$. Theorem II. If K is a Jordan curve consisting of points of D, and $I(K)$ its interior, then $\lim u_n(I(K))/n = m(I(K))$. Theorem III. If D is bounded by a finite number of analytic Jordan curves, and E is a closed set bounded from C, then $u_n(E)<M$, a constant independent of n. Theorem IV. If E is a closed set bounded from C, then $\lim \int_E \left| \left| T_n(z) \right|^{1/n} - \exp (U(z)) \right| dA = 0$. Theorem V. If D is bounded by a finite number of analytic Jordan curves, then $\lim \left| T_n(z) \right|^{1/n} = \exp (U(z))$, except for a set of measure zero. (Received March 18, 1954.)

516. F. M. Wright: *Some results relative to indeterminate Stieltjes moment sequences.*

The author has stated in earlier papers that an S-fraction transformation theorem previously presented but as yet unpublished can be used to obtain rather easily and extend somewhat the results proved by H. S. Wall (Trans. Amer. Math. Soc. vol. 31 (1929) pp. 91–116) relative to the backward extension of a given indeterminate Stieltjes moment sequence; in this paper it is first shown exactly how this can be accomplished. The author then uses the S-fraction transformation theorem mentioned above to obtain a necessary condition for a determinate Stieltjes moment sequence $\{\lambda_n\}$ such that the S-fraction expansion of the formal power series $\sum_{n=0}^{\infty} \lambda_n w^{n+1}$ is nonterminating to be the first backward extension of an indeterminate Stieltjes moment sequence $\{\mu_n\}$. (Received March 11, 1954.)

Applied Mathematics

517. A. A. Blank: *The non-euclidean geometry of binocular visual space.*

According to Luneburg (*Mathematical analysis of binocular vision*) the binocular sensory space is Lobachevskian. This conclusion follows in part from his inference upon experimental evidence that the space possesses a complete six-parameter group of rigid transformations. The conclusion is valid since the space appears to be Desarguesian, every triple of points being spanned by a convex two-dimensional subspace. However, the phenomena which led Luneburg to assume free movability in visual space are better explained by a replacement of his transformation equations for the mapping of physical space onto visual sensory space. An individual's binocular space is completely characterized once this mapping is known. The determination of the mapping of physical space onto visual sensory space is reduced to the specification of visual radial distance as a function of convergence disparity. (Received March 6, 1954.)

518t. H. D. Block: *A remark on integral invariants.*

Let the $2n$ variables $q_1, q_2, \cdots, q_n, p_1, p_2, \cdots, p_n$ be related to the $2n$ variables $Q_1, Q_2, \cdots, Q_n, P_1, P_2, \cdots, P_n$ by a canonical transformation. Let σ be the unit square: $0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$, and let $q_i = f_i(u, v)$, $p_i = g_i(u, v)$ $(i = 1, 2, \cdots, n)$, where f_i and g_i have continuous derivatives on σ. This induces the relationships $Q_i = F_i(u, v)$, $P_i = G_i(u, v)$. Let $s_i = \bigcup_{(u,v) \in \sigma} (f_i(u, v), g_i(u, v))$ and $S_i = \bigcup_{(u,v) \in \sigma} (F_i(u, v), G_i(u, v))$, i.e. the maps of σ on the (q_i, p_i) and (Q_i, P_i) planes respectively. It is widely believed that, under the conditions stated, $\sum_{i=1}^{n} \iint_{s_i} dq_i dp_i = \sum_{i=1}^{n} \iint_{S_i} dQ_i dP_i$. It is the purpose of this paper: (1) to show by a simple example that this is not true, and (2) to discuss the sources of the mistake and point out correct formulations. The details will appear in the Quarterly of Applied Mathematics. (Received March 8, 1954.)

519t. H. D. Block: *On the minimality of the variational principles of classical particle mechanics.*

This paper deals principally with the variational principles which are associated with the names of Hamilton, Jacobi, Maupertuis, and Hilbert. Each principle involves a functional in the form of an integral. The requirement of the vanishing of the first variation of the functional provides the differential equations characterizing the motion. For this reason these principles are called variational or stationary principles. If, in addition, the actual motion minimizes the functional, then the principle is called a minimum principle. We show that some of the classical variational principles are minimum principles under certain conditions and only stationary principles under other conditions; others are always minimum principles, while still others are always only stationary principles. Some facts of this type are well known, but it is hoped that the treatment presented here not only provides more complete results, but also is more elementary in method, simpler in development, and yields results in more convenient form. (Received March 8, 1954.)

520. P. C. Hammer (p) and Jack Hollingsworth: *Numerical treatment of differential equations.* I.

There is proposed here a general class of methods of integrating ordinary differential equations numerically. These methods extend from the concept of differential equivalence introduced by Hammer. The class includes many methods such as those of Adams, Milne, and Moulton but does not directly include the Runge-Kutta type methods. In particular this class of methods includes all those which approximate the solution function and/or its derivatives by means of selecting from classes of functions, depending on a finite number of parameters to each class, particular functions for a designated range of independent variable. The selections may be made on the basis of explicit or implicit numerical determination of a finite number of parameters in any manner. Specific new methods resulting from this general attack show promise for difficult problems. Test calculations are now being carried out on a variety of methods at the University of Wisconsin Numerical Analysis Laboratory. (Received March 10, 1954.)

521. P. C. Hammer and Jack Hollingsworth (p): *Numerical treatment of differential equations.* II.

Given a system of differential equations whose behavior one wishes to study numerically, the question of what constitutes a good method is significant. The answer cannot be given categorically, but must be modified for example by the equipment and personnel available. The class of methods proposed in I is quite general and can often be made to yield a method tailored to fit the problem. Various specializations to meet certain situations are discussed. Let an nth order differential equation with suitable continuity in a closed region be given. Then there exists a family of functions of $n+2$ parameters from which arcs of any member (or from several members) can be joined to form a curve which uniformly approximates a solution. Moreover a sequence of such approximate solutions can be constructed, a subsequence of which converges uniformly to a solution. If a Lipschitz condition is met, this solution is unique. (Received March 10, 1954.)

522. W. C. Hoffman: *Scattering of electromagnetic waves from a random surface.*

Suppose that $z(x, y)$ is real and continuous in the mean over a finite region D, with mean value zero and covariance function $r(u, x; v, y)$. As such $z(x, y)$ defines a random surface. Then by Karhunen's theorem on the representation of a stochastic process, $z(x, y)$ has an expansion in the eigenfunctions of the covariance function. Employing this expression in the Stratton-Chu formulation of the electromagnetic field then leads to an approximate expression for the radiation scattered from the random surface, from which mean and covariance of the scattered field may be determined. (Received March 10, 1954.)

523. R. S. Ledley: *Digitalization, systematization, and formulation of the theory and methods of the propositional calculus.*

The procedures of the propositional calculus previously used almost exclusively in the set theoretical proofs of pure mathematics have recently become of fundamental importance in applied mathematics as well. In many scientific, social, and industrial fields, complex logical non-numerical problems arise more frequently than do problems of a numerical nature. "Logic machines" capable of handling such problems must be based on systematized and digitalized manipulations for carrying out computations of the propositional calculus. Consequently, the need arises for a systematic method of formulating, analyzing, and solving propositional functions and equations. The author has obtained methods for: (1) expressing the propositional calculus uniquely in a manner amenable to easy mechanization; (2) systematically generating all true and false implications of given propositions; (3) systematically generating all possible changes of variables (i.e. substitution of new variables) that can be made on a propositional calculus while preserving the logical meaning of the calculus (these substitutions being subject to arbitrary restrictions); and (4) the systematic generation of all solutions to any number of simultaneous propositional equations in any number of unknowns. The solution to this latter problem imples the ability to perform manipulations with propositional variables in a straightforward manner analogous to manipulations of ordinary algebraic equations. (Received March 9, 1954.)

524. W. S. Loud: *Existence and uniqueness of a periodic solution for Duffing's equation with damping and a forcing term.*

The equation considered is (1) $\ddot{x}+c\dot{x}+x+\beta x^3=\cos \omega t$. Analogue computer studies suggest that for certain values of the parameters c, β, and ω, (1) has more than one periodic solution of period $2\pi/\omega$, with the stable ones not odd-harmonic. Using results of Levinson (Journal of Mathematics and Physics vol. 22 (1943)) and Cartwright (Annals of Mathematics Studies, no. 20) it is shown that for c positive, and β non-negative and sufficiently small, this can not occur, and that (1) has a unique, odd-harmonic, solution of period $2\pi/\omega$, to which all solutions converge. An explicit, but not best possible, range of allowable values of β is given. (Received March 10, 1954.)

525. Y. L. Luke: *Numerical integration near a logarithmic singularity.*

Purpose of paper is to present coefficients to facilitate computation of $\int_0^{nh} \log x f(x)dx$. Integration by parts yields the integrals $\int_0^{nh} f(x)dx$ and $C_{nr}=\int_0^{nh} x^{-1}\int_0^x f(u)du$. The former can be integrated by several well known techniques. Let f be represented by a polynomial through the $(n+1)$ points $x=mh$, $m=0(1)n$. Paper gives exact coefficients of the Lagrangian type for the evaluation of C_{nr}; $n=1(1)10$, $r=1(1)(n+1)$. The remainder is studied after the manner of W. E. Milne (*The remainder in linear methods of approximation*, Journal of Research, National Bureau of Standards, vol. 43 (1949) pp. 501–511). Write $R_{nr}=\int_0^{nh} f^{(n-1)}(S)G_{nr}(S)dS$. Mean value theorem is applicable if $G(S)$ is one signed (that is, definite) in $(0, c)$ where c is the larger of n and r. $G_{nr}(S)$ is definite for all reported values of r and n such that $r\leqq n$. If $r=n+1$, $G_{n,n+1}(S)$ is definite (indefinite) if n is even (odd). In the indefinite cases, the function has one zero in $(0, n+1)$ open. Let ζh be the vanishing point. Also tabulated are $\int_0^c G_{nr}(S)dS$, ζh, and $\int_0^{\zeta h} G_{n,n+1}(S)dS$ for all r and n. (Received March 8, 1954.)

526. Hillel Portisky and R. P. Jerrard (p): *Expression of a wave function over a half-space in terms of its values over the boundary plane.*

A solution of the wave equation in the half-space $z>0$ is expressed in terms of its values $u=f(x,y)$ over the plane $z=0$ as follows: $u|_{P_0}=(1/2\pi)(\int[f]d\Omega-(1/c)[\partial f/\partial t]Rd\omega)$, where the integration is carried out over the plane $z=0$. Here R is the distance between the field point P_0 and a point on the plane of integration, $d\Omega$ is the element of solid angle subtended by the plane element $dx'dy'$ at P_0, and brackets indicate "retarded" values; in other word $[f]$ denotes the value of f not at the time t but at the time $t'=t-R/C$ such that a spherical wavelet starting from the point in question at the time t', with a radius expanding at the rate c, will reach the point P_0 at the time t. Two proofs of this relation are outlined. The first is based on Fourier time resolution; the second utilizes the proof used by Kirchhoff in his well known expression of the solution of the non-homogeneous wave equation in terms of retarded potentials of the sources. Several examples are given by way of illustration. (Received March 11, 1954.)

GEOMETRY

527t. John DeCicco: *An inverse theory of conservative fields of force.*

The extremals of the variation problem: $\int_1^2 g(x, y, \tau, \mathfrak{T}, t)ds=$ minimum possess an integral of the form: $v^2/2=\mathfrak{T}(x,y, \tau, t, E)$, where $E=$constant, if and only if $g=g(x, y, \tau, \mathfrak{T}(x, y, \tau, t, E), t)$ is independent of the time t. The rectangular components ϕ and ψ of a field of force such that a constrained motion is possible along every extremal of this variation problem depend on $\mathfrak{T}$, g, and an arbitrary resistance R. If these extremals form a system S_k, then ϕ and ψ are determined in terms of $\mathfrak{T}$, g, and k. These results are applied to the case when $g=v^{1+k}H$ and $H=H(x, y, \tau, \mathfrak{T},$

t). In particular, when $H=1$, ϕ and ψ are determined in terms of $\mathfrak{T}$ and k. This field of force is independent of k if and only if the energy equation is of the form $v^2/2 = \mathfrak{T}(x, y, E)$. Similar results are obtained for velocity systems S_∞. For these generalized conservative fields of force, the Lagrangian and Hamiltonian equations of a system S_k, including $k=\infty$, are obtained. (Received March 5, 1954.)

528. John DeCicco: *Some theorems in the integral theory of polygenic functions.*

For a clock congruence whose central and phase point vectors are the polygenic functions $f(z)$ and $f(z)+g(z)$, the vector $\lambda = f_{\bar z} - g_z$ has the phase point and center of the derivative clock congruences of $f(z)$ and $f(z)+g(z)$ as its initial and terminal points. If $I(\Gamma) = \int_\Gamma f(z)dz + g(z)d\bar z$, then $|I(\Gamma)| \leqq 2RA$, where $R = \text{l.u.b. } |\lambda|$ for z within or on Γ, and A is the area enclosed by Γ. The generalized areolar derivative is $\lim I(\Gamma)/A = 2i\lambda$. For $f(z)$ of class $2q$, $(1/2\pi i)\int_\Gamma f(z)dz/(z-z_0) = \sum_{k=0}^{q-1}(r^{2k}/2^2\cdot 4^2 \cdots (2k)^2)\nabla^{k}f(z_0)+R$, where $|R| \leqq (r^{2q}/2^2\cdot 4^2\cdots(2q)^2)\ \text{l.u.b.}_{|z-z_0|\leqq r}|\nabla^{q}f(z)|$, and Γ is $|z-z_0| = r$. If $f(z)$ is a nonconstant real q-harmonic function in $|z-z_0| \leqq r$, and if $\nabla f(z_0) = 0, \cdots, \nabla^{q-1}f(z_0)=0$, then z_0 is not a maximum or minumum of $f(z)$. If $f(z)$ is real, nonconstant, and of class $2q$ in $|z-z_0|\leqq r$, and if $\nabla f(z_0)=0, \cdots, \nabla^{m-1}f(z_0)=0$, $\nabla^m f(z_0) \neq 0$, where $1 \leqq m < q$, then in $|z-z_0| < S \leqq r$, $f(z)$ can not have a minimum or maximum at z_0 according as $\nabla^m f(z_0)$ is <0 or >0. Finally the integrals $(1/2\pi i) \cdot \int_\Gamma f(z)dz/(z-z_0)^{m+1}(\bar z - \bar z_0)^n$ are studied in detail. (Received March 11, 1954.)

529t. John DeCicco: *Some theorems in the mapping theory of surfaces.*

In a cartogram Γ between two surfaces S and $\bar S$, the metric scale $\sigma = d\bar s/ds$ and the projective scale $\tau = d\bar\phi/d\phi$ are studied. There always exist the principal metric scales σ_1 and σ_2, and the principal projective scales τ_1 and τ_2 for a cartogram Γ. If Γ is not conformal, σ_1 and σ_2 are assumed along the unique orthogonal net Ω which is preserved by Γ, and if Γ is not quasi-projective, τ_1 and τ_2 are assumed along the unique conjugate net Π which is preserved by Γ. The ratio of surface elements of S and $\bar S$ is equal to $\sigma_1\sigma_2$, and the ratio of the Gaussian curvatures of S and $\bar S$ is equal to $(\tau_1\tau_2/\sigma_1\sigma_2)^2$. The totality of all cartograms which carry a given orthogonal net Ω or a given conjugate net Π of a surface S into an orthogonal net or a conjugate net of any surface $\bar S$ forms a group $G(\Omega)$ or a group $G(\Pi)$. Characterizations of the spherical representation of a surface S are obtained. Finally are studied cartograms for which the lines of curvature on S correspond to those on $\bar S$. (Received March 4, 1954.)

530t. Edward Kasner and John DeCicco: *Physical systems of curves in general fields of force.*

Physical systems S_k of curves of a field of force in the plane, where the force depends not only on the position of the point but also on the velocity and the time, are studied. A system S_k consists of ∞^4 trajectories or else, in the directional case, it consists of ∞^3 trajectories. In a general field of force, there are ∞^3 lines of force unless the field of force is directional together with a tangential resistance, in which case there are only ∞^1. The rest trajectory of a system S_k at a point P is initially tangent to a line of force at P, and the ratio of the initial curvatures of the rest trajectory of S_k and the line of force at P is $(1-\lambda)/((3+k)/(1+k)-\lambda)$, where λ is a certain angular rate at P. Generalizations of this formula are obtained for higher order contact. This generalizes a well known theorem of Kasner. Fields of force which retain some of the properties of positional fields of force are also studied. (Received March 5, 1954.)

STATISTICS AND PROBABILITY

531. Miriam C. Ayer, H. D. Brunk (p), G. M. Ewing, W. T. Reid, and Edward Silverman: *An empirical distribution function for sampling with incomplete information.*

For $i=1, 2, \cdots, n$, let N_i independent trials be made of an event with probability p_i, and suppose that the probabilities p_i are known to satisfy the inequalities $p_1 \geqq p_2 \geqq \cdots \geqq p_n$. Particular examples of this situation are found in bio-assay (cf. C. H. Goulden, *Methods of statistical analysis*, New York, Wiley, 1952) and in the proximity fuze problem discussed by M. Friedman (E. Eishenhart, M. W. Hastay, W. A. Wallis (Editors), *Techniques of statistical analysis*, New York, McGraw-Hill, 1947, Chap. 11). The maximum likelihood estimators, $\bar{p}_i$, of the probabilities p_i are determined and their consistency is proved. Let a_i denote the number of successes in the ith trial, and set $A(r, s)=\sum_{\nu=r}^{s} a_\nu / \sum_{\nu=r}^{s} N_\nu$. One finds that $\bar{p}_i=\min_{1 \leqq r \leqq i} \max_{i \leqq s \leqq n} A(r, s)$ $=\max_{i \leqq s \leqq n} \min_{1 \leqq r \leqq i} A(r, s)=\min_{1 \leqq r \leqq i} \max_{r \leqq s \leqq n} A(r, s)=\max_{i \leqq s \leqq n} \min_{1 \leqq r \leqq s} A(r, s)$. The first of these expressions is useful in proving the consistency of the estimators, but other, simpler, means are available for the rapid calculation of the $\bar{p}_i$. (Received March 10, 1954.)

532t. G. E. Baxter: *Stationary "space" state of the Brownian motion of continuous one-dimensional media.*

Let a one-dimensional mechanical system have kinetic energy $2^{-1}\int_a^b r(x)(\partial\xi/\partial t)^2 dx$ and potential energy $(\alpha/2)\ \xi^2(a)+2^{-1}\int_a^b \{p(x)(\partial\xi/\partial x)^2+q(x)\xi^2\} dx+(\beta/2)\xi^2(b)$ where $\alpha, \beta \geqq 0$, $r(x)>0$ and continuous, $q(x) \geqq 0$ and continuous, $p(x)>0$ and continuously differentiable. Under the assumption that this system is the limit of n particles, it is shown that the stationary "space" state of its Brownian motion is described by a Gaussian stochastic process $\{\xi_x, x \in [a, b]\}$ with $E\{\xi_x\}$ zero and $E\{\xi_x \xi_y\}$ a multiple of the Green's function of the system. It is also shown that the exponent of the n-variate density function of $\xi_{x_1}, \xi_{x_2}, \cdots, \xi_{x_n}$ $(a<x_1<x_2<\cdots<x_n<b)$ for this process is equal to a multiple of $\sum_{j=1}^{n+1} \min\ \{\int_{x_{j-1}}^{x_j} \{p(x)(\partial\xi/\partial x)^2+q(x)\xi^2\} dx+(\alpha/2)\xi^2(a)\delta_{1j}$ $+(\beta/2)\xi^2(b)\delta_{n+1,j}\}$ where the minimization of the ith subintegral is subject to the conditions $\xi(x_j)=\xi_j$, $\xi(x_{j-1})=\xi_{j-1}$. Finally, there is considered the effect of additional terms in the potential energy which correspond to constant forces acting on the system. The only change is to make the mean function $u(x)=E\{\xi_x\}$ the rest position of the system. (Received March 15, 1954.)

533t. J. H. Curtiss: *Sampling from a lot which is itself a probability sample.*

First it is proved that virtually any form of probability sampling *without* replacements from a lot which is itself a random sample, yields a random sample from the process. The sample and the uninspected remainder (u.r.) of the lot will be distributed independently. It is shown that in spite of this independence it is possible to draw inferences from sample to u.r. Various elementary inequalities linking parameter of the u.r. to those of the lot and of the sample are derived. For the normal case and attributes case, various confidence intervals for parameter of the u.r. are also derived. Then conditional inferences into over-all lot quality are considered. To develop this point of view, various conditional distributions of sample statistics relative to hypotheses on the corresponding lot statistics are derived for the normal, binomial, Poisson cases. (E.g. the conditional distribution of the sample variance for fixed lot variance is in the normal case a beta distribution.) Other results include general formulas for the unconditional distribution of the sample mean in sampling *with* replacements, and for conditional mean values in sampling without replacements. (Received March 15, 1954.)

534*t*. Ernest Ikenberry: *The convergence of Type* A *Gram-Charlier series.*

It is shown that the partial sums $f_R(x)=\alpha(t)\sum_{r=0}^{R}c_rH_r(t)$, where $t=k(x-X)$, X and $1/k^2$ are the mean and variance of a sample in which the relative frequency of occurrence of x_j is $f(x_j)$, $\alpha(t)=(1/(2\pi)^{1/2})\exp.(-t^2/2)$, $\alpha(t)\cdot H_r(t)=(-D)^r\alpha(t)$, and the coefficients c_r are evaluated from $c_r=(1/r!)\sum_j f(x_j)H_r(t_j)$, converges to zero as R approaches infinity, except for $x=x_j$, at which points there is divergence. The sum function as R approaches infinity may conveniently be written in the form $f(x)=\sum_j f(x_j)[\alpha(t)/\alpha(t_j)]^{1/2}\cdot\delta(x-x_j)$ where $t_j=k(x_j-X)$ and $\delta(x-x_j)$ is the Dirac delta function. This result adds to the list of disadvantages of Type A Gram-Charlier series given by Kendall (*The advanced theory of statistics*, London, Griffin, 2d ed., rev., 1945, vol. 1, pp. 151–154). (Received March 8, 1954.)

535*t*. J. H. B. Kemperman: *Isotropic random flight in k-dimensional space.*

Consider a sequence X_i $(i=1,2,\cdots)$ of independent k-dimensional random vectors $(k\geqq 1)$, each having the same isotropic distribution. Let $F(r)=\Pr(|X_i|\leqq r)$ and suppose that $F(0)<1$. Let $Z_j=X_1+X_2+\cdots+X_j$ and let n be the random variable defined by $|Z_j|<R$ $(j=1,\cdots,n-1)$ and $|Z_n|\geqq R$, where R is a positive constant. Finally, let $p_N=\Pr(n=N)$ $(N=1,2,\cdots)$. Neglecting the excess $|Z_n|-R$ over the boundary (causing only small errors if $E(|X_i|^2)\ll R$), the following generating function is derived by means of Wald's fundamental identity. (*) $\sum_{N=1}^{\infty}p_Nw^{-N}=g(R^2t)^{-1}$ for $0\leqq t\leqq\epsilon$, ϵ being a sufficiently small positive quantity. Here, $w=\int_0^\infty g(r^2t)dG(r)$ and $g(t)=\Gamma(k/2)\sum_{h=0}^{\infty}(-t)^h/\Gamma(h+k/2)\Gamma(h+1)$. Provided that $E(|X_i|^{2h})<\infty$ one may differentiate (*) h times term by term. Differentating only once gives $E(n)=R^2/E(|X_i|^2)$. Differentiating twice gives $E(n^2)$ which, however, depends on the dimension k. (Received March 11, 1954.)

536. H. G. Landau: *Distribution of solution times for random communication in a task-oriented group.*

A task-oriented group consists of n individuals, $i=1,2,\cdots,n$, each having initially one piece of information which must be transmitted to all the others to complete the task. At every sending time, $t=1,2,3,\cdots$, each individual sends all the information received previously to one of his possible recipients chosen at random. The communication net, specifying the possible paths for messages, is given by the matrix $S=(s_{ij})$, where $s_{ii}=1$, and for $i\neq j$, $s_{ij}=1$ if i can send to j, $s_{ij}=0$ otherwise. The information state at any time is given by a matrix $C_\alpha=(C_{\alpha,ij})$ where $C_{\alpha,ii}=1$, and for $i\neq j$, $C_{\alpha,ij}=1$ if j has received the information which i had initially, $C_{\alpha,ij}=0$ otherwise. The information state after each sending time is given by a Boolean matrix product. The solution time is the time when C_α first becomes (1), all elements equal unity. The C_α can be considered as the states of a finite, stationary Markov chain, which is particularly simple because the C_α can be ordered so that the transition probability matrix has only zeros below the principal diagonal. The generating function for the distribution of solution times can then be calculated very simply from the transition probability matrix. The transition probabilities themselves can be obtained by a straightforward calculation. Some special cases are considered. (Received February 22, 1954.)

TOPOLOGY

537. Maurice Auslander: *Compound group extensions.* IV.

Let $\psi: K \to G$ be a homomorphism of the group K into the group G with kernel X. By a continuation of this homomorphism we mean a pair (E, ϕ) where E is a group containing K as a subgroup and $\phi: E \to G$ is a homomorphism onto satisfying the conditions (1) $\phi^{-1}(1) = X$. (2) $\phi | K = \psi$. It is shown that every continuation of $\psi: K \to G$ induces a homomorphism $\theta: G \to A(X)/I(X)$ satisfying certain conditions. We define (E_1, ϕ_1) to be isomorphic to (E_2, ϕ_2) if there is an isomorphism $\varpi: E_1 \approx E_2$ such that (1) $\varpi | K =$ identity, (2) $\phi_2 \varpi = \phi_1$. It is then shown that the relative cohomology group $H^2(G, \psi(K); Z_X)$, where Z_X is the center of X, operates simply transitively on the isomorphism classes of continuations of $\psi: K \to G$ which induce the same homomorphism $\theta: G \to A(X)/I(X)$. An obstruction to the existence of continuations is defined to be a certain element of $H^3(G, \psi(K), Z_X)$. Finally, the results of J. H. C. Whitehead on operator extensions (Quart. J. Math. Oxford Ser. (2) vol. 1 (1950)) are obtained as an application of above results. (Received March 8, 1954.)

538*t*. Maurice Auslander: *Relative cohomology groups of groups.* Preliminary report.

Let G be a group operating on an abelian group A and N a subgroup of G. Let $r: C^q(G, A) \to C^q(N, A)$ be the restriction map of the group of q-dimensional cochains on G onto the q-dimensional cochains on H. Define $C^q(G, N; A) = r^{-1}(0)$ for $q > 0$ and $C^0(G, N; A) = 0$. We designate by $H^q(G, N; A)$ the cohomology groups of the cochain complex $\{C^q(G, N; A), \delta\}$, where δ is induced by the coboundary operator of $C^q(G, A)$. The following results are obtained: (1) there is an exact sequence $\cdots \to H^q(G, N; A) \to H^q(G, A) \to H^q(N, A) \to H^{q+1}(G, N; A) \to \cdots$. (2) For all $q > 0$, $H^q(G, N; A) \approx H^q(G, C)$ for a well determined C. (3) If N is a proper normal subgroup such that for all A, $H^q(G, N; A) = 0$, then $H^i(G, N; A) = H^i(G, A) = H^i(N, A) = \{0\}$ for all A and all $i \geq q$. (4) If G contains an element of finite order and N is a proper normal subgroup, then the restriction map $H^q(G, A) \to H^q(N, A)$ can not be an isomorphism for two successive values of q and all A. (Received March 8, 1954.)

539*t*. R. H. Bing: *Monotone upper semicontinuous decompositions of E^3.*

Suppose G is a monotone upper semicontinuous decomposition of E^3 such that G has only a countable number of nondegenerate elements and the complement of each is topologically equivalent to the complement of a point. It is shown that the decomposition space G is topologically equivalent to E^3 if either (a) each element of G is starlike, (b) each nondegenerate element of G is a tame arc, or (c) the sum of the nondegenerate elements of G is a G_δ set. An example is given of an upper semicontinuous decomposition H of E^3 such that each nondegenerate element of H is a tame arc but where it is not known whether or not the decomposition space H is topologically equivalent to E^3. However, it is shown that such a decomposition space H is topologically equivalent to E^3 if each element of the decomposition H is a subset of a vertical line. (Received March 11, 1954.)

540*t*. R. H. Bing: *Some monotone decompositions of a cube.*

Suppose G is a monotone upper semicontinuous decomposition of a cube such that each nondegenerate element of G is a subset of the boundary of the cube. The decomposition space G is shown to be topologically equivalent to the closure of the complement of a cactoid in S^3. The topological character of the decomposition space is determined by using the characterization of S^3 given in *A characterization of 3-space by partitionings*, Trans. Amer. Math. Soc. vol. 70 (1951) pp. 15–27. (Received March 11, 1954.)

541. H. J. Curtis: *A metrisation problem concerning lattices.* Preliminary report.

Let L be a semi-modular, atomistic lattice with the following properties: the point space, P, of L is a metric space; every element of L, considered as a point set, is closed in the topology of P; the g.l.b. of distances from a given point to points of any nonzero element of L is continuous in any set of independent points determining the element. Wilcox (Duke Math. J. vol. 8 (1941) pp. 273–285) extended the topology of P to L in such a way that L is a Hausdorff space. In this paper a sufficient condition that L be metrisable is established. It is that L satisfy the following three conditions: P is separable; no element of dimension greater than one, considered as a point set, has arbitrarily small diameter; a generalization of Wilcox's axiom pertaining to the g.l.b. of distances from a point to points of a nonzero element of L. This result is achieved by the use of the theory of compact spaces. Non-isomorphic examples of lattices which satisfy the sufficient condition are the real euclidean and projective spaces. (Received March 22, 1954.)

542. O. G. Harrold, H. C. Griffith (p), and E. E. Posey: *A characterization of tame curves in 3-space.*

Let J be a 1-manifold in 3-space. If J lies in the combinatorial boundary of a disk (closed, topological 2-cell) K which is locally polyhedral at each point of $K-J$, say J has *property* Q. If for each point x of J there is a disk K_x such that (i) K_x is locally polyhedral at points of K_x-J, (ii) $K_x \cap J$ is an arc, and (iii) $K_x \cap J$ is the closure of a neighborhood of x relative to J, say J has property Q at x. There are examples of arcs and curves having property Q that are not tamely imbedded. By combining property Q with a property P given earlier (Bull. Amer. Math. Soc. Abstract 59-4-434) the following results are obtained. If a homeomorphism h on 3-space maps J into a plane, J has properties P and Q. If J is tame, J has property P and the local form of property Q at each point. If J has property P and the local form of property Q, J is tame. If J has property P and Q, J is tame and unknotted. (Received March 8, 1954.)

543. G. R. Livesay: *A separation theorem.*

Let A be a compact subset of E_n, B closed subset not intersecting A. Then for any open set $U \supset A$, there exist a finite number of compact manifolds, disjoint from one another, and from $A \cup B$, such that the union of these manifolds is contained in U, and separates A from B in E_n. Further, if A is compact and connected, and B is closed and contained in a single component of $E_n - A$, then there is a single compact manifold satisfying the above conditions. The manifolds are triangulable. (Received March 8, 1954.)

544. R. J. Nunke: *Two theorems on the cohomology groups of a chain complex.* Preliminary report.

For abelian groups A and C let Ext (A, C) be the group of abelian group extensions of C by A. If Hom $(A, Z)=0=$ Ext (A, Z), where Z is the group of integers, then A is trivial. This result is used to prove the following two theorems: I. If K is a chain complex composed of free abelian groups and if, for every integer $q>n$, $H^q(K; Z)=0$, then for all $q>n$, $H^q(K; Z)=0$. II. Let K and L be chain complexes composed of free abelian groups and let $f: K\rightarrow L$ be a chain transformation. In order that f be a chain equivalence, it is necessary and sufficient that the induced homomorphism $f^*: H^q(L; Z) \rightarrow H^q(K; Z)$ be an isomorphism for all dimensions q. (Received March 10, 1954.)

545. D. E. Sanderson: *On isotopy of homeomorphisms in E_3 and S_3.* Preliminary report.

The principal results obtained in this paper are (i) given $\epsilon>0$, every homeomorphism of Euclidean 3-space, E_3, onto itself is ϵ-isotopic to a piecewise linear homeomorphism, and (ii) the space of homeomorphisms of the 3-sphere, S_3, onto itself is uniformly locally connected. Both results are obtained by applications of Moise's piecewise linear approximation theorem for homeomorphisms on 3-manifolds (Ann. of Math. vol. 56 (1952) p. 97), Alexander's deformation theorem for an n-cell (Proc. Nat. Acad. Sci. U.S.A. vol. 9 (1923) pp. 406–407) and a theorem on fitting together homeomorphisms previously announced by the author (Bull. Amer. Math. Soc. Abstract 59-6-717). By a slight modification of the proof of (i) it can be shown that the space of homeomorphisms of E_3 onto itself under the compact-open topology defined by Fox (Bull. Amer. Math. Soc. vol. 51 (1945) p. 429 ff.) is locally connected. (Received March 11, 1954.)

546t. C. T. Yang: *A generalization of Borsuk-Ulam's theorem.*

Let R^{n+1} be the euclidean $(n+1)$-space and 0 its origin. Let X be a compact subset of $R^{n+1}-\{0\}$ symmetric with respect to 0. It is proved that the following are equivalent: (i) X separates 0 and ∞. (ii) Every mapping of X into the euclidean n-space maps some pair of symmetric points into a single point. (iii) X has a symmetric n-cycle (mod 2), with respect to which the order of 0 is not zero. That (i) implies (ii) is a generalization of a well known theorem of Borsuk-Ulam (Borsuk, Fund. Math. (1933)). (Received March 9, 1954.)

547. C. T. Yang: *Mappings from spheres to euclidean spaces.* Preliminary report.

In this paper it is proved that for any positive integers m, n there exists a positive integer k such that every mapping of a k-sphere into the euclidean m-space maps the end points of some n mutually orthogonal diameters into a single point. Denote by $k(m, n)$ the smallest of such integers k. In addition to the known results that $k(m, 1)=m$ (Borsuk, Fund. Math. (1933)) and $k(1, n)=n$ (Yang, Bull. Amer. Math. Soc. Abstract 59-6-720), the author has proved (i) $k(m, n)\leqq mn+n-1$, (ii) $k(m, 2)\leqq 2m$ and (iii) $k(2, 2)=4$. These results are actually established in a more general form as in an early work of the author (Bull. Amer. Math. Soc. Abstract 60-1-144). (Received March 9, 1954.)

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THE MAY MEETING IN YOSEMITE

The five hundred third meeting of the American Mathematical Society was held at Yosemite National Park on Saturday, May 1, 1954. Attendance was approximately 125, including 82 members of the Society.

All meetings were held in the office and recreation room of Camp Curry. By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor P. A. White of the University of Southern California presented an hour address entitled *Regular convergence*. Professor White was introduced by Professor J. L. Kelley. The four sessions for contributed papers were presided over by Professors C. B. Morrey, Ivan Niven, Roy Dubisch, and Leon Henkin.

Abstracts of papers presented at the meeting follow. Mr. Maschler and Dr. Bremermann were introduced by Professor Stefan Bergman, Mr. Miller by Professor R. H. Sorgenfrey, and Dr. Fraïssé by Professor Alfred Tarski. Abstracts whose numbers are followed by the letter "*t*" were presented by title. In the case of joint papers, (p) following one of the authors' names indicates the one who actually read the paper.

ALGEBRA AND THEORY OF NUMBERS

548. S. P. Avann: *Dual symmetry of projective sets in a finite modular lattice.*

Let Q be any complete set of projective complemented quotient sublattices of a finite modular lattice L. Let $V_{Q,k}(L)$ be the set of elements x for which there are precisely k quotients in Q that are minimal in the complemented quotient lattice x^*/x, generated above x by the elements covering x. Define $W_{Q,k}(L)$ dually. The orders of $V_{Q,k}(L)$ and $W_{Q,k}(L)$ are equal. For $k=1$, the number of maximal quotients of Q equals the number of minimal quotients of Q. When the orders of $V_{Q,k}$ and $W_{Q,k}$ are summed over all complete sets of projective prime (two-element) quotients, one obtains a recent result of Dilworth. The number of elements of L covered by precisely k elements equals the number of elements of L covering precisely k elements. If, furthermore, $k=1$, the number of meet-irreducibles and the number of join-irreducibles in L are equal. The abstract Moebius function is employed to obtain the results. (Received March 10, 1954.)

549*t*. J. L. Brenner: *The direct product of simple groups.*

Let G_n be the simple group $SL_{4n}[F]$, and let G be the direct product of the G_n taken in order. G has certain obvious normal subgroups which arise from the general properties of the direct product. It is shown that G has still other normal subgroups, and all the normal subgroups of G are found. (Received January 18, 1954.)

550. Chen-Chung Chang and Anne C. Davis (p): *Some theorems on ordinal products of relations.*

The ordinal product $R \cdot S$ of binary relations R and S is the set of ordered pairs $\langle\langle r, s\rangle, \langle r', s'\rangle\rangle$ where r, r' belong to the field of R, s, s' to the field of S, and either (i) $s \neq s'$ and $\langle s, s'\rangle \in S$ or (ii) $s = s'$ and $\langle r, r'\rangle \in R$. (Related definitions are found in Whitehead and Russell, *Principia Mathematica*, vol. II, 2d ed., 1927, p. 310.) If α, β are the (isomorphism) types of relations R, S respectively, then the type of the relation $R \cdot S$ is denoted by $\alpha \cdot \beta$, the power of the field of the relation R by $\mathfrak{p}(\alpha)$. Let $\alpha^1 = \alpha$, $\alpha^{n+1} = \alpha^n \cdot \alpha$ for every natural n. The following theorems hold for types of nonempty reflexive relations. I. If $\alpha \cdot \gamma = \beta \cdot \delta$ and if $\mathfrak{p}(\alpha) = \mathfrak{p}(\beta) < \aleph_0$, then $\gamma = \delta$. An immediate consequence of I is: II. If $\alpha^n = \beta^n$ and if $\mathfrak{p}(\alpha) < \aleph_0$, then $\alpha = \beta$. III. If $\alpha \cdot \gamma = \beta\ \gamma$, $\mathfrak{p}(\alpha) < \aleph_0$, $\mathfrak{p}(\gamma) < \aleph_0$, then $\alpha = \beta$. From I and III one easily obtains: IV. If $\alpha \cdot \beta = \gamma \cdot \delta$, $\mathfrak{p}(\alpha) < \aleph_0$, $\mathfrak{p}(\beta) < \aleph_0$, and if either $\mathfrak{p}(\alpha) = \mathfrak{p}(\gamma)$ or $\mathfrak{p}(\beta) = \mathfrak{p}(\delta)$, then $\alpha = \gamma$ and $\beta = \delta$. (Received March 10, 1954.)

551. C. J. Clark: *Introduction to C-R algebras.*

This paper introduces the concept of closure-relation (C-R) algebras described by a collection of axioms which prove to be satisfied if one takes the elements of the algebra to be the set of all binary relations between elements of a compact first countable Hausdorff space, and the fundamental operations of this algebra to be the operations of join, meet, product, converse, and closure. This abstract algebraic system, which is an extension of the notion of closure algebra, is a Boolean algebra with operators. In this system the closure operation is extended from elements of a Boolean algebra to those of a relation algebra. This necessitates additional axioms on operations and elements peculiar to relation algebras. Definitions are given for the operations of taking the domain of an element and the image of one element under another. Both of these are abstractions and generalizations of the notions of domain of a function and image of a set under a function. Also the closed element of C-R algebra can be considered a generalization of the notion of continuous function. (Received March 4, 1954.)

552t. J. H. Curtiss: *A generalization of the method of conjugate gradients for solving systems of linear algebraic equations.*

In the description that follows, all matrices are real and $n \times n$, all vectors are real and n-dimensional, and a prime denotes transposition. Let A be the matrix of the system to be solved; we assume that A is arbitrary but nonsingular. Let B be an arbitrary nonsingular matrix and let T be a positive definite matrix. The generalization in question consists of an algorithm for generating a sequence of scalars $a_0, a_1, \cdots$ and a sequence of $BATA'B'$-orthgoonal vectors $\delta_0, \delta_1, \cdots$ such that the solution of the given system can be expressed as a linear combination of the first n vectors δ_N with coefficients given by the scalars a_N. The absolute value of the determinant of the system is given by $(a_0\, a_1 \cdots a_{n-1} \det T)^{-1/2} |B|^{-1}$. The generalization is specialized in various ways to yield practical results. In particular it is shown that the specialization $B = I$, $T = I$ leads to a certain conjugate gradient algorithm proposed in his MIT doctoral thesis by E. J. Craig for the case of an arbitrary A, which seems to be simpler than that proposed by Hestenes and Stiefel (N.B.S. Journ. Res. vol. 49 (1952) pp. 409–436) for this case. (Received February 19, 1954.)

553. L. A. Henkin: *Metamathematical theorems equivalent to the prime ideal theorems for Boolean algebras.* Preliminary report.

For notations see abstracts 562 and 564. *The following three statements M_1, M_2, M_3 are equivalent to each of B_1-B_5.* M_1: Let Π be a set of formulas of sentential logic containing arbitrarily (possibly nondenumerably) many sentential symbols. If Π is formally consistent, then truth-values can be assigned to the symbols so that all elements of Π become true. M_2: Let Σ be a set of first-order sentences with arbitrarily many nonlogical constants. If Σ is consistent, then it has a model. M_3: Let Σ be as in M_2. If $\mathfrak{S}$ is a system formed by a nonempty set A and by finitary relations and operations on A, and if every finite subsystem of S can be imbedded in a model of Σ, then $\mathfrak{S}$ can be imbedded in a model of Σ. Using the equivalence of M_1-M_3 with B_1-B_5, further purely mathemarical results are obtained. Thus, B_5 *proves to be equivalent to the following statement* B_6 (a sharper form of the representation theorem): B_6. Every Boolean algebra $\mathfrak{B}$ is isomorphic to a set algebra formed by subsets of a set which has the same power as $\mathfrak{B}$. Furthermore B_5 *implies the following ordering principle*: O_1. Every set can be simply ordered. Hence *a fortiori* B_5 implies AC for families of finite sets (a result previously obtained in a different way by Anne Davis). (Received March 8, 1954.)

554. Edwin Hewitt (p) and H. S. Zuckerman: *Finite-dimensional convolution algebras.*

Let G be a finite semi-group (i.e., a set with an associative operation, written as xy for $x, y \in G$; the existence of a unit is not assumed). The set $\mathcal{L}_1(G)$ consisting of all formal linear combinations $\sum_{x\in G}\alpha_x x$, where the α_x are complex numbers, forms an associative algebra over the complex number field under the operations $(\sum_x\alpha_x x)+(\sum_x\beta_x x)=\sum_x(\alpha_x+\beta_x)x$, $\lambda(\sum_x\alpha_x x)=\sum_x(\lambda\alpha_x)x$; $(\sum_x\alpha_x x)(\sum_y\beta_y y)=\sum_x\sum_y\alpha_x\beta_y xy$. The following theorems are proved. I. $\mathcal{L}_1(G)$ is not a simple algebra if the order of $G=o(G)>1$. II. If G is Abelian, then $\mathcal{L}_1(G)$ is semisimple if and only if, for every $x\in G$, there exists an integer $k>0$ such that $x^{k+1}=x$. III. The radical of $\mathcal{L}_1(G)$ may have any dimension $<o(G)$, but never $=o(G)$. IV. If $xy=xz$ implies $y=z$, for all $x, y, z\in G$, then $\mathcal{L}_1(G)$ is the Kronecker product of a group algebra and an algebra of all complex matrices $(a_{ij})_{i,j=1}^{n}$ for which $a_{ij}=0$ for $i>1$. The noncommutative semisimple case is discussed. All $\mathcal{L}_1(G)$ are identified for $o(G)=2$ and 3. (Received March 15, 1954.)

555. R. D. James and Ivan Niven (p): *Unique factorisation in multiplicative systems.*

In discussing unique factorization various writers cited certain multiplicative systems of positive integers having multiple factorings. For example, in the multiplicative system of all positive integers of the form $1+7k$, the number 792 can be factored as $22\cdot 36$ and also as $8\cdot 99$. We examine all multiplicative systems composed of arithmetic progressions and decide the question of unique factorization. For a fixed positive integer n, let M be a set of positive integers which are closed under multiplication, and such that if $x\in M$ and $y\equiv x \pmod{n}$ with $y>0$, then $y\in M$. There is no loss of generality in assuming that n is the smallest integer which can be used to define the set M. We prove that M has the property of unique factorization if and only if M consists of all positive integers relatively prime to n. (Received March 3, 1954.)

556t. J. H. McKay: *Group extensions by left loops.*

If F is a subgroup of a group G, any set of representatives for the right cosets of F in G is a left loop (binary operation, two-sided identity, unique solutions on the left) under a natural operation. In this paper the procedure for finding all extensions of a group F by a left loop R is separated into two parts by using the fact that any extension G of F by R is homomorphic to a transitive permutation group on the elements of R such that the kernel of the homomorphism is the largest subgroup of F which is normal in G. In the first part (called a simple extension), this homomorphism is an isomorphism, while the other part is a Schreier extension. The set of all groups which can be simply extended by a given left loop is characterized. A definition is given for equivalence of left loops and its relationship to extensions is shown. Also, it is shown that the classes of equivalent left loops of degree n are in 1-1 correspondence with the transitive permutation groups of degree n. (Received March 10, 1954.)

557. Rimhak Ree and R. J. Wisner (p): *A note on torsion-free nil groups.*

A nil group $\mathfrak{G}$ is an (additively written) abelian group such that every ring $\mathfrak{R}$ with $\mathfrak{G}$ as its additive group has trivial multiplication, i.e., $AB=0$ for every $A, B \in \mathfrak{R}$. Szele [*Zur Theorie der Zeroringe*, Math. Ann. vol. 121 (1949)] completely characterized torsion nil groups and showed that there are no mixed nil groups. The question of the structure of torsion-free nil groups is considered here. Using a characterization of the additive rationals due to Beaumont and Zuckerman [Pacific Journal of Mathematics vol. 1 (1951) pp. 169–177], one finds sufficient conditions for torsion-free groups to be nil. The construction of examples of nil groups of arbitrary finite rank can be obtained easily. (Received March 11, 1954.)

558. Herman Rubin (p) and Dana Scott: *Some topological theorems equivalent to the Boolean prime ideal theorem.* Preliminary report.

For notation see abstracts 562–564 of Tarski. *The following five theorems are equivalent to the prime ideal theorem B_2 for Boolean algebras*: T_1. Tychonoff's compactness theorem for products of Hausdorff spaces. T_2. Alaoglu's theorem that the unit sphere of the adjoint of a Banach space is a compact Hausdorff space (cf. Ann. of Math. vol. 41 (1940) pp. 252–267.) T_3. The Stone-Čech compactification theorem. T_4. The theorem that every complete and totally bounded uniform space is compact. T_5. Alexander's lemma that a space is compact if there is a subbase for the open sets enjoying the Heine-Borel property. (Cf. Proc. Nat. Acad. Sci. U.S.A. vol. 25 (1939) pp. 296–298.) Each of T_1-T_5 implies that products of nonempty compact Hausdorff spaces are nonempty, and hence products of nonempty finite sets or nonempty closed subsets of a given compact Hausdorff space are nonempty. That the full Tychonoff theorem implies AC was shown by Kelley (cf. Fund. Math. vol. 37 (1950) pp. 75–76) who also raised the question as to whether T_1 is equivalent to AC. In view of the above results it appears a positive answer to this question is possible only if B_2 implies AC. (Received March 8, 1954.)

559. R. L. San Soucie: *Right alternative rings of characteristic two.*

Let F be the free nonassociative ring generated by x_1 and x_2 and suppose R is any right alternative ring. Elements t, u, v in R form an alternative triple (Kleinfeld, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 939–944) if (i) there exist elements $\alpha[x_1, x_2]$, $\beta[x_1, x_2]$, $\gamma[x_1, x_2]$ in F and elements r_1, r_2 in R such that $t=\alpha[r_1, r_2]$, $u=\beta[r_1, r_2]$, $v=\gamma[r_1, r_2]$ and (ii) if s_1 and s_2 are elements from an arbitrary alternative ring, and if $t'=\alpha[s_1, s_2]$, $u'=\beta[s_1, s_2]$, $v'=\gamma[s_1, s_2]$, then $(t', u', v')=0$. R is said to have property (P) if t, u, v an alternative triple in R and $(t, u, v)^2=0$ imply $(t, u, v)=0$. Kleinfeld (loc. cit.) proves that a right alternative ring of characteristic not two having property (P) is alternative, thus generalizing a theorem of Skornyakov that a right alternative division ring of characteristic not two is alternative. The author has generalized the Skornyakov theorem to the characteristic two case (Bull. Amer. Math. Soc. Abstract 59-4-451). This paper provides a generalization of the Kleinfeld result by proving that a right alternative ring R of characteristic two having property (P) is alternative if and only if R satisfies the identity $w(xy\cdot x)=(wx\cdot y)x$ for all w, x, y in R. (Received February 26, 1954.)

560. Dana Scott: *Prime ideal theorems for rings, lattices, and Boolean algebras.* Preliminary report.

For notation see abstracts 562–564 of Tarski. *The prime ideal theorem B_3 for Boolean algebras proves to be equivalent to each of the following statements P_1 and P_2.* (Each of these comprehends B_3 as a particular case and seems to be much stronger than B_3.) P_1. In every commutative ring with unit, every proper ideal is included in some prime ideal. P_2. Let $\langle L, \cup, \cap\rangle$ be a lattice with unit 1. In order that every proper ideal of this lattice be included in some prime ideal, it is necessary and sufficient that, for any elements x, y, z in L, the formulas $x\cup y=1$ and $x\cup z=1$ imply $x\cup(y\cap z)=1$. Consider the following statements. P_1': In every commutative ring with unit every proper ideal is included in some maximal ideal. P_2': In every lattice with unit every proper ideal is included in some maximal ideal. P_2' is known to be equivalent to AC (cf. the author's abstract 60-1-172). It is not known whether P_1' is equivalent either to P_1 or to AC. (Received March 8, 1954.)

561. J. D. Swift: *Sets of covering congruences.*

A covering congruence is defined as a finite set of ordered pairs (r_i, m_i) such that $m_i>1$ for all i; $m_i\neq m_j$, $i\neq j$; and such that any integer n satisfies (at least) one congruence of the form $n\equiv r_i \pmod{m_i}$. If min $m_i=M$, one says that the covering congruence is of order M. Sets up to order 6 are known. Infinite classes of covering congruences for small M may be given of which the simplest is: Let p be an arbitrary odd prime, g an arbitrary singly even primitive root of p, then the set $(g^{k-1}, 2^k)$, $(g^{k-1}, 2^{k-1}p)$, $(0, 2^{p-1}p)$, $1\leq k\leq p-1$, is a covering congruence. As a corollary, we may deduce: The sum of the reciprocals of the moduli of a covering congruence may be less than $1+\epsilon$ for any positive ϵ. M may be increased in various ways beginning with these sets but they do not appear to lead directly to an answer to the question: Does a covering congruence exist for arbitrary M? (Received March 10, 1954.)

562. Alfred Tarski: *Prime ideal theorems for Boolean algebras and the axiom of choice.* Preliminary report.

A statement S is said here to ***imply*** a statement T if T is derivable from S by means of usual set-theoretical axioms excluding AC (the axiom of choice). S and T are ***equivalent*** if S implies T and T implies S. Boolean algebras are regarded as systems with two operations yielding joins and meets. Set algebras are Boolean algebras where joins coincide with unions, and meets with intersections. *The following statements B_1–B_5 are mutually equivalent*: B_1. Every Boolean algebra (with $0\neq 1$) has a prime ideal. B_2. Every infinite Boolean algebra has a non-principal prime ideal. B_3. In every Boolean algebra every proper ideal is included in some prime ideal. B_4. In every Boolean algebra every principal ideal is isomorphic to a subalgebra containing 1. B_5. Every Boolean algebra is isomorphic to a set algebra. The equivalence of B_1, B_3, B_5 is a known result of Stone. As is known, AC implies B_1–B_5. The problem is open whether the converse holds. However, B_1–B_5 *cannot be proved without using AC*. For Anne

Davis showed (1948) that B_3 *implies* AC *restricted to families of finite sets*; and (by results of Mostowski) AC thus restricted is still not derivable from remaining set-theoretical axioms. (Received March 8, 1954.)

563t. Alfred Tarski: *Prime ideal theorems for set algebras and ordering principles*. Preliminary report.

For notation see abstracts 562, 564, 553. By a result of Henkin, B_1–B_5 imply the ordering principle O_1. E. W. Beth and the author found a purely mathematical proof of this implication by deriving O_1 directly from S_3. Using a similar method the author showed that S_3 *implies two further ordering principles*, O_2 *and* O_3 (of algebraic character): O_2. Every Abelian group $\langle G, +\rangle$ in which all nonzero elements are of infinite order can be simply ordered; i.e., there is a relation $\leq$ which simply orders G and under which $+$ is monotonic. O_3. Every field $\langle F, +, \cdot\rangle$ in which -1 is not a sum of squares can be simply ordered. Scott showed that S_3 *implies the following*: O_4. Every partial ordering of a set A can be extended to a simple ordering of A. O_1–O_4 were originally established by means of AC. For O_2 see Tarski, Ann. Soc. Pol. Math. vol. 7 (1929) p. 267 ff.; for O_3 see Artin and Schreier, Abh. Math. Sem. Hamburg vol. 5 (1926) pp. 83–115; for O_4 see Szpilrajn, Fund. Math. vol. 16 (1930) pp. 386 ff. The problem remains open whether any of the statements O_1–O_4 is equivalent to S_3. (Received March 8, 1954.)

564t. Alfred Tarski: *Prime ideal theorems for set algebras and the axiom of choice*. Preliminary report.

For notation see abstract 562. A set algebra is called complete if it is formed by all subsets of a certain set. Let S_1–S_4 be respectively the statements B_1–B_4 restricted to complete set algebras. Let S_5 be B_5 restricted to those Boolean algebras which are homomorphic images of complete set algebras. S_1 and S_4 are obviously provable without using AC. The author showed that S_3 *and* S_5 *are both equivalent to* B_1-B_5 (and hence *cannot be proved without using* AC). Scott noticed that, by analyzing the proof of this equivalence, *the following can be established without applying* AC: Every Boolean algebra is a homomorphic image of a set algebra. S_3 implies S_2; whether the converse holds is not known. It is unlikely that S_2 is independent of AC, for (by a result of Sierpinski) S_2 implies the existence of sets non-measurable (L). S_1–S_3 are equivalent to certain measure-theoretical statements; e.g., S_3 is equivalent to the following: Let $\mathcal{J}$ be a family of sets, $X \subseteq A$ such that A is not a union of finitely many sets $X \in \mathcal{J}$; then there is a two-valued finitely additive measure m on all sets $X \subseteq A$ such that $m(A) = 1$ and $m(X) = 0$ for every $X \in \mathcal{J}$. (Received March 8, 1954.)

565. Morgan Ward: *Mappings of the positive integers into themselves which preserve division.*

Let ϕ: $\phi(n)$ denote any mapping of the integers $0, 1, 2, \cdots$, into themselves such that $\phi(n) = 0$ if and only if $n = 0$; $\phi(1) = 1$; $\phi(n)$ divides $\phi(m)$ if n divides m. Necessary and sufficient conditions are given for ϕ to admit a rank function ρ: $\rho(m)$ such that for every modulus m, $\phi(n) \equiv 0 \pmod{m}$ if and only if $n \equiv 0 \bmod \rho(m)$. The results generalize and extend from properties of the Lucas numbers $(\alpha^n - \beta^n)/(\alpha - \beta)$. (Received April 6, 1954.)

566. P. S. Wolfe: *Some games of infinite length.*

Gale and Stewart (*Infinite games of perfect information*, Contributions to the theory of games II, Princeton, 1953, pp. 245–266) have considered zero-sum two-person games with no umpire moves having no stop rule, and ask whether such a game is strictly determined in pure strategies when the payoff is the characteristic function of a set of plays which is a G_δ in a certain topology. An affirmative answer is given by proving the strict determinateness of a related game in which the payoff from the play s is 1 when an infinite number of the positions constituting s belong to a certain subset of positions, and 0 otherwise. This result yields the strict determinateness of the two games having payoffs $\limsup_{i\to\infty} h(s(i))$ and $\alpha \sup_i h(s(i)) + \beta \inf_i h(s(i))$, where h is any bounded function of positions and $s(i)$ is the ith position in the play s. (Received March 18, 1954.)

ANALYSIS

567. H. J. Bremermann: *Complex convexity. I. Regions of holomorphy.*

Let z denote the point $(z_1, \cdots, z_n)$ in the space of n complex variables C^n, and let $\delta_D(z)$ denote the Euclidean distance of the point $z \in D$ from the boundary of D $(D \subset C^n)$. A region of holomorphy H is a region such that there exists a function holomorphic in H but singular at every boundary point. Let H be a region of holomorphy, let S, T be point sets, $S \cup T \subset\subset H$, such that for every function $h(z)$, holomorphic in H, $\max_{z \in S\cup T} |h(z)| = \max_{z\in T} |h(z)|$. Let $\chi(z)$ be a fixed function, holomorphic and different from zero in H. Then $\min_{z\in S\cup T} |\chi(z)| \delta_H(z) = \min_{z \in T} |\chi(z)| \delta_H(z)$. Using this theorem it is easy to derive the holomorph-convexity of Cartan-Thullen as well as certain new results concerning the holomorph convexity. Also a new and simpler proof of the following theorem is given: If H is a region of holomorphy, $-\log \delta_H(z)$ is a plurisubharmonic function in H. The function $-\log \delta_H(z)$, being plurisubharmonic, generates a positive semidefinite metric form: $ds^2 = \sum_{\mu,\nu=1}^{n} (\partial^2(-\log \delta_H(z))/\partial z_\mu \partial \bar{z}_\nu), dz_\mu d\bar{z}_\nu$. Other metrics and distance functions are studied and compared with the Bergman kernel function and the holomorphically invariant metric form generated by it. (Received May 1, 1954.)

568t. H. J. Bremermann: *Complex convexity. II. Pseudo-convex regions.*

A region $D \subset C^n$ which has the property that $-\log \delta_D(z)$ is plurisubharmonic in D is called "pseudo-convex." Thus any region of holomorphy is pseudo-convex. (The converse has also been proved by the author.) It is shown that the notions "pseudo-convex region" and "plurisubharmonic function" are an extension of the notions "convex region" and "convex function" from the real space R^n to complex space C^n (in the same sense as the notion "Hermitian form" is an extension of the notion "quadratic form" from the R^n to C^n). Also, by associating with every convex region a pseudo-convex tube region and with every convex function a plurisubharmonic function defined in a tube region, it is proved that any theorem that is true for all pseudo-convex regions and for all plurisubharmonic functions is in a corresponding form true for all convex regions and convex functions. (Received May 1, 1954.)

569t. H. J. Bremermann: *Complex convexity. III. Bergman kernels and other kernel functions.*

The correspondence theorem in II leads the author to consider several complex variables methods in the convex case. In particular, he interprets the Bergman kernel function and invariant metric in the convex case. It is also shown that the non-negative convex functions as well as the non-negative plurisubharmonic functions form a Hilbert space with a reproducing kernel, if one introduces the inner product $(f, f) = \int_D f^2 d\omega$ ($d\omega$ the volume element). The existence of such a kernel follows immediately from the fact that the inequality $f^2(x^{(0)}) \leqq v_n(r)^{-1}\int_{|x-x^{(0)}|<r} f^2 d\omega_n$ holds in the convex case ($v_n(r)$ the volume of the n-sphere), and the corresponding inequality $f^2(z^{(0)}) \leqq v_{2n}(r)^{-1}\int_{|z-z^{(0)}|<r} f^2 d\omega_{2n}$ holds in the pseudo-convex case. Finally, he finds the analogue to "Bergman's extended class of functions" in the convex case, and by reversing the method comes to define a different extended class for (pseudo-convex) analytic polyhedra in the C^n. (Received May 1, 1954.)

570. M. A. Dengler: *Laplace transforms related to the theory of transverse waves in elasticity.*

The object of the investigation is the derivation of certain Laplace transforms which are directly related to the theory of transverse waves in elastic media as deduced by Goland and the author in *Transverse impact of long beams, including rotatory inertia and shear effects*, Proceedings of the First U. S. National Congress for Applied Mechanics, Chicago, 1950. The theory referred to here is based on the Timoshenko bending mechanism which, as well known, accounts for the influences of rotatory inertia and transverse shear and therefore permits the calculation of the transient response of elastic structures with satisfactory accuracy. The Laplace transforms presently reported must be considered as the central domain of these theoretical studies, when seen from the mathematical point of view. Relations between transforms and inverse transforms are derived by contour integrations and appear in closed and real form. (Received March 1, 1954.)

571. Solomon Leader: *Convergence topologies for measures.*

In the norm topology for signed measures there may exist functionals which cannot be represented as integrals in the original space. This defect is removed by introducing weaker topologies defined by convergence of nets, so that functionals can be expressed as integrals and convergence of positive measures is equivalent to weak convergence. The space S of real-valued, additive functions on a Boolean algebra of sets with the topology of simple convergence [$F_\alpha \to 0$ if $F_\alpha(E) \to 0$ for each E] has as dual space all step functions. The space B of bounded functions in S with the topology of bounded convergence [$F_\alpha \to 0$ if $F_\alpha(E) \to 0$ and $\lim \sup_\alpha \|F_\alpha\|$ is finite] has as dual space uniform limits of step functions. Similar results hold for countably additive measures. The space R of bounded, regular, signed measures on a normal space with the topology of regular convergence [for positive measures, $F_\alpha \to F$ if $F_\alpha(E) \to F(E)$ wherever $F(\text{bdy } E) = 0$] has as dual space bounded, continuous functions. Continuous operators are characterized through the invariance of the dual space under the adjoint operators. This gives an existence theorem for transition probabilities generating a stochastic semigroup. This work was supported by the Research Council of Rutgers University. (Received March 1, 1954.)

572t. A. R. Lovaglia: *Locally uniformly convex Banach spaces.*

In this paper a normed linear space is defined to be *locally uniformly convex* (l.u.c.) if given $\epsilon > 0$ and an element x with $\|x\| = 1$, there exists $\delta(\epsilon, x) > 0$ such that $\|x+y\|/2 \leqq 1-\delta(\epsilon, x)$ whenever $\|x-y\| \geqq \epsilon$ and $\|y\| = 1$. Let $\{B_n\}$ be a sequence of l.u.c. Banach spaces. Let $P_p(B_n)$ be the space of all sequences $x = \{x^n\}$, $x^n \in B_n$, for which $\sum_{n=1}^{\infty}\|x^n\|_n^2$ is convergent. ($\|\ \|_n$ is the norm in B_n.) Let $\|x\|=(\sum_{n=1}^{\infty}\|x^n\|_n^2)^{1/2}$. Then $P_2(B_n)$ is a l.u.c. Banach space. It is shown that there exist l.u.c. Banach spaces not isomorphic to any uniformly convex Banach space. (J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. vol. 40 (1936) pp. 396–414.) From this it follows that the notion of l.u.c. is essentially different from M. M. Day's notion of *local uniform convexity near a point b_0*. (M. M. Day, *Uniform convexity*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 745–750.) If B is a l.u.c. Banach space and linear functionals attain their maximum on the unit sphere, then the norm in the adjoint space $\bar{B}$ is strongly differentiable. (V. Smulian, *Sur la dérivabilité de la norme dans l'espace de Banach*, C. R. (Doklady) Acad. Sci. URSS vol. 27 (1940) pp. 643–648.) If the adjoint space $\bar{B}$ is l.u.c. then the norm in B is strongly differentiable. If B has a basis $\{x_i\}$ with the property that for each number $c>0$ there exists a number $r_c>0$ such that $\|\sum_{i=1}^{n}a_ix_i\|=1$ and $\|\sum_{i=n+1}^{\infty}a_ix_i\|\geq c$ imply $\|\sum_{i=1}^{\infty}a_ix_i\|\geq 1+r_c$, then B is isomorphic to a l.u.c. space. (Received March 18, 1954.)

573. Michael Maschler: *On properties of minimal domains.*

By using the method of the kernel function, S. Bergman introduces the mapping of a domain onto a "minimal domain." A minimal domain D in the z-plane, with respect to $z=t\in D$, is a bounded domain such that any function $f(z)$ $[f(t)=0, f'(t)=1]$, regular in D, maps it onto a domain whose area is not smaller than the area of D. While every simply-connected minimal domain is a circle, a multiply-connected minimal domain is in general, according to an unpublished result of M. Schiffer, non-schlicht. A minimal domain is defined in the space of n complex variables in a similar way. The results stated below are special cases of analogous theorems in that space: A necessary and sufficient condition for a domain D to be a minimal domain w.r.t. t is that its Bergman's kernel function $K_D(z, t)=$constant. This constant must be $1/S$ where S is the area of D. For a minimal domain D w.r.t. t, and only for such a domain, $S\cdot f(t)=\iint_D f(z)dxdy$ $(z=x+iy)$, for every $f(z)\in L^2(D)$. These results lead to the determination of various properties of minimal domains and have applications in the theory of conformal and pseudo conformal transformations. (Received March 10, 1954.)

574. E. G. Straus: *On integer-valued entire functions.*

It is shown that an entire function $f(z)$ which satisfies $f(z)=O(e^{|z|^{m-1}})$, and all whose derivatives at m points $z_1, \cdots, z_m$ are rational integers, must satisfy a linear differential equation with integral constant coefficients. More precisely, if one sets $z_1=0$, then $f(z)$ is either (i) a polynomial with rational coefficients, or (ii) $\sum c_ke^{kz}$, c_k rational, or (iii) $\sum(c_k\exp(km^{1/2}z)+\bar{c}_k\exp(-km^{1/2}z))$ where m is a rational integer and c_k, $\bar{c}_k$ are conjugate elements of $R(m^{1/2})$. On the other hand there is a continuum of entire functions of any order $\rho\geq m$ which have integral derivatives of all orders at m prescribed points. Generalizations, such as replacing "rational integer" by "algebraic integer" or even "algebraic number" are possible. (Received May 1, 1954.)

APPLIED MATHEMATICS

575. K. O. Friedrichs and D. H. Hyers (p): *The existence of solitary waves.*

Recently M. Lavrentieff (Akad. Nauk Ukrain RSR Sbirnik Prac. Inst. Mat. vol. 1946 no. 8 (1947) pp. 13–69; cf. Math. Revs. vol. 14, p. 102) has given a proof of the existence of two-dimensional solitary gravity waves of permanent type by considering them as limiting cases of periodic waves as the wavelength tends to infinity. In the present paper their existence is proved in a direct manner. Introducing the dimensionless parameter $\gamma = ghU^{-2}$ where g is the acceleration of gravity, h the depth of liquid at infinity, and U the velocity of propagation, it is shown that for each value of $\gamma < 1$ and sufficiently close to one, a symmetric solitary wave exists. A decisive step in the method is the introduction of new independent variables involving a stretch in one direction, analogous to the change of variables used by Friedrichs in his derivation of the shallow water theory (Com. Pure and Appl. Math. vol. 1, p. 81) and by Keller in his approximate treatment of solitary and periodic waves (ibid., vol. 1, p. 323). Thus, in a sense, the result affords a justification of the second order shallow water theory for this problem. (Received March 10, 1954.)

576*t*. C. H. Papas: *An application of Sommerfeld's complex order wave functions to antenna theory.*

In the past wave functions of integral order have been used advantageously in the solution of boundary-value problems. However, in some instances these wave functions are completely alien to the problem and introduce difficulties which, indeed, can be resolved but only at the expense of logical simplicity. To place in evidence the usefulness and the "naturalness" of complex order wave functions (A. Sommerfeld, *Partial differential equations*, New York, Academic Press, 1949), the input impedance of a boss antenna is examined theoretically with the aid of these functions. Following Schwinger's formalism (H. Levine and C. H. Papas, Journal of Applied Physics vol. 22, pp. 29–43) the input impedance is cast in variational form. This form involves a Green's function and an aperture electric field, both of which are expressed in terms of complex order wave functions. A physically plausible choice of trial functions yields a rather neat expression for the input impedance. (Received January 18, 1954.)

577*t*. Gabor Szegö: *Inequalities for certain eigenvalues of a membrane of given area.*

Let the domain D with an analytic boundary curve C represent the membrane in its position of equilibrium. We denote by λ_1 the least positive eigenvalue of D with boundary condition $u = 0$ and by μ_2 its least positive eigenvalue with boundary condition $\partial u/\partial n = 0$. If A is the area of D, we have the Rayleigh-Faber-Krahn inequality $\lambda_1 \geqq j(A/\pi)^{-1/2}$ where $j = 2.40$ is the least positive zero of $J_0(x)$. In the present paper an inequality of opposite nature is proved for μ_2 which was announced without proof by E. T. Kornhauser-I. Stakgold (Journal of Mathematics and Physics vol. 31 (1952) p. 45). This is the following: $\mu_2 \leqq p(A/\pi)^{-1/2}$ where $p = 1.84$ is the least positive zero of $J_1'(x)$. In the remaining part of the paper the next two eigenvalues λ_2, λ_3 and μ_2, μ_3 for the boundary problems mentioned above are evaluated in a certain approximate sense provided C is a "nearly circular" curve. (Received February 4, 1954.)

578. W. R. Wasow: *Discrete approximations to elliptic differential equations.*

The method of Collatz [Zeitschrift für Angewandte Mathematik und Mechanik vol. 13 (1933) pp. 56–57] for the appraisal of the truncation error in the solution of boundary value problems for elliptic differential equations by means of finite differences is generalized so as to apply to general types of nets and to a general class of interpolation schemes at curved boundaries. Following Petrovsky the whole class of discrete approximations considered can be treated in a unified fashion by using

Stieltjes integral notation. The theory applies also to the discretization error connected with solutions of boundary value problems by random walks that are not restricted to the points of any net. With the help of the results obtained the extrapolation technique of L. F. Richardson [Phil. Trans. (A) vol. 210 (1911) pp. 307–357] for the improvement of approximations obtained by finite difference methods can be analyzed mathematically. It turns out that this method is applicable to problems in domains with curved boundaries only if the boundary values are interpolated by a formula so precise that the resulting contribution to the truncation error is small of higher order than the contribution originating in the interior of the domain. (Received February 16, 1954.)

LOGIC AND FOUNDATIONS

579. Chen-Chung Chang: *A remark on convex classes.*

For notation see Tarski, Abstract 59-4-500. A class $\mathcal{K}$ of non-empty algebraic systems (with operations and relations) is called (i) *convex*, or (ii) *weakly convex*, if for every $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C} \in \mathcal{K}$ such that $\mathfrak{B}$, $\mathfrak{C}$ are subsystems of $\mathfrak{A}$, (i) the intersection $\mathfrak{B} \cap \mathfrak{C}$ belongs to $\mathcal{K}$, or (ii) $\mathfrak{B} \cap \mathfrak{C} \in \mathcal{K}$ whenever $\mathfrak{B} \cap \mathfrak{C} \neq 0$. Theorems: I. If $\mathcal{K} \in AC_\delta$ and $\mathcal{K}$ is (i) convex, or (ii) weakly convex, then for any nonempty subclass $\mathcal{L}$ of $\mathcal{K}$ consisting of subsystems of a system $\mathfrak{A} \in \mathcal{K}$, (i) the intersection $\bigcap \mathcal{L}$ belongs to $\mathcal{K}$, or (ii) $\bigcap \mathcal{L} \in \mathcal{K}$ whenever $\bigcap \mathcal{L} \neq 0$. II. If $\mathcal{K} \in AC_\delta$ and $\mathcal{K}$ is convex, or even weakly convex, then, for every non-empty subclass $\mathcal{L}$ of $\mathcal{K}$ which is simply ordered by the subsystem relation, the union $\bigcup \mathcal{L}$ belongs to $\mathcal{K}$. Theorems I and II remain true if the formula $\mathcal{K} \in AC_\delta$ is replaced by the condition that $\mathcal{K}$ is quasi-arithmetical in the sense of Henkin (Trans. Amer. Math. Soc. vol. 74, p. 418 ff.) or, more generally, pseudo-arithmetical in the sense of Tarski (*Contributions to the theory of models*, to appear in Proc. Dutch Acad.). Theorem II improves a result in Robinson, *On the metamathematics of algebra*, 1951, p. 117, 9.1.3., who uses a much stronger definition of convexity. (Received March 8, 1954.)

580*t*. Roland Fraïssé: *Some elementary properties of universal classes.*

Given two relational systems $\mathfrak{A}$, $\mathfrak{B}$, the formula $\mathfrak{B} < \mathfrak{A}$ means that $\mathfrak{B}$ can be isomorphically embedded in $\mathfrak{A}$. A relational system $\mathfrak{A} = \langle A, R_1, \cdots, R_n \rangle$ will be called *finite* if the set A is finite. The following are proved (for notation, see Tarski, Bull. Amer. Math. Soc. Abstract 59-4-500): (1) A class K of relational systems is a UC (a UC_δ) iff there exists a finite (a denumerable) set $\{\mathfrak{F}_1, \cdots, \mathfrak{F}_m, \cdots\}$ of finite relational systems, such that $\mathfrak{A} \in K$ iff $\mathfrak{A} \not> \mathfrak{F}_i$ $(i = 1, \cdots, m)$. (2) One can always assume the $\mathfrak{F}_i$'s are mutually incomparable ($\mathfrak{F}_i \not< \mathfrak{F}_j$ when $i \neq j$); then the $\mathfrak{F}_i$'s are all the finite relational systems (defined up to isomorphism) which do not belong to K and such that every proper subsystem of each of them belongs to K. These $\mathfrak{F}_i$ will be called *associated* with K. (3) If $K, L \in UC_\delta$, then $L \subseteq K$ iff, for every $\mathfrak{F}_i$ associated with K, there exists $\mathfrak{G}_j < \mathfrak{F}_i$ ($\mathfrak{G}_j$ associated with L). (4) The formula $\mathfrak{B} \prec \mathfrak{A}$ means that every finite subsystem of $\mathfrak{B}$ can be isomorphically imbedded in $\mathfrak{A}$; it is equivalent to say that every UC which contains $\mathfrak{A}$ contains also $\mathfrak{B}$. Assuming $K \in UC$, K is indecomposable for $\bigcup$ (i.e. it is not any union of UC's different from K) iff there exists a relational system $\mathfrak{K}$ such that $\mathfrak{A} \in K$ iff $\mathfrak{A} \prec \mathfrak{K}$. (Received March 8, 1954.)

581. Roland Fraïssé: *Some relational systems which generalise simply ordering relations.*

A relational system $\mathfrak{A} = \langle A, R_1, \cdots, R_n \rangle$ will be called *p-monotypic* (n, p natural numbers) if all subsystems of $\mathfrak{A}$ having p elements are isomorphic. $\mathfrak{A}$ will be called *monotypic* if it is p-monotypic for every p (A being infinite). The following are proved: (1) If, A being infinite, $\mathfrak{A}$ is p-monotypic, then it is q-monotypic for $q<p$. This result fails if A is finite. (2) If $\mathfrak{A}=\langle A, R_1, \cdots, R_n\rangle$ is monotypic, there exists an ordering (i.e. simply ordering) relation I defined on A, such that $\langle A, R_1, \cdots, R_n, I\rangle$ is monotypic. (3) Let I, J be two ordering relations such that $\mathfrak{B}=\langle A, R_1, \cdots, R_n, I\rangle$ and $\mathfrak{C}=\langle A, R_1, \cdots, R_n, J\rangle$ are monotypic. Either, for every p, the subsystems of $\mathfrak{B}$ and $\mathfrak{C}$ having p elements are all isomorphic, or this is true when J is replaced by its converse. (4) If $\mathfrak{A}$ is monotypic and I an ordering relation as in (2), then each relation R_i $(1\leqq i\leqq n)$ is in the least class K of relations, such that the identity relation, I, and A^k $(k=1, 2, \cdots)$ are in K, and such that K is closed under the following operations: complementation, formation of union of two elements, permutation of arguments, passing from an m-ary relation R to an $(m+1)$-ary relation R' where $R'(x_1, \cdots, x_m, x_{m+1})$ holds iff $R(x_1, \cdots, x_m)$ holds. (Received March 8, 1954.)

582*t*. Herman Rubin and P. C. Suppes: *A note on two-place predicates and fitting sequences of measure functions.*

The main purpose of this note is to show by means of a counterexample that Carnap's measure function m^* (*Logical foundations of probability*) does not have the intuitively desirable property of fittingness when we consider finite languages using two-place predicates. Following Carnap, the notation: $1m^*$, $2m^*$, $\cdots$ is used for the measure function m^* in languages $\mathcal{L}_1$, $\mathcal{L}_2$, $\cdots$, where the only predicate in $\mathcal{L}_i$, for $i=1, 2, \cdots$, is the two-place predicate 'R'. Let 'a' and 'b' be two individual names and let i be the sentence: *Raa* & *Rab* & *Rba* & *Rbb*. We easily verify that $2m^*(i)=.1$, and with somewhat more trouble that $3m^*(i)=23/312$. Hence the sequence $1m^*$, $2m^*$, $\cdots$ is not fitting. A group-theoretic theorem which guides the construction of such counterexamples is stated and proved, and is used in conjunction with results of Davis (Proc. Amer. Math. Soc. vol. 4 (1953) pp. 486–495) to show that $4m^*(i)$ is neither of the previous values. (Received March 5, 1954.)

583. A. C. Sugar: *A foundational analysis of Lagrange's equations.*

Beginning with a set theoretic definition of function and a modified form of Menger's notation the ambiguity of the conventional notation for functions is eliminated. Subsequently the question is raised as to what is meant in the standard derivations of Lagrange's equations by $\partial x/\partial\dot{q}$ and $\partial\dot{x}/\partial q$, where $\dot{x}=f'(q)\dot{q}$. Since the above notation indicates that $\dot{x}$ is to be differentiated, then "$\dot{x}$" is being used to designate a function, say F. On the basis of the above indicated operation of partial differentiation, F must be a function of more than one argument and therefore is not a function of time only. Furthermore F cannot be a function of q and $\dot{q}$ because these arguments are not independent as required by the definition of partial differentiation. These difficulties concerning $\dot{x}$ will appear of course also for kinetic energy. To illustrate (still using conventional notation) let the position $x=q^2$ and the generalized coordinate $q=e^t$; then the kinetic energy may, among others, be given by $T=2mq^2\dot{q}^2=2mq^4=2m\dot{q}^4$. It is interesting to ask what $\partial T/\partial q$ represents in this case. The physicist would take a "derivative" of the function defined by the first of the three expressions. A precise derivation of Lagrange's equations may be based on either of two points of view. We are either differentiating a generalized kinetic energy function or carrying out a process of symbolic partial differentiation of a specified kinetic energy function of two dependent arguments. (Received March 8, 1954.)

584*t*. P. C. Suppes: *Axiomatic analysis of the method of equal-appearing intervals.*

In this paper a set of independent axioms is given which characterizes the procedures of measurement that psychologists lump together under the title of the method of equal-appearing intervals. The axioms are based on three primitive notions: (1) a finite set K, ordinarily a set of tones or colors, or in the case of utility, economic goods; (2) a binary relation P of precedence whose field is K; and (3) a quaternary relation E whose fields is K such that in the intended interpretation $x, y \mathrm{E} u, v$ only if the difference between x and y is equivalent to that between u and v. The expected adequacy theorem is established, namely that: (A) there exists a real-valued function ϕ defined on K such that (i) $x \mathrm{P} y$ if and only if $\phi(x)<\phi(y)$, and (ii) $x, y \mathrm{E} u, v$ if and only if $\phi(x)-\phi(y)=\phi(u)-\phi(v)$; (B) ϕ is unique up to a linear transformation. It is also proved that it is not possible to add further axioms based on these primitive notions so as to strengthen (B) of the adequacy theorem to: (B′) ϕ is unique up to a similarity transformation. Due to the assumption that K is finite and that differences between adjacent elements are equivalent, the axiomatization and proof of adequacy given here are considerably simpler than that given for more general situations in Suppes and Winet (Bull. Amer. Math. Soc. Abstract 60-1-167). (Received March 8, 1954.)

TOPOLOGY

585*t*. C. E. Burgess: *Some properties of certain types of homogeneous continua.*

A continuum M is said to be n-homogeneous if for any n points $x_1, x_2, \cdots, x_n$ of M and any n points $y_1, y_2, \cdots, y_n$ of M there is a homeomorphism of M onto itself carrying $x_1+x_2+\cdots+x_n$ onto $y_1+y_2+\cdots+y_n$. In the author's paper *Some theorems on n-homogeneous continua*, Bull. Amer. Math. Soc. Abstract 60-3-309, it was shown that if $n>1$ and the bounded plane continuum M is n-homogeneous, then M is a simple closed curve. If this hypothesis is changed so that M is an *unbounded* proper subcontinuum of the plane, then M is homeomorphic with a straight line. The following theorems are also proved. (1) If every proper subcontinuum of the compact metric continuum K is homogeneous, then K is hereditarily indecomposable. (2) If K is a decomposable bounded plane continuum such that for any two nondegenerate proper subcontinua K_1 and K_2 of K there is a homeomorphism of K onto itself carrying K_1 onto K_2, then K is a simple closed curve. (Received March 1, 1954.)

586. W. S. Miller: *A non-Hausdorf, finest bicompact space.*

A space is constructed from the closed unit circle with the origin deleted, and two points O and O'. Neighborhoods of the points O and O' are defined so that they have no disjoint neighborhoods, so that the space is bicompact, and such that every bicompact set in the space is closed. (Received March 10, 1954.)

587*t*. Barrett O'Neill: *A comparison fixed point theorem.*

A topological space X is called a Lefschetz space provided the Lefschetz theorem holds for X in this sense: if f is a self-map of X for which the Lefschetz number exists and is nonzero, then f has a fixed point. Let X and Y be compact spaces. Write

$X < Y$ provided: given any element U of the uniform structure on X there is a homology equivalence, $f: X \to Y$, $g: Y \to X$, such that $g(f(x)) \in U(x)$ for all $x \in X$. This relation is an order relation on the class of compact spaces. This theorem is proved: if $X < Y$ and Y is a Lefschetz space, then so is X. The criterion is applied to give examples of non-HLC spaces which are Lefschetz spaces and, in particular, have the fixed point property. (Received March 9, 1954.)

J. W. GREEN,
Associate Secretary

RESEARCH PROBLEMS

19. Walter Rudin: *Maximum modulus algebras.*

Let D be a domain bounded by a simple closed curve C, and let $K = D \cup C$. In Duke Math. J. vol. 20 (1953) pp. 449–458, the following theorem is proved: Let A be an algebra of complex-valued functions continuous on K and suppose (1) for every $f \in A$ there is a point $z_0 \in C$ such that $|f(z)| \leq |f(z_0)|$ $(z \in K)$; (2) A contains a non-constant function which is analytic in D; (3) A contains a schlicht (that is, one-to-one) function. Then every $f \in A$ is analytic in D. Is the conclusion valid if (3) is omitted from the hypotheses? Is it possible to weaken (3), for instance by requiring that A separates points? (Received April 6, 1954.)

20. Walter Rudin: *Radial limits of analytic functions.*

If f is analytic in the interior U of the unit circle, and if there exists a set E of positive measure such that $f(re^{i\theta})$ is bounded for $0 \leq r < 1$, $\theta \in E$, does $\lim_{r \to 1} f(re^{i\theta})$ necessarily exist for almost all $\theta \in E$? The same question may be asked about functions meromorphic in U. (Received April 6, 1954.)

BOOK REVIEWS

Stability theory of differential equations. By Richard Bellman. New York, McGraw-Hill, 1953. 13+166 pp. $5.50.

This is the first book in the English language on the topic. In fact, while many aspects of stability theory have been summarized, notably by Russian authors, no mathematically satisfactory treatise on the subject has appeared since the classical work of Liapounoff. It is the reviewer's feeling, therefore, that any new book on stability must, at least partially, be judged on how it summarizes and organizes the highlights of the work done in this field since Liapounoff.

Bellman's book is organized in seven chapters. The first and third chapters are introductions to the theory of linear and nonlinear differential equations, respectively. Matrix and vector notation is used throughout these two well-written chapters. Chapter two deals with questions of stability, boundedness, and asymptotic behavior of linear differential equations, many of them originally investigated by Bellman himself. Chapter four is the heart of the book. Three different proofs are given for the stability theorem for the case where the linearized system has constant coefficients. The proofs are clear. A clever counterexample is provided to show that one cannot hope to extend the theorem without modification to the case where the linearized system does not have constant coefficients. In chapter five the asymptotic behavior of certain first order nonlinear equations is considered. Chapter six deals with linear second order equations. After an introduction to the more elementary properties of linear equations the author proves certain theorems about the boundedness in norm of solutions using various norms. Chapter six closes with a study of the oscillatory and asymptotic behavior of the solutions. Finally, chapter seven discusses the linear second order equation,

$$\frac{d}{dt}\left(t^{\rho}\frac{du}{dt}\right) \pm t^{\sigma}u^{\mu} = 0.$$

It is shown that certain conclusions with regard to the nature of solutions can be drawn from a knowledge of ρ, σ and μ.

The book as a whole is well-written and quite readable. There are many exercises to supplement the text. However, it seems unfortunate that the author excluded certain aspects of stability theory which, in a book dedicated to this field, seem more important than chapters five and seven. There is hardly anything in the book, for instance, on the stability of periodic solutions, or in the sixth chapter on the second order linear equation with periodic coefficients. There is comparatively little reference to work done in the last ten years either in this country or abroad. Aside from these omissions, however, Bellman's book is a pleasant and interesting contribution to the theory of differential equations.

F. HAAS

Discontinuous automatic control. By I. Flügge-Lotz. Princeton University Press, 1953. 8+168 pp. $5.00.

Although self-regulating devices have been in operation since the days of the governor on Watt's steam engine, it is only in recent years that the subject of automatic control has assumed a central position in the engineering and industrial world.

From the mathematical side, the control problem leads to systems of nonlinear differential equations in the following way. If we assume that the state of the physical system is specified at time t by the vector $x(t)$, the study of small displacements from equilibrium gives rise, in a system without control, to a linear vector-matrix equation $\dot{x}=Ax$. If we now consider a system with control, where the control is manifested by a forcing term and the magnitude of the control is dependent upon the state of the system, the resulting equation for x has the form $\dot{x}=Ax+f(x)$, and is, in general, nonlinear.

The term "continuous control" will be used to describe situations in which $f(x)$ is a continuous function of x. In many cases, it was found that continuous control was far too expensive to use. In place then of control devices which gave rise to forcing terms of continuous type, it was far cheaper to design control devices yielding forcing terms whose components are step-functions of x. The simplest version of this type of control system is one with a simple on-and-off control mechanism. This type of control is called "discontinuous automatic control."

A simple example of the mathematical equations which result is the following second order equation, $\ddot{u}+a\dot{u}+bu=c \operatorname{sgn} (\dot{u}+ku)$, where u is now a scalar function. This equation has the form $\ddot{u}+a\dot{u}+bu=c$, over the region of phase space described by $\dot{u}+ku>0$, and the form $\ddot{u}+a\dot{u}+bu=-c$, over the region of phase space described by $\dot{u}+ku<0$. If $\dot{u}+ku=0$, the forcing term is taken to be zero.

We observe then the interesting fact that while the equation itself is nonlinear, over the regions $\dot{u}+ku\gtrless 0$, u may be determined as a solution of a linear equation, albeit a different linear equation over different regions.

This reduces the study of the asymptotic behavior of the solutions of the original nonlinear equation to the study of the iteration of two explicit transformations. These are of simple enough analytic form to permit the use of graphical analysis with great effect. It is this approach which the author has exploited.

Equations of this quasi-linear type are of great interest from the theoretical point of view since they furnish a vital link between the well-regulated world of linearity and the chaotic universe of nonlinearity. It is therefore a valuable contribution to the theory of nonlinear differential equations to have the behavior of the solutions of an important class of these equations presented in as complete and systematic a fashion as is done by the author. References to rigorous proofs of results used in the text, which is aimed at the engineer who must use mathematics, rather than the mathematician who is poaching in the domain of the engineer, are given throughout, particularly to papers of Bilharz, Klotter, Hodapp, and Scholz.

The occurrence of retarded control, which introduces a time-lag in the exertion of the forcing term, gives rise to differential-difference equations in place of the conventional differential equation. There is a brief treatment of this phenomenon in this volume. Those interested in further discussion of the mathematical and engineering consequences of retardation may wish to refer to the papers of Minorsky, cf. Journal of Applied Physics vol. 19 (1948) pp. 332–338, where further references may be found.

The last part of the volume treats the problem of the control of a missile, a problem involving more than one degree of freedom.

The editors of the Princeton University Press are to be congratulated upon adding another attractive and interesting volume to their series on nonlinear mechanics.

RICHARD BELLMAN

Complex variable theory and transform calculus. By N. W. McLachlan. 2d ed. Cambridge University Press, 1953. 11+388 pp. $10.00.

This book is the second edition of a text first published in 1939 (reviewed in Bull. Amer. Math. Soc. vol. 47 (1941) pp. 8–10). The principal changes are in the early, function-theoretic part. The author says that his exposition should now be "rigorous enough for all but the pure mathematician (to whom the book is not addressed)." On the whole this claim seems justified, in some instances more than justified, as on p. 116 where the continuity of a particular entire function is elaborately discussed. There are still mathematical obscurities. For instance, the definition of regular makes $z^{3/2}$ regular at the origin; the explanation on p. 341 why a closed interval is used in the definition of uniform convergence is just plain wrong.

There are serious objections against the first three chapters of the book. These concern the choice of material and the manner of presentation. It is misleading to call "Introduction to Complex Variable Theory" what is really an introduction to the technique of Contour Integration. Even the student primarily interested in technical applications should be treated to a less narrow approach (e.g. conformal mapping should be discussed; it would even be useful in the present text). The beginner must find the book very hard to follow. It is badly organized: definitions of technical terms are sometimes not given ("phase," "index") or given in passing after their first use (behavior at ∞, O-notation). Basic results are barely mentioned in small print (Cauchy-Riemann equations, regularity of power series) or used without explicit mention (differentiability of rational functions) or hinted at obscurely (behavior near a pole).

On the other hand there are unnecessary repetitions, because results are not formulated explicitly for further reference, or are not referred to, even if formulated (discussion of change of path of integration). In the reviewer's opinion the book would be improved by replacing the first three chapters by a paragraph stating the principal theorems used and referring for their complete statement and proof to one of the many good textbooks of Complex Variables.

As soon as the author comes to his subject proper, the Laplace Transform, the book rapidly improves. The systematic treatment of the complex inversion formula for functions with branch points and the discussion of asymptotic series and approximations will have points of interest and novelty even for skilled contour-integrators.

The last two parts of the book deal, mathematically, with ordinary, linear differential equations with constant coefficients, with the telegraph equation and the diffusion equation. The main interest here lies in the very well chosen, technical examples, their complete solution and comparison with experiment. The "unaddressed pure mathematician," especially if he is connected with the teaching of engineers, should derive real enjoyment and benefit from eavesdropping on these discussions of engineering applications. The excellence of these portions of the book more than compensates for the shortcomings of the beginning.

It is a pity that the application of the Laplace transform to differential equations with variable coefficients or to other types of functional equations has not been illustrated at all.

W. H. J. FUCHS

Curvature and Betti numbers. By K. Yano and S. Bochner. (Annals of Mathematics Studies, no. 32.) Princeton University Press, 1953. 10 and 3–190 pp. $3.00.

This book is a unified and systematic account of a number of contributions to differential geometry in the large which have appeared in papers of Bochner over the past several years and in some recent papers of Yano, Lichnerowicz, and others. The well-spring of all these results is the theorem of E. Hopf-Bochner (an application by Bochner of a theorem of E. Hopf) which states that if ϕ, a function defined on a compact Riemannian manifold, has positive (negative) semidefinite Laplacian, i.e., using semicolons to denote covariant differentiation, $\Delta\phi = g^{ij}\phi_{;i;j} \geqq 0$ $(\leqq 0)$, then $\Delta\phi$ vanishes and ϕ is a constant. Then this is applied to obtain necessary conditions for the existence of various types of tensor fields by considering $\Delta\phi$ for the function $\phi = \xi^{i_1 \cdots i_p}\xi_{i_1 \cdots i_p}$, where $\xi^{i_1 \cdots i_p}$ are the contravariant components and $\xi_{i_1 \cdots i_p}$ the covariant components of the same tensor, and deriving conditions under which $\Delta\phi \geqq 0$ and $\phi = 0$. Although this in barest outline is the basic approach, in each application a study of the expression $\Delta\phi$ is required, and many technical difficulties arise in the search for conditions on the metric sufficient to insure definiteness of $\Delta\phi$ for various classes of tensors. Since $\Delta\phi$ involves the second covariant derivatives, interchange of the order of differentiation via the Ricci formula introduces the curvature tensor. Thus for example in the first case treated by the authors, that of vector fields ξ_i, they obtain easily

$$\text{(i)} \qquad \frac{1}{2}\Delta\phi = \xi^{i;j}\xi_{i;j} + R_{ij}\xi^i\xi^j$$

if ξ_i is a harmonic vector field, i.e., if $\xi^i{}_{;i} = 0 = \xi_{i;j} - \xi_{j;i}$, and

$$\text{(ii)} \qquad \frac{1}{2}\Delta\phi = \xi^{i;j}\xi_{i;j} - R_{ij}\xi^i\xi^j$$

if ξ_i is a Killing vector field, i.e., if $\xi_{i;j} + \xi_{j;i} = 0$. From (i) follows Myers' theorem that the first Betti number of a compact manifold of positive Ricci curvature is zero, and from (ii) follows that there exists no one-parameter group of isometries on a compact manifold of negative Ricci curvature. These results, although for the simplest case, are quite typical. Thus in the following chapters tensors of higher order are considered and more complicated statements generalizing (i) and (ii) are obtained. It is not always possible to give simple criteria for the definiteness of $\Delta\phi$, but the special cases of flat manifolds and manifolds of constant curvature are considered and results on the existence of harmonic and Killing tensors obtained. Several measures of deviation from constancy of curvature are defined and it is shown that if this deviation remains within certain limits the Betti numbers are unaltered. To the reviewer these relations between curvature and Betti numbers are among the most interesting consequences of the theory.

A chapter is devoted to the special case of semi-simple group manifolds, and here the deviation from flatness is explicitly calculated. Following this is a chapter on Riemannian manifolds carrying additional structure in the form of an affine connection with torsion (as in the case of the group space). A chapter on Kähler manifolds completes the main body of the book. This latter chapter includes important applications of the theory; for example, it is shown that if the deviation of the curvature of a Kähler manifold from constant positive holomorphic curvature remains within prescribed limits, then there are no effective harmonic tensors, and hence the Betti numbers are those of the complex projective space.

A final chapter by S. Bochner entitled "Supplements" is perhaps the most significant since it contains indications of new directions in which the theory is proceeding. Important as it is, however, the topics are so diverse as to make a brief summary impossible. The titles of the eight sections are: (1) Symmetric Manifolds, (2) Convexity, (3) Minimal Varieties, (4) Complex Imbedding, (5) Sufficiently Many Vector or Tensor Fields, (6) Euler-Poincaré Characteristic, (7) Noncompact Manifolds and Boundary Values Zero, (8) Almost Automorphic Vector and Tensor Fields.

The book contains in addition to the above a chapter outlining the relevant differential geometry and tensor analysis and a brief introduction to complex-analytic manifolds. It should be remarked that in addition to interesting theoretical contributions, Yano should be commended for the careful, readable exposition he has given here of this topic in global differential geometry.

WILLIAM M. BOOTHBY

Higher transcendental functions. By A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. Based, in part, on notes left by Harry Bateman and compiled by the Staff of the Bateman Manuscript Project. New York, McGraw-Hill, 1953. Vol. I, 26+302 pp., $6.50. Vol. II, 17+396 pp., $7.50.

These two volumes compiled by the "Bateman Manuscript Project" represent a stupendous accomplishment. Under the able directorship of A. Erdélyi, the thirteen chapters of the two volumes were written by W. Magnus, F. Oberhettinger, and F. G. Tricomi, with the aid of a number of research assistants. The project was inspired by the idea of the late H. Bateman, from the California Institute of Technology, to prepare a compilation on a grand scale of the "special functions" (the definition of this term is somewhat arbitrary) occurring in mathematical analysis. When Bateman died in 1946, he left a considerable material of notes, system-cards, and drafts pertaining to such a plan. In the present undertaking they were used only to a limited extent, but in a way they served as an inspiration and the basis of the whole plan.

The work on the "Bateman Manuscript Project," that is, on the writing of the present volumes, extended with varying intensity from 1948 to 1952. It was the result of a joint effort on the part of the Office of Naval Research and the California Institute of Technology. Its history is told in two Prefaces to the first volume, written by Mina Rees and E. C. Watson, representing the institutions mentioned. The difficulties of such a compilation as was planned by Bateman and is carried out in the present work are enormous. They are due not so much to the vastness of the pertinent material but rather to the intrinsic difficulty of formulating and following clear and consistent principles in organizing it. (The term "Higher Transcendental Functions" appears just as arbitary as the term "Special Functions.") In whatever way one designs the underlying viewpoints, there will be ample interconnections and overlappings resulting in the occurrence of the same subject matter, or part of it, under different headings. (For instance, in the present system, Legendre polynomials appear in Chapter III as a special section and in Chapter X likewise.)

The single chapters have been compiled and written by various authors; see below, where we shall indicate the authors by M, O, and T, respectively. The Foreword is modestly silent about the share of Professor Erdélyi which, judging from diverse signs, must have been considerable—not only in terms of organization and coordination but also in actual contributions. By and large it seems to be a good policy that unity of style, form, and procedure were not too rigidly enforced. In some chapters more details (proofs) are presented than in other chapters which are more concise with respect to derivations.

References are given in each chapter, following the principle of usefulness rather than completeness. The symbols employed were chosen with particular care. Other notations occurring frequently in the literature are listed. An important feature is the listing of exhaustive tables and systematic collection of formulas which will greatly increase the efficiency of the work.

The books produced by this imposing effort make delightful reading with much color and usefulness at the same time. As is pointed out in the Foreword to the first volume, they remind one somewhat of a new-fashioned version of the *Modern Analysis* of Whittaker and Watson, except for the absence of interesting problems and exercises (which were of course outside the scope of the Project).

We now list briefly the contents of the single chapters and their outstanding features.

Chapter I (O), The Gamma functions. We find here sections dealing with the generalized zeta function, with the Riemann zeta function, and with the Bernoulli and related polynomials.

Chapter II (M), The hypergeometric function. We find here a rich collection of well known formulas and relations. The 15 relations of Gauss between contiguous functions and Kummer's 24 solutions of the hypergeometric equation are given, moreover 15 integral representations. In addition, this chapter lists the less generally known quadratic and higher transformations (47 items) and the degenerate cases (29 items) which are hardly available except in original papers.

Chapter III (O), Legendre functions. The representations of $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ (first and second solution of the Legendre differential equation) are given in terms of hypergeometric functions of various arguments; this list contains 18 items for each function. Further useful sections deal with toroidal and conical functions.

Chapter IV (M), The generalized hypergeometric series.

Chapter V, Further generalizations of the hypergeometric function, dealing with MacRobert's E-function, Meijer's G-function (with a list of 75 special formulas) and with hypergeometric functions of several variables. Horn's list is given, containing 34 items; moreover, an extensive list of systems of partial differential equations, transformations, etc.

Chapter VI (T), The confluent hypergeometric function, involving many special functions under this heading which receive systematic and more extensive treatment in other parts of the work. The sections dealing with the asymptotic behavior (if the argument or some of the parameters become large), and those relating to the position of the zeros, deserve particular mention.

Chapter VII (O), Bessel functions. Asymptotic expansions (for large values of the argument and the order) are treated with particular exhaustiveness listing numerous results which do not appear in the Treatise of Watson (1922); they are due to Watson, van der Corput, Langer, and others. The same is true for the section on the zeros. The second part contains a huge system of formulas.

Chapter VIII (T), Functions of the parabolic cylinder and of the paraboloid of revolution.

Chapter IX (M, T), The incomplete gamma functions and related functions.

Chapter X (T), Orthogonal polynomials. Various parts of this chapter go considerably beyond the material dealt with in the reviewer's book on the same topic: for instance, the work of Tricomi on the asymptotic behavior and the zeros of Laguerre polynomials and the properties of the important polynomials of Pollaczek.

Chapter XI (M), Spherical and hyperspherical harmonic polynomials, a very elegant and important chapter much of which was taken from unpublished notes of a course given by G. Herglotz.

Chapter XII, Orthogonal polynomials in several variables.

Chapter XIII (T), Elliptic functions and integrals.

Both volumes have a subject index and index of notations which will greatly increase the usefulness of the work.

The mathematical public will be indebted to the collaborators and to the editor of this project for their accomplishment.

G. SZEGŐ

The algebraic theory of spinors. By C. Chevalley. New York, Columbia University Press, 1954. 8+131 pp.

Most of the results of the theory of spinors are due to its founder E. Cartan; and, until this year, the only place where they could be found in book form was E. Cartan's own *Leçons sur la théorie des spineurs*, published in 1938. Strangely enough, the deep and unerring geometric insight which guided Cartan's researches, and places him among the greatest mathematicians of all time, is too often smothered in his books under complicated and seemingly gratuitous computations: witness, for instance, his fantastic definition of spinors (at the beginning of the second volume of the work quoted above) by means of the coefficients of a system of (non-independent) linear equations defining a maximal isotropic subspace! The reason for this is most probably to be found in the fact that E. Cartan's generation did not have at its disposal the geometric language which modern linear algebra has given us, and which now makes it possible to express in a clear and concise way concepts and results which otherwise would remain hopelessly buried under forbidding swarms of matrices.

The remarkably skillful way in which this language is used is certainly the most conspicuous feature of Chevalley's book. It goes without saying that, as usual in modern algebra, the basic field K of the theory is arbitrary (whereas E. Cartan considered only the real and complex number fields of classical analysis); the specialist will, however, be amazed to see that the author has succeeded in pushing this generality so far that in the greater part of the book, no special treatment is necessary for fields of characteristic 2; and indeed, so great is the author's virtuosity that the non-specialist will need a very thorough reading of the book to realize that this case actually exhibits special features at all.

The first chapter begins with the fundamental properties of the orthogonal groups: as mentioned above, characteristic 2 is included from the start, and all the necessary results are proved in 14 pages, including the best proof of Witt's theorem known to the reviewer, and a new proof of the generation of the orthogonal groups by symmetries (the author extends that name to the orthogonal transvections in the case of characteristic 2; the justification for this is of course that both can be given the same definition and handled in exactly the same way). The second half of the chapter is devoted to the study of the representations of the orthogonal group on the p-vectors, their decompositions into simple components and the classification of these with regard to equivalence: this is done not only for the orthogonal group, but also for the subgroup of rotations and the group of commutators; for characteristic 2, only the case $p=1$ is considered (the representation being no longer completely reducible for $p>1$).

The first part of chapter II gives a complete study of the Clifford algebra of a quadratic form, and can be considered as the first such study in the literature, for all other books on spinors or quadratic forms are in such a hurry to reach their main theme that they are content with giving the Clifford algebra the most cursory treatment, brought down to the minimum number of properties they really need. Chevalley's presentation of the theory is entirely original; the main novelty consists in exhibiting a fundamental connection between the Clifford algebra C and the exterior algebra E of the underlying vector space M (it has long been noticed that the two algebras exhibit very similar features, but this had remained very vague until now). The quadratic form $Q(x)$ being written as $B_0(x, x)$, where B_0 is a symmetric bilinear form, Chevalley shows that C, as a vector space, can be identified with E, and that the multiplication in C can be obtained from the multiplication in E and the form B_0 in the following explicit way: it is sufficient to define multiplication on the left by an element $x \in M$ (since they generate C); the operator $L'_x : s \rightarrow xs$ can then be written $L'_x = L_x + \delta_x$, where L_x is the operator of left multiplication by x in E, and δ_x the (unique) antiderivation of E such that $\delta_x \cdot y = B_0(x, y)$ for every $y \in M$. This is the cornerstone on which all subsequent developments are based. First the simplicity of C when M has even dimension $n = 2r$ and Q is nondegenerate (and nondefective when the characteristic is 2) is proved by reducing the question to the case in which Q has maximal index r, and then exhibiting a faithful representation $w \rightarrow \rho(w)$ of C onto the ring of vector-space endomorphisms of an isotropic subspace N of M, of maximal dimension r (a method which will acquire fundamental importance in chapters III and IV). From this follow easily the structure of the subalgebra C^+ consisting of elements of even order in C, relations between C and the Clifford algebras of the restrictions of Q to two supplementary orthogonal subspaces of M, the structure of C and C^+ when n is odd, and finally the determination of the radical of C when Q is degenerate or defective.

The classical connection between C and the orthogonal group $O_n(K, Q)$ (Q is henceforth taken as nondegenerate and nondefective) is then developed: to avoid trouble with such ill-defined concepts as "many-valued representations," Chevalley starts with the group Γ of invertible elements $s \in C$ such that $sMs^{-1} = M$, and shows that $x \rightarrow sxs^{-1}$ is a transformation $\chi(s)$ of the orthogonal group O_n, χ being a mapping of Γ onto O_n (with a single exception, when K has 2 elements, $n = 4$ and the index of Q is 2). To the subgroup Γ^+ of even elements of Γ corresponds the group of rotations O_n^+; on the other hand, if $t \rightarrow t^J$ is the natural anti-automorphism of C (associating to a product $x_1 x_2 \cdots x_p$ of elements of M the product $x_p x_{p-1} \cdots x_1$ taken in the reverse order) the elements $s \in \Gamma^+$ have a "norm" $\lambda(s) = ss^J$ in K, and those having norm 1 form a normal subgroup Γ_0^+ of Γ^+, which is mapped onto a normal subgroup O'_n of O_n^+, containing the commutator subgroup Ω_n of O_n. Following Eichler, it is proved that for forms Q of index $\nu > 0$, $O'_n = \Omega_n$ and O_n^+/Ω_n is isomorphic to the multiplicative group of elements of K, modulo the squares in K.

Spinors are next introduced, as forming a space in which (for n even) acts a simple representation ρ of C; the restriction ρ^+ of ρ to C^+ is either simple or splits into two simple nonequivalent representations, the *half-spin* representations. Similar definitions are given in the odd-dimensional case, and the following sections study the restriction of the spin representation of C to the Clifford algebra of the restriction of Q to a non-isotropic subspace, and its extension when the base field K is extended. The chapter ends with a study of the classical case of quadratic forms over the real field (with special emphasis on the relationship between the Clifford algebra and the Lie algebra of the orthogonal group), and with a very elegant proof of Hurwitz's theorem on quadratic forms "permitting composition," using the simplicity of the Clifford algebra.

Chapters III and IV are restricted to the special case of quadratic forms of maximal index $[n/2]$; no attempt is made to extend the results obtained in that case (in particular, the principle of triality) to more general ones, and this is probably the only part of E. Cartan's theory which is not covered by the book. Chapter III begins with the exposition of the theory of *pure spinors*, one of the most beautiful discoveries of E. Cartan, which unfortunately also constitutes one of the most obscure parts of his book. Here everything is neatly cleared up by Chevalley; the dimension $n=2r$ being even, the space M is decomposed into a direct sum of two totally isotropic subspaces N and P, and the space S of spinors is identified with the subalgebra C^N of C generated by N (and isomorphic to the exterior algebra of N); if f is an r-vector representing P, $Cf=C^Nf$ is a minimal ideal, and the spin representation ρ is defined by $vuf=(\rho(v)\cdot u)f$ for $u\in C^N=S$, $v\in C$. Now for every maximal isotropic subspace Z, let f_Z be the product in C of the elements of a base of Z; f_ZC is a minimal right ideal of C, and its intersection with Cf is a 1-dimensional vector subspace; any element of that space can be written u_Zf where u_Z is a spinor well determined up to a scalar factor, and these spinors are the *pure spinors* associated to Z. Such a spinor entirely determines Z, as the set of vectors x such that $\rho(x)\cdot u_Z=0$, and conversely this condition is characteristic for the pure spinors associated to Z. Pure spinors play for maximal isotropic subspaces a part similar to the one which decomposable p-vectors play for p-dimensional vector spaces in exterior algebra. Their study is developed in great detail: they are always half-spinors, and the two families of pure half-spinors correspond to the two intransitivity classes of maximal isotropic spaces under the group of rotations; a sum $u+u'$ of two pure spinors is pure if and only if the intersection of their corresponding subspaces has dimension $r-2$. An interesting feature, which is an original contribution of the author, is an expression of the elements $s\in\Gamma$ such that $\chi(s)$ leaves all elements of N invariant; s can be written uniquely in the form exp (u), where $u=\sum_{i<j}a_{ij}x_ix_j$ is a 2-vector in N (the x_i's being a base of N), and exp $(u)=\prod_{i<j}(1+a_{ij}x_ix_j)$ by definition. Using this, the author can show that a pure spinor corresponding to a maximal

isotropic subspace Z can be written $\exp(u)x_1x_2 \cdots x_k$, where the x_i's form a base for $Z \cap N$.

Next there is introduced, after Cartan, the bilinear invariant $\beta(u, v)$ on $S \times S$, as being the scalar such that $(uf)^J vf = \beta(u, v)f$; its invariance is expressed by the equation

$$\beta(\rho(s) \cdot u, \rho(s) \cdot v) = \lambda(s)\beta(u, v)$$

for any $s \in \Gamma$. It is shown that β is a nondegenerate bilinear form, which is either symmetric or antisymmetric according to the parity of $r(r-1)/2$; and $\beta(u, v) = 0$ for pure spinors u, v is the condition for their corresponding subspaces to have an intersection not reduced to 0.

The following sections are devoted to the study of the tensor product of the spin representation ρ by itself. First the tensor product $S \otimes S$ of the space S by itself can be identified to C, by the linear mapping $\phi(u \otimes v) = ufv^J$, and this immediately shows that the tensor product $\rho \otimes \rho$ can be identified with the representation which, to each $s \in \Gamma$, assigns the endomorphism $w \rightarrow \lambda(s)sws^{-1}$ of the vector space C. The most complete results are obtained in the case of characteristic $\neq 2$; then one can choose the form B_0 in an intrinsic way, as $B(x, y) = \frac{1}{2}(Q(x+y) - Q(x) - Q(y))$, and it can be shown that with this particular identification of C to the exterior algebra E, any automorphism of C is also an automorphism of E. In particular, an inner automorphism of C leaves invariant the subspaces of p-vectors when it leaves M invariant, i.e. when it is determined by an element $s \in \Gamma$. This gives immediately the decomposition of $\rho \otimes \rho$ in a direct sum of representations in the spaces of multivectors, studied in chapter I. To this decomposition corresponds a decomposition of ufv^J, for u and v in S, into the sum $\sum_{k=0}^{n} \beta_k(u, v)$, where β_k is a bilinear mapping of $S \times S$ into the space E_k of k-vectors, which is covariant under the representation $\rho \otimes \rho$. The study of these mappings enables one to describe completely the decomposition of the representation $\rho \otimes \rho$ when restricted to the group Γ^+; they also yield a characterization of pure spinors, and a criterion giving the dimension of the intersection $Z \cap Z'$ of two maximal isotropic subspaces, in terms of the corresponding pure spinors.

The remaining sections of chapter III are taken up by the relations between the half-spin representations of Γ^+ and their restrictions to the subgroup leaving invariant the elements of a nonisotropic plane, the determination of the kernels of the half-spin representations, the extension of the theory to odd dimensional spaces (by imbedding the space as a hyperplane in an even-dimensional space), and finally an application of spinor theory to get the classical description of the orthogonal group in 6 variables when the index is 3 (as isomorphic to a linear group in 4 variables): this does not seem to the reviewer to bring any information which may not be obtained in a quicker and more natural way by the classical method.

Chapter IV develops the famous "principle of triality." The dimension being $2r=8$, and the index equal to 4, the spaces S_p, S_i of half-spinors have the same dimension 8 as M. Following Cartan, the direct sum $A=M+S_p+S_i$ is considered, and on it are defined: 1° a symmetric bilinear form $\Lambda(x+v, x'+v')=B(x, x')+\beta(v, v')$ for x, x' in M, v, v' in S; 2° a trilinear symmetric form $\phi(\xi, \eta, \zeta)$ such that $\phi(x, u, u')=\beta(\rho(x)\cdot u, u')$ for $x\in M$, $u\in S_p$, $u'\in S_i$. From these one defines a (non-associative, but commutative) multiplication $\xi\circ\eta$ in A, by the condition $\phi(\xi, \eta, \zeta)=\Lambda(\xi\circ\eta, \zeta)$. All these definitions are invariant under the group Γ_0 (subgroup of the $s\in\Gamma$ such that $\lambda(s)=ss^J=1$) and conversely any automorphism of A which leaves invariant each of the subspaces M, S is produced by an element of Γ_0. But in addition, there is an automorphism j of A, of order 3, which permutes M, S_p, and S_i cyclically, and the existence of such an automorphism constitutes the principle of triality; it can be shown that for $x\in M$, $j(x)$ is of the form $u_1\circ x\in S_i$ and $j^{-1}(x)=u_1'\circ x\in S_p$, where u_1 is a fixed semi-spinor in S_p and u_1' a fixed semi-spinor in S_i. Beautiful geometric interpretations of the multiplication $\xi\circ\eta$ and of the automorphism j can be given when they act on pure spinors. Finally, the mapping $(x, y)\to x*y=(x\circ u_1')\circ(y\circ u_1)$ defines a nonassociative multiplication in M itself, which is shown to be that of the Cayley-Dickson algebra of *octonions*; on the other hand, j defines in a natural way an automorphism J of the commutator subgroup Ω_8, and the subgroup of Ω_8 consisting of the invariant elements under that automorphism constitutes the group of automorphisms of the algebra of octonions. At this point the stage is set for the geometric study of the exceptional Lie groups, in which the author has recently made such remarkable progress (in work unfortunately still partly unpublished); and it is to be hoped that in the near future, taking up the task where he breaks it off here, he will lead us into this fascinating new geometry and thus add to the thanks he has deserved from all mathematicians for the splendid job he has done in this volume.

The proofreading has not been too careful, and a list of corrections would be welcome, as also an index of notations.

J. DIEUDONNÉ

BRIEF MENTION

Opere. Vol. 2. *Applicabilità e problemi di deformazione*. By L. Bianchi. Ed. by the Unione Matematica Italiana with the assistance of the Consiglio Nazionale delle Ricerche. Rome, Cremonese, 1953. 4+327 pp. 3000 lire.

For the two parts of vol. 1 cf. this Bulletin vol. 59, p. 416; vol. 60, p. 288.

Tables of circular and hyperbolic sines and cosines for radian arguments. (National Bureau of Standards Applied Mathematics Series, vol. 36.) Washington, Government Printing Office, 1953. 12+407 pp. $3.00.

This is a third edition of MT3 of the New York Mathematical Tables Project, first published in 1939.

Four-place tables of transcendental functions. By W. Flügge. New York, McGraw-Hill; London, Pergamon, 1954. 4+136 pp. $5.00.

Functions tabulated: trigonometric and hyperbolic, exponential and logarithmic, Bessel, Thomson, elliptic integrals, erf, Fresnel integrals, Ei, Si, Ci, Γ.

Simultaneous linear equations and the determination of eigenvalues. Ed. by L. J. Paige and O. Taussky. (National Bureau of Standards Applied Mathematics Series, no. 29.) Washington, Government Printing Office, 1953. 4+126 pp. $1.50.

This volume contains, in 19 papers, the proceedings of a symposium held in 1951.

Tables of 10^x. (National Bureau of Standards Applied Mathematics Series, no. 27.) Washington, Government Printing Office, 1953. 8+543 pp. $3.50.

This is a rearrangement of J. Dodson's Antilogarithmic Canon (London, 1742), giving 10 decimal places at interval 0.00001.

Table of natural logarithms for arguments between zero and five to sixteen decimal places. (National Bureau of Standards Applied Mathematics Series, no. 31.) Washington, Government Printing Office, 1953. 10+501 pp. $3.25.

This is reissue of MT 10 of the New York Mathematical Tables Project, originally issued in 1941.

APPROXIMATE SOLUTIONS OF FREDHOLM-TYPE INTEGRAL EQUATIONS

A. T. LONSETH[1]

1. **Introduction.** The integral equation whose approximate solution will be discussed here is the linear nonhomogeneous equation of Fredholm type and second kind

$$x(s) - \int_0^1 K(s, t)x(t)dt = y(s), \qquad 0 \leqq s \leqq 1, \tag{1.1}$$

where the functions $K(s, t)$—the "kernel"—and $y(s)$ are known. The question posed is this: Given an approximate method for solving (1.1), how can one estimate the error?

Since late in the nineteenth century the importance of (1.1) in mathematical physics has been recognized, along with that of

$$x(s) - \lambda \int_0^1 K(s, t)x(t)dt = 0, \qquad 0 \leqq s \leqq 1, \tag{1.2}$$

in which one is to determine values of λ (proper values) such that a continuous solution $x(s) \neq 0$ exists. Typical problems leading to equations like (1.1) are the Dirichlet and Neumann problems of potential theory; to (1.2), time-dependent problems in elastic vibration and heat flow, by "separating out" the time.

It is now fifty years since Fredholm published his distinguished paper in Acta Mathematica [10],[2] in which he gave the first detailed account of the existence and multiplicity of solutions of (1.1) and (1.2). Few mathematical publications have stimulated so much further work. Several papers appeared in which physical problems were set up in terms of integral equations (elasticity, gas theory, etc.); some were on numerical solutions; most of the many theoretical papers followed the now familiar trend toward greater generality and abstraction. Hilbert recognized analogies with Euclidean geometry, except that now space was infinite-dimensional [13]; through Fréchet and F. Riesz this led to Banach spaces and more rarefied

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

concepts. Such contemporary objects as "abstract spaces," "spectral theory," and "normed rings" are among the grandchildren of the Fredholm theory.

Despite their apparent remoteness from numerical work, some of these abstract ideas lie very close to our question of error-limitation. This is true of such simple metric notions as the norm (or length) of a function and the bound of an integral transformation K defined by

$$Kx = \int_0^1 K(s, t)x(t)dt.$$

It is convenient to write (1.1) in the abbreviated form

$$(I - K)x \equiv x - Kx = y$$

and to speak of x and y as "vectors" in some "linear space," such as the totality of functions which are continuous on $0 \leqq s \leqq 1$, or of those whose squares are integrable thereon. More details may be found in [2], [15], or [19].

It will be assumed throughout that (1.1) has a unique solution $x(s)$ for given $y(s)$. Thus we may speak of the inverse $(I-K)^{-1}$ of operation $I-K$. All functions and numbers will be assumed real, but everything here extends to the complex case with only trivial alterations.

2. **Three classes of methods.** The three categories for which error bounds will be derived do not cover all ways of attacking (1.1), nor are they mutually exclusive. However, most of the frequently used procedures will be included.

Class I. *The kernel $K(s, t)$ is replaced by an approximation of such form that the resulting equation can be solved exactly.* Usually the new kernel is one of "finite rank" such as

$$(\alpha) \sum_{i=1}^{n} \phi_i(s)\psi_i(t) \qquad \text{or} \qquad (\beta) \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\phi_i(s)\psi_j(t),$$

where the ϕ's and ψ's are known functions, preferably tractable; with such a kernel (1.1) reduces to an ordinary system of linear equations [11; 26; 3]. For example, with $K(s, t)$ replaced by (α) the integral equation becomes

$$x_n(s) - \sum_{i=1}^{n} \phi_i(s) \int_0^1 \psi_i(t)x_n(t)dt = y(s). \tag{2.1}$$

We multiply in turn by $\psi_1(s), \cdots, \psi_n(s)$ and integrate over (0, 1), getting n linear equations for the numbers

$$p_i^{(n)} = \int_0^1 \psi_i(t)x_n(t)dt, \qquad i = 1, \cdots, n.$$

If this system is nonsingular, its solution substituted into (2.1) gives exactly

$$x_n(s) = y(s) + \sum_{i=1}^{n} p_i^{(n)}\phi_i(s).$$

The problem then is to get some kind of limitation for the difference $x(s)-x_n(s)$. Various such bounds have been published by Tricomi [27], Akbergenov [1], Kantorovich [15; 16] and Lonseth [19]. These error-limitations do not require that the approximating kernel be of finite rank, but merely that it be sufficiently close to $K(s, t)$ in some sense.

Class II. The equation is unchanged; instead, *the "best" near-solution of form*

$$x_n(s) = \phi_0(s) + \sum_{i=1}^{n} a_i\phi_i(s)$$

is determined, where the ϕ's are known functions. (Boundary conditions imposed by $K(s, t)$ and $y(s)$ may be absorbed in $\phi_0(s)$, while the other ϕ's satisfy the corresponding "homogeneous" conditions.) Everything now depends on the meaning of "best," and with different criteria we have different methods. One criterion for choice of the a's is to minimize the integral

$$Q(a_1, \cdots, a_n) = \int_0^1 \left\{x_n(s) - \int_0^1 K(s, t)x_n(t)dt - y(s)\right\}^2 ds.$$

This "method of least squares" leads to a system of linear equations for $a_1, \cdots, a_n$ and is quite effective. Another method is based on the principle of moments (Galerkin's method): the a's are chosen so that

$$x_n(s) - \int_0^1 K(s, t)x_n(t)dt - y(s)$$

is orthogonal to $\phi_1(s), \cdots, \phi_n(s)$ on (0, 1). That is, the a's satisfy n linear equations

$$\int_0^1 \phi_i(s)\left\{x_n(s) - \int_0^1 K(s, t)x_n(t)dt - y(s)\right\} ds = 0, \quad i = 1, \cdots, n.$$

If $K(s, t) \equiv K(t, s)$, the method of moments coincides with a minimum principle of Rayleigh and Ritz. Various generalizations have been considered. We refer those who are interested to the books by Kravchuk [17] and by Kantorovich and V. I. Krylov [16], and to journal articles by N. M. Krylov [18], Picone [25], and Oberg [22].

Class III. *Iterative methods.* Some iterative methods are schemes for solving (1.1) as it stands, with given $y(s)$; others yield the inverse of the operator $I-K$ which in (1.1) transforms the unknown x into the known y. C. Neumann's expansion is the prototype of methods of both sorts. Formally it says that, whatever y, the solution is given by

$$x = y + Ky + K^2y + \cdots ,$$

where K^ny is the result of operating n times on y with K; this is the same as saying that the inverse

$$(I - K)^{-1} = I + K + K^2 + \cdots .$$

Thus x is approximated by x_n which is recursively defined by

$$\begin{aligned} x_0 &= y, \\ x_{n+1} &= y + Kx_n \qquad \text{if } n = 0, 1, \cdots . \end{aligned}$$

Likewise, $(I-K)^{-1}$ is approximated by S_n where

$$\begin{aligned} S_0 &= I, \\ S_{n+1} &= I + KS_n \qquad \text{if } n = 0, 1, \cdots . \end{aligned}$$

Faster schemes for getting $(I-K)^{-1}$ will also be described in §6.

Perhaps the most important method not explicitly discussed here is that in which the integral

$$\int_0^1 K(s, t)x(t)dt$$

is replaced by an approximation (trapezoidal, Simpson's, Gauss', etc.) and $x(t)$ is approximately determined only at finitely many points. This idea was Fredholm's heuristic guide in [10]. It has been used by Nyström in [21] and elsewhere, and the error has been investigated by Ostrowski [24] and Kantorovich [15; 16]. Gradient methods (such as "steepest descent") may also prove useful, particularly for symmetric kernels.

Actual application of these methods may require numerical integrations, or solution of a system of linear equations, or both. In the illustrations at the end of this paper the necessary quadratures could be carried out exactly, and the linear systems were of such low orders that a desk-computer was quite adequate. Such favorable examples cannot always be expected, however, and a modern, high-speed automatic calculator may be desirable, or even necessary. Solving a high-order system of linear equations can be quite a problem. It is discussed comprehensively in G. E. Forsythe's paper [29].

3. **Metric notions.** The length or *norm* $\|x\|$ of a function $x(s)$, $0 \leqq s \leqq 1$, may be defined in various ways. A natural adaptation of the ordinary Euclidean length of an n-dimensional vector is

$$\|x\| = \left(\int_0^1 x^2(s)ds\right)^{1/2}. \tag{3.1}$$

With this quadratic norm, a linear combination of two functions with finite norm also has finite norm; furthermore,

(i) $\|x\| \geqq 0;$

(ii) $\|ax\| = |a| \cdot \|x\|$ if a is a constant;

(iii) $\|x + y\| \leqq \|x\| + \|y\|.$

These properties characterize norms generally: they hold if one defines, for $x(s)$ continuous on $0 \leqq s \leqq 1$,

$$\|x\| = \max_{0 \leqq s \leqq 1} |x(s)|, \tag{3.2}$$

or, for $x(s)$ such that $|x(s)|^p$ is integrable,

$$\|x\| = \left(\int_0^1 |x(s)|^p ds\right)^{1/p}, \qquad p \geqq 1. \tag{3.3}$$

Error-estimates will be found in terms of norms. It is important to know what a linear transformation does to the norm of a function. Suppose that the functions $x(s)$ under consideration constitute a (real) normed linear space (i.e., if x_1 and x_2 are in the space, so is $a_1x_1 + a_2x_2$, where a_1 and a_2 are any real numbers; for every x, $\|x\|$ exists). Suppose further that T is a linear transformation of the type known as *linear operators* (i.e., additive, homogeneous, such that $\|x\|$ finite implies $\|Tx\|$ finite, continuous in the sense that $\|x_n - x\| \to 0$ implies $\|Tx_n - Tx\| \to 0$). Then there exist [2, p. 54] non-negative numbers $M(T)$ and $m(T)$,

$$M(T) = \underset{\|x\| \neq 0}{\text{l.u.b.}} \frac{\|Tx\|}{\|x\|}, \qquad m(T) = \underset{\|x\| \neq 0}{\text{g.l.b.}} \frac{\|Tx\|}{\|x\|}, \tag{3.4}$$

which we call respectively the upper and the lower bounds of T. Evidently, whatever x,

$$m(T)\|x\| \leqq \|Tx\| \leqq M(T)\|x\|. \tag{3.5}$$

If the sum T_1+T_2 and the product T_1T_2 of two linear operators T_1 and T_2 are operators defined so that, for every x, $(T_1+T_2)x=T_1x+T_2x$ and $(T_1T_2)x=T_1(T_2x)$, they are also bounded and

$$M(T_1 + T_2) \leqq M(T_1) + M(T_2), \quad M(T_1T_2) \leqq M(T_1)M(T_2). \tag{3.6}$$

If T has the bounded inverse T^{-1}, then [19]

$$M(T^{-1})m(T) = 1. \tag{3.7}$$

If $T=I-K$ and $M(K)<1$,

$$1/\{1 + M(K)\} \leqq M\{(I - K)^{-1}\} \leqq 1/\{1 - M(K)\}, \tag{3.8}$$

inequalities which we shall use later.

The exact calculation of $M(K)$ is usually out of the question. If $K(s, t)$ is symmetric and the norm is the quadratic (3.1), $M(K)$ is the reciprocal of the smallest λ solving the characteristic problem (1.2). With the same norm and a nonsymmetric kernel, $M(K)$ is $\mu^{-1/2}$, where μ is the smallest number such that there is a $w(s)\neq 0$ satisfying

$$w(s) - \mu\int_0^1 \left\{\int_0^1 K(r, t)K(r, s)dr\right\} w(t)dt = 0.$$

These may not be easy to find. An upper bound to $M(K)$ comes from the Schwarz inequality:

$$M(K) \leqq \left(\int_0^1 \int_0^1 K^2(s, t)dsdt\right)^{1/2}. \tag{3.9}$$

With $\|x\| = \max |x(s)|$,

$$M(K) \leqq \max_{0\leqq s\leqq 1} \int_0^1 | K(s, t) | dt. \tag{3.10}$$

With $\|x\| = \int_0^1 |x(s)| ds$,

$$M(K) \leqq \int_0^1 \max_{0\leqq t\leqq 1} | K(s, t) | ds. \tag{3.11}$$

With the Minkowski norm (3.3) for $p>1$, Hölder's inequality yields

$$M(K) \leqq \left\{\int_0^1 \left[\int_0^1 | K(s, t) |^q dt\right]^{p/q} ds\right\}^{1/p}, \tag{3.12}$$

where $p^{-1}+q^{-1}=1$.

4. **Error-bound for kernel-replacement.** Suppose that $K(s, t)$ is

replaced by $K(s, t)+\kappa(s, t)$, and write $x(s)+\xi(s)$ for the solution of the new equation. Then, briefly, we have

$$x + \xi - (K + \kappa)(x + \xi) = y$$

while

$$x - Kx = y;$$

so

$$\xi - K\xi = \kappa x + \kappa\xi. \tag{4.1}$$

If now $(I-K)^{-1}=I+R$,

$$\xi = (I + R)(\kappa x + \kappa\xi).$$

If $M\{(I+R)\kappa\}<1$, *we can limit* $\|\xi\|$; for

$$\|\xi\| \leqq M\{(I + R)\kappa\}(\|x\| + \|\xi\|),$$

and consequently

$$\|\xi\| \leqq M\{(I + R)\kappa\}\|x\|/[1 - M\{(I + R)\kappa\}]. \tag{4.2}$$

Since

$$M\{(I + R)\kappa\} \leqq M(I + R)M(\kappa),$$

the foregoing will be valid for sufficiently small $M(\kappa)$. And if $M(\kappa)<1-M(K)$, we find that

$$\|\xi\| \leqq M(\kappa)\|x\|/\{1 - M(K) - M(\kappa)\}. \tag{4.3}$$

We can also write down a point-wise bound for $|\xi(s)|$, no matter which of the norms in §3 is used. For instance, with the Euclidean norm (3.1) we see by applying the Schwarz inequality to (4.1) that

$$\begin{aligned} |\xi(s)| \leqq {} & \left(\int_0^1 K^2(s, t)dt\right)^{1/2}\|\xi\| \\ & + \left(\int_0^1 \kappa^2(s, t)dt\right)^{1/2}(\|x\| + \|\xi\|). \end{aligned} \tag{4.4}$$

Into (4.4) we must substitute the bound (4.2) for $\|\xi\|$ and use

$$\|x\| \leqq M(I + R)\|y\|,$$

in which $M(I+R)\leqq 1/\{1-M(K)\}$ if $M(K)<1$.

A quite different bound, based on Fredholm's solution-formula and the Hadamard determinant inequality, was published in 1924 by Tricomi [27]. Akbergenov [1] in 1935 got essentially the results of this section for the norms (3.1) and (3.2), the latter of which is given by Kantorovich and Krylov [16, p. 157]. Related but complicated expressions are to be found in [15] as well. The present discussion is based on [19].

5. **Error-bound for methods of the second class.** Here we are looking for the "best" near-solution to (1.1) that lies in a certain finite-dimensional functional manifold. Let

$$x_n(s) = \phi_0(s) + \sum_{i=1}^{n} a_i\phi_i(s)$$

be an arbitrary function in this manifold; let the selected values of $a_1, \cdots, a_n$—determined by least squares, moments, or any other method—be $\bar{a}_1, \cdots, \bar{a}_n$. The corresponding "solution" $\bar{x}_n$ does not necessarily satisfy (1.1) but rather

$$\bar{x}_n - K\bar{x}_n - y = \bar{r}_n, \tag{5.1}$$

in which the residual $\bar{r}_n(s)$ can be calculated a posteriori. Subtracting from (5.1) the true equation satisfied by the still unknown $x(s)$, we get

$$\bar{x}_n - x - K(\bar{x}_n - x) = \bar{r}_n; \tag{5.2}$$

so that with $(I-K)^{-1} = I+R$ as before,

$$\bar{x}_n - x = (I + R)\bar{r}_n$$

and for an appropriate norm

$$m(I + R)\|\bar{r}_n\| \leqq \|\bar{x}_n - x\| \leqq M(I + R)\|\bar{r}_n\|. \tag{5.3}$$

If $M(K) < 1$, (5.3) becomes, because of (3.8),

$$\|\bar{r}_n\|/\{1 + M(K)\} \leqq \|\bar{x}_n - x\| \leqq \|\bar{r}_n\|/\{1 - M(K)\}. \tag{5.4}$$

Thus in this case we not only get the desired upper bound for the error-norm, but also a positive lower bound, if $\|\bar{r}_n\| > 0$. (The lower bound holds even if $M(K) \geqq 1$.) That there should be such a lower bound may at first seem surprising, but actually it is to be expected: we are trying to pick out of a subspace a vector which cannot be expected to be there, and whose "distance" from each vector of the subspace (usually) exceeds some positive quantity. The same considerations apply more generally to solution of $Tx = y$, where T is linear (possibly a differential or integro-differential operator): if T is itself bounded, $m(T^{-1}) = 1/M(T)$ and $\|\bar{x}_n - x\|$ is bounded away from zero; if T has a bounded inverse, we get an upper bound for $\|\bar{x}_n - x\|$.

The local error can be limited much as in §4 (see 4.4). Or it may be better to observe from (5.2) that

$$x(s) - \tilde{x}_n(s) + \tilde{r}_n(s) = -\int_0^1 K(s, t)[\tilde{x}_n(t) - x(t)]dt. \tag{5.5}$$

Now if, e.g., the quadratic norm is being used, we have by the Schwarz inequality

$$|x(s) - \tilde{x}_n(s) + \tilde{r}_n(s)| \leqq k(s)\|\tilde{x}_n - x\|, \qquad 0 \leqq s \leqq 1, \tag{5.6}$$

where

$$k(s) = \left(\int_0^1 [K(s, t)]^2 dt\right)^{1/2}$$

and (5.3) can be used to limit $\|\tilde{x}_n - x\|$. Thus we have $x(s)$ lying between the two functions

$$\tilde{x}_n(s) - \tilde{r}_n(s) \pm k(s)\|\tilde{x}_n - x\|, \qquad 0 \leqq s \leqq 1. \tag{5.7}$$

If the quadratic norm is used, the method of least squares is seen to have unique advantages. One is that the criterion for goodness is precisely that $\|\tilde{r}_n\|$ be the minimum; accordingly inequalities (5.3) would be most favorable. This does not prove, of course, that least squares is actually more accurate than any other similar method. Probably no such categorical statement is possible. Nevertheless, a method for which one can claim a certain accuracy has an advantage over methods for which one must claim less.

Another advantage which "least squares" has over other methods in this class consists in the relative ease of computing $\|\tilde{r}_n\|$—still the quadratic norm. In principle, $\tilde{r}_n(s)$ can always be computed from (5.1) after x_n has been determined; it must then be squared and integrated, which may involve some rather nasty computation. But for least squares there is an alternative method based on "Bessel's identity":

$$\|\tilde{r}_n\|^2 = \|y - \phi_0\|^2 - \sum_{i=1}^{n} \tilde{a}_i(y - \phi_0, T\phi_i), \tag{5.8}$$

in which the inner products

$$(y - \phi_0, T\phi_i) = \int_0^1 (y - \phi_0)(\phi_i - K\phi_i)ds, \qquad i = 1, \cdots, n,$$

are the constant right members of the equations determining $\tilde{a}_1, \cdots, \tilde{a}_n$. Thus the only new computation is that of $\|y - \phi_0\|^2$.

It should be pointed out that the simple considerations of this section are actually quite general; the $\tilde{x}_n$ in (5.1) can be any kind of approximation to the true x, and (5.3) holds provided that the symbols have meaning.

6. Error-bounds for iterative methods. If one writes (1.1) as

$$\begin{aligned} x &= y + Kx \\ &= y + K(y + Kx) = y + Ky + K^2x \\ &= y + Ky + K^2y + K^3x, \text{ etc.,} \end{aligned}$$

it is not hard to guess that if $M(K) < 1$, so that $\|K^nx\| \to 0$, the sequence of vectors $\{x_n\}$, defined by

$$x_n = y + Ky + \cdots + K^{n-1}y,$$

will converge to the solution x of (1.1), in the sense that $\|x - x_n\| \to 0$. In fact,

$$(6.1) \qquad \|x - x_n\| \leqq M^n(K)\|y\|/\{1 - M(K)\}.$$

Also, if $S_n = I + K + \cdots + K^{n-1}$,

$$(6.2) \qquad M\{(I - K)^{-1} - S_n\} \leqq M^n(K)/\{1 - M(K)\}.$$

Instead of such a purely iterative approach, one might get an initial approximate solution by least squares or some other procedure and then wish to improve it by iteration. Suppose $\bar{x}(s)$ is such an approximation, and write

$$(6.3) \qquad \bar{x} - K\bar{x} = y + \bar{r}.$$

The point-wise bounds in (5.7) suggest that $\bar{x}(s) - \bar{r}(s)$ may be a fair approximation to the true $x(s)$; we define $\bar{x}_1(s) = \bar{x}(s) - \bar{r}(s)$, so that

$$\bar{x}_1 - \bar{x} = -\bar{r},$$

then define a sequence $\{\bar{x}_n\}$ of further approximations by

$$\bar{x}_{k+1} - \bar{x}_k = K^k(-\bar{r}), \qquad k = 1, \cdots,$$

the idea being to reduce $\|\bar{r}\|$, already small, by operating with K, $M(K) < 1$. Adding all such differences through $k = n - 1$, we see that

$$(6.4) \qquad \bar{x}_n - \bar{x} = \sum_{k=0}^{n-1} K^k(-\bar{r}), \qquad K^0 = I.$$

On the other hand, by subtracting (1.1) from (6.3) we observe that $\bar{x} - x$ satisfies equation

$$\bar{x} - x - K(\bar{x} - x) = \bar{r},$$

whence

$$\bar{x} - x = \sum_{k=0}^{\infty} K^k \bar{f}. \tag{6.5}$$

Addition of (6.4) and (6.5) leads to the inequality

$$\|\bar{x}_n - x\| \leqq M^n(K)\|\bar{f}\|/\{1 - M(K)\}. \tag{6.6}$$

For approximating the inverse $(I-K)^{-1}$, a more rapidly convergent iterative scheme than Neumann's can be based on an identity which Euler [8, p. 335] used in a problem on partitions, namely

$$1/(1 - x) = (1 + x)(1 + x^2)(1 + x^4) \cdots (1 + x^{2^k}) \cdots ,$$

which holds if $|x| < 1$. The nth partial product

$$\prod_{k=0}^{n-1} (1 + x^{2^k}) = \sum_{k=0}^{2^n-1} x^k,$$

so that for $I-K$, with

$$H_n = (I + K)(I + K^2) \cdots (I + K^{2^{n-1}}), \tag{6.7}$$

we have

$$(I - K)^{-1} - H_n = K^{2^n} + K^{2^n+1} + \cdots , \qquad M(K) < 1,$$

and [20]

$$M\{(I - K)^{-1} - H_n\} \leqq M^{2^n}(K)/\{1 - M(K)\}. \tag{6.8}$$

The product expression (6.7) for H_n can be replaced by a recursion formula. Clearly $H_{n+1} = H_n(I + K^{2^n}) = H_n\{2I - (I - K^{2^n})\}$, or

$$H_{n+1} = H_n\{2I - (I - K)H_n\}, \tag{6.9}$$

the final expression coming immediately from (6.7) on pre-multiplying each member by $I-K$. This technique was suggested by Ostrowski [23] for equations of Volterra type, and has been used for matrices by several authors.

A hierarchy of still more rapidly convergent iterative schemes can be constructed along the lines of Euler's identity.[3] Next in line would be the identity

$$\begin{aligned} 1/(1 - x) &= (1 + x + x^2)(1 + x^3 + x^6)(1 + x^9 + x^{18}) \cdots \\ &= \prod_{k=0}^{\infty} (1 + x^{3^k} + {}^{2\cdot 3^k}), \end{aligned} \tag{6.10}$$

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[3] Added later. I am grateful to R. D. James and Leo Moser for helpful suggestions.

valid if $|x| < 1$. Generally, we have

$$1/(1 - x) = \prod_{k=0}^{\infty} \left(\sum_{j=0}^{m-1} x^{j m^k} \right). \tag{6.11}$$

With $m = 3$ and operators instead of numbers, we write

$$H_n^{(3)} = \prod_{k=0}^{n-1} (I + K^{3^k} + K^{2\cdot 3^k}). \tag{6.12}$$

Then one can show that

$$(I - K)^{-1} - H_n^{(3)} = K^{3^n} + K^{3^n+1} + \cdots ; \tag{6.13}$$

so

$$M\{(I - K)^{-1} - H_n^{(3)}\} \leq M^{3^n}(K)/\{1 - M(K)\}. \tag{6.14}$$

A recursion formula like (6.9) is also obtainable, but it is not so simple. From

$$H_{n+1}^{(3)} = H_n^{(3)}(I + K^{3^n} + K^{2\cdot 3^n})$$

and

$$(I - K)H_n^{(3)} = I - K^{3^n}$$

(the latter identity follows immediately from (6.13)), we get

$$\begin{aligned} H_{n+1}^{(3)} &= H_n^{(3)}\{I + [I - (I - K)H_n^{(3)}] + [I - (I - K)H_n^{(3)}]^2\} \\ &= H_n^{(3)}\{3I - 3(I - K)H_n^{(3)} + [(I - K)H_n^{(3)}]^2\}, \end{aligned}$$

which is considerably more complicated than (6.9) or the simple relation

$$H_{n+1}^{(1)} = I + KH_n^{(1)} \qquad (H_n^{(1)} = S_n)$$

which characterizes Neumann's expansion. The higher-order methods suggested by (6.11) with $m > 3$ would be increasingly cumbersome.

Bodewig [4; 5] has also discussed such rapidly convergent processes. It is interesting to observe that in [5] he refers to Euler's extension [9, p. 422] of Newton's method.

A different modification of the Neumann scheme was proposed by Wiarda and extended by H. Bückner and C. Wagner; details may be found in Bückner's monograph [6]—which contains much on the general subject of solving integral equations, particularly (1.2)—and Wagner's paper [28].

7. Numerical illustrations.[4] Some of the methods described above have been tried out on the equation

$$x(s) + \int_0^1 K(s, t)x(t)dt = s^2, \qquad 0 \leqq s \leqq 1,$$

where

$$K(s, t) = \begin{cases} s(1-t) & \text{if } 0 \leqq s \leqq t \leqq 1, \\ t(1-s) & \text{if } 0 \leqq t \leqq s \leqq 1. \end{cases} \tag{7.1}$$

This integral equation corresponds to the simple boundary value problem

$$x''(s) - x(s) = 2, \qquad x(0) = 0, \qquad x(1) = 1.$$

However, the exact solution will not be used in checking the approximate solutions for accuracy.

Kernel (7.1) has characteristic values $n^2\pi^2$ $(n = 1, 2, \cdots)$; so with the quadratic norm $M(K) = 1/\pi^2$; but even without this information it follows from the Schwarz inequality that $M(K) \leqq 1/(90)^{1/2} < 1/9$. Hence

$$9/10 < m(I + K) < M(I + K) < 9/8. \tag{7.2}$$

To $n^2\pi^2$ corresponds $\sin n\pi s$ as characteristic function; hence the spectral representation of $K(s, t)$ is its Fourier sine series

$$K(s, t) = 2\sum_{n=1}^{\infty} \frac{\sin n\pi s \sin n\pi t}{n^2\pi^2}. \tag{7.3}$$

Four solution-procedures will be summarized: (a) $K(s, t)$ is replaced by one, two, or three terms of (7.3); (b) least squares; (c) Galerkin's method of moments; (d) least squares followed by iteration. It should be remembered that all the upper bounds are pessimistic.

EXAMPLE (a). With $K(s, t)$ replaced by just the first term from its series representation (7.3), the solution

$$x_1(s) = s - 0.034832 \sin \pi s$$

is found. From inequality (4.3), with $M(K)$ computed from Bessel's identity, it is found that with the quadratic norm

$$\|x - x_1\| \leqq 0.017.$$

The point-wise bound, from (4.4), satisfies in $0 \leqq s \leqq 1$

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[4] The computations in this section were carried out by Mr. Thomas L. Glahn.

$$|x(s) - x_1(s)| \leqq 0.0098(s - s^2) + 0.520\left(\frac{s^2(1 - s)^2}{3} - \frac{2}{\pi^4}\sin^2 \pi s\right)^{1/2}$$

$$\leqq 0.027.$$

Similar computations for $n = 2$ and $n = 3$ were carried out. The norm-errors are bounded as follows:

$$\|x - x_2\| \leqq 0.00822, \qquad \|x - x_3\| \leqq 0.00501.$$

Kantorovich [16, pp. 160–161] applies Akbergenov's error bound with norm max $|x(s)|$ to the equation

$$x(s) - \int_0^{1/2} (\sin st)x(t)dt = 1 + s^{-1}\left(\cos\frac{s}{2} - 1\right),$$

and with kernel sin st replaced by $st - s^3t^3/6$ shows that the pointwise error does not exceed 10^{-4}. This excellent result is possible mainly because the kernel is analytic, with alternating power series in st, and because the interval is rather short. It is not obvious how his inequality could be applied to the kernel (7.1). Kernels which are Green's functions for differential operators are not analytic, but frequently their squares are integrable, so the quadratic norm has a chance of being useful.

Example (b). "Least squares" was applied with $\phi_0(s) \equiv 0$, $\phi_k(s) = s^{k-1}$ if $k \geqq 1$. With $n = 2$, it was found that

$$x_2(s) = -0.186785 + 0.983500s;$$

with (5.4)

$$0.0764 < \|x - x_2\| < 0.0955.$$

With $n = 3$, it was found that

$$x_3(s) = 0.006903 - 0.144901s + 1.128397s^2,$$

and now

$$0.00280 \leqq \|x - x_3\| \leqq 0.00349.$$

It is clear that the norm of the error is 0.003 to three place accuracy

Example (c). The moment method with $n = 3$ gave

$$x_3(s) = 0.006853 - 0.144819s + 1.128423s^2,$$

with

$$0.00286 \leqq \|x - x_3\| \leqq 0.00358.$$

Example (d). A single iteration of the second sort described in §6 was applied to the $x_2(s)$ found in (b) above. This led to the approximation

$$\tilde{x}_2(s) = -0.070524s + 0.906607s^2 + 0.163917s^3,$$

with upper bound (6.6)

$$\|x - \tilde{x}_2\| < 0.0106.$$

Added in proof (August 23, 1954). Since this paper was written, three related error limitations have been found on the Oregon State College project. The first, by T. L. Glahn, is for a recursive scheme which determines successive coefficients in Fredholm's two series for the resolvent kernel. The second, by L. B. Rall, applies to the iterative processes of G. Wiarda, H. Bückner, C. Wagner, and P. A. Samuelson. The third, by G. T. Thompson, gives a close bound for Bateman's method as applied to boundary problems for self-adjoint ordinary linear differential equations of second order.

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THE OREGON STATE COLLEGE

REGULAR CONVERGENCE

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1. **Introduction.** The notion of convergence is certainly one of the most important ones in all mathematics. In analysis if a sequence of functions converges to a limit function, we ask the question what properties enjoyed by the members of the sequence are carried over to the limit function. With no restrictions on the type of convergence, of course very little can be said. We, therefore, impose conditions on the convergence which will allow some conclusions to be made. The same situation prevails in topology if we consider the concept of convergence of point sets. We say that the sequence of points sets (G_i) converges to the point set G, where all sets belong to a Hausdorff space, if the following is true. Every point with the property that each of its neighborhoods contains points from infinitely many G_i lies in G and each point of G has the property that each of its neighborhoods contains points from all but a finite number of the (G_i). It is easily seen that G will always be closed regardless of whether the G_i are or not. This notion was first introduced by Zarankiewicz [1]. An equivalent definition in a compact metric space is as follows. If we call the spherical neighborhood of a point set X with radius ϵ the set of all points x whose distance from some point in X is less than ϵ, then (G_i) converges to G if G is closed and for every ϵ the spherical neighborhood about G with radius ϵ contains all but a finite number of the G_i and the spherical neighborhood with radius ϵ about all but a finite number of the G_i contains G. For example, the sets $G_i = \{(x, y) \mid x = 1/i, (0 \leq y \leq 1)\}$ converge to $G = \{(x, y) \mid x = 0, (0 \leq y \leq 1)\}$. The second definition is equivalent to saying that G is closed and the Hausdorff distances [2] from G_i to the G converge to 0. Very few properties are carried over to the limit set by convergence of this general type. The reason for this is that two sets can be close to each other without being at all similar. For example in the above mentioned example the sets G_i could be replaced by the points of the line forming G_i that are rational with denominator i, and the limit set would still be the same. The reverse situation is not true, however, i.e. if all the members of the sequence in a compact metric space are closed and connected, then the limit set will also be closed and con-

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nected. Thus under convergence several components may be put together, but not torn apart.

2. **Regular convergence.** By requiring more than just closeness in the convergence, a much more satisfactory situation can be achieved. In 1935 G. T. Whyburn gave such a definition [3] in which he made use of the definition of local connectivity. In his definition a sequence (A_i) of closed sets is said to *converge regularly* to a limit set A if for each $\epsilon>0$ there exist positive numbers δ and N such that for $n>N$, any two points in A_n whose distance apart is $<\delta$ lie in a connected subset of A_n with diameter $<\epsilon$. If all the sets A_n in a sequence coincide with the limit set A, this is the definition of uniform local connectivity. Thus we require the members of the sequence to approximate each other more and more closely in the sense of local connectivity as the sequence progresses. With this type of convergence it is possible to prove the following results.

If (A_i) converges to A regularly, where all sets are contained in a compact metric space, then:

(1) A is locally connected (whether any of the A_i are or not);

(2) if in addition all sets A_i are locally connected continua, then every simple closed curve J and every simple arc ab in A (which is also a locally connected continuum) is the limiting set of a regularly convergent sequence of simple closed curves $J_1, J_2, \cdots$ or arcs $a_1b_1, a_2b_2, \cdots$ respectively, where $J_n, a_nb_n\subset A_n$;

(3) if each A_i is an arc a_ib_i, then A is an arc ab (or a single point), and by a proper choice of notation $(a_i)\rightarrow a$, $(b_i)\rightarrow b$;

(4) if each A_i is a simple closed curve, then A is a simple closed curve (or a single point);

(5) if each A_i is a topological sphere, then A is a cactoid [4];

(6) if each A_i is a 2-cell with boundary C_i such that $d(C_i)\rightarrow 0$ ($d(C_i)$ = diameter of C_i), then A is a cactoid;

(7) if each A_i is a 2-cell and their boundaries J_i converge to J, then J is a boundary curve—furthermore a necessary and sufficient condition for the convergence to be regular is that J be a simple closed curve;

(8) if the A_i are 2-cells with boundaries J_i that converge to J, then A is a hemicactoid [4] whose base set is bounded by J.

3. **r-regular convergence.** The type of convergence described in §2 can be thought of as the zero-dimensional case of r-dimensional regular convergence. In order to give this definition it is necessary to make use of some kind of r-dimensional homology theory. In a

compact metric space, perhaps the most convenient theory is the one introduced by Vietoris [5].

3.1 DEFINITION. An *ϵ-simplex of dimension r* is a collection of $(r+1)$-points whose diameter is less than ϵ.

3.2 DEFINITION. An *ϵ-cycle (chain) of dimension r* is a cycle (chain) of ϵ-simplices of dimension r.

3.3 DEFINITION. A cycle z^r is said to be *ϵ-homologous to* 0 $(z^r \sim_\epsilon 0)$ if z^r is the boundary of an ϵ-chain C^{r+1} $(z^r = \partial C^{r+1})$.

3.4 DEFINITION. Two cycles z_1^r and z_2^r are *ϵ-homologous* to each other $(z_1^r \sim_\epsilon z_2^r)$ if $z_1^r - z_2^r \sim_\epsilon 0$.

3.5 DEFINITION. An r-dimensional *Vietoris cycle* V^r is a collection of r-dimensional δ_i-cycles (z_i^r) called its *coordinate cycles* such that

(1) $(\delta_i) \to 0$,

(2) for each $\epsilon > 0$ there is an integer N such that for $m, n > N$, $z_m^r \sim_\epsilon z_n^r$.

3.6 DEFINITION. A Vietoris cycle V^r is *homologous to* 0 $(V^r \sim 0)$ if for each $\epsilon > 0$ all but a finite number of its coordinate cycles are ϵ-homologous to 0.

3.7 DEFINITION. Two Vietoris cycles V_1^r and V_2^r are *homologous* to each other if for each $\epsilon > 0$ all but a finite number of the corresponding coordinate cycles are ϵ-homologous.

The coefficient group for the coordinate cycles is assumed to be the mod 2 group. However, many results hold with more general groups which will be noted from time to time. In no case will the coefficient group be more general than a commutative ring with a unit element.

We can now give the general definition of regular convergence of any dimension.

3.8 DEFINITION. The sequence of closed sets (A_i) is said to converge *r-regularly* to A if corresponding to each $\epsilon > 0$ there is a positive integer N and a $\delta > 0$ such that if $n > N$, any Vietoris cycle V^j, $j \leqq r$, in a subset of A_n of diameter $< \delta$ is ~ 0 in a subset of A_n of diameter $< \epsilon$.

Here again if all the sets A_i were to coincide with A, then the above definition would become the usual one for uniform local-j-connectedness for all $j \leqq r$ in terms of Vietoris cycles.

This definition of r-regular convergence was also introduced by G. T. Whyburn [3] except that he required the condition only on cycles of the one dimension r. However so little results without imposing the same conditions on the lower-dimensional cycles that it is easier to include all of them in the one definition.

It is easily seen that in the 0-dimensional case this definition is equivalent to the regular convergence of §2, since in a compact metric space a V^0 can be interpreted as a pair of points and being ~ 0 is equivalent to the pair's lying in a connected subset. Thus we have a true generalization of the earlier definition.

In this type of convergence it is impossible to close up "holes" with r-dimensional boundaries that exist in the members of the sequence. For example suppose the set A_i is the set of polar coordinate points $\{(r, \theta) \mid r=1 \ (1/i \leqq \theta \leqq 2\pi-(1/i))\}$. The missing arc between $\theta=1/i$ and $\theta=-1/i$ would be called a hole with a zero-dimensional boundary, i.e. the two end points of the arc. This hole is gradually closed up as i increases so that the limit set is the entire circle which contains no hole with 0-dimensional boundary. This could happen because the convergence was *not* 0-regular. This follows by considering the pair of end points $(1/i, 1)$ and $(2\pi-(1/i), 1)$ which may be thought of as 0-dimensional Vietoris cycles. The diameters of these cycles $\to 0$ as $i \to \infty$, but although each is ~ 0 in its set A_i, the diameter of the smallest set in which this cycle is ~ 0 (namely all of A_i) is always 2—violating the definition of 0-regular convergence. Next consider the sets A_i in the 3-dimensional space described in spherical coordinates as follows: $1/i \leqq \rho \leqq 1$ (i.e. all the points between and on the 2 concentric spheres $\rho=1/i$ and $\rho=1$). As i increases the hole inside bounded by the 2-dimensional boundary $\rho=1/i$ is gradually shrunk so that the limit set is the solid sphere $0 \leqq \rho \leqq 1$. Here the 2-regular convergence is violated by the 2-cycles for the sphere $\rho=1/i$ on A_i can be thought of as 2-dimensional Vietoris cycles. Now these cycles are contained in subsets of A_i whose diameters converge to 0, yet none is ~ 0 at all—contrary to the definition of 2-regular convergence. Notice, however, in this case that the convergence is 1-regular. Next consider the sets A_i in 3-dimensional space described in Cartesian coordinates as all points (x, y, z) such that $x^2+y^2+z^2 \leqq 2$, and $x^2+i^2y^2+i^2z^2 \geqq 1$. Again as i increases the sets A_i converge to a solid sphere $A: x^2+y^2+z^2 \leqq 2$. However this time there are no nonbounding 2-cycles whose diameters converge to 0 as i increases. The convergence is still not 2-regular since the 1-cycles correspond to the circles $i^2y^2+i^2z^2=1$ converge to 0 as i increases, but none is ~ 0 on a subset of the corresponding A_i with diameter <1. This time the convergence is 0-regular. Finally let $A_i=$ the set of points (x, y, z) such that $x^2+y^2+z^2 \leqq 2$ and $x^2+y^2+z^2i^2 \geqq 1$, as i increases again (A_i) converges to the solid sphere $x^2+y^2+z^2 \leqq 2$. This time the 2-regular convergence is *violated* because the 0-cycles correspond to the pairs of points $(0, 0, 1/i)$, $(0, 0, -1/i)$ have diameters that converge to 0 as i increases, but again do not bound on subsets with diameters <1. In this case the 2-regular convergence is *not* violated because of the behavior of either the 1 or 2 cycles. Thus we see that a hole with a 2-dimensional boundary can be closed up if 2-regular convergence is violated by cycles of just one of the 3 dimensions 0, 1, or 2.

4. **The Betti numbers and local connectivity.** A more general and precise description of the behavior referred to in the previous section can be obtained in terms of the Betti numbers.

4.1 DEFINITION. In a compact metric space M the r-dimensional Betti number ($p^r(M)$) is the maximum number of linearly independent Vietoris cycles (i.e. a number k such that $a_1Z_1^r + \cdots + a_sZ_s^r \sim 0$ for any $s > k$ Vietoris cycles $Z_1^r, \cdots, Z_s^r$ and elements $a_1, \cdots, a_s$ from the coefficient group; but there exist k cycles $Z_1^r, \cdots, Z_k^r$ such that $a_1Z_1^r + \cdots + a_kZ_k^r \sim 0$ implies $a_1 = \cdots = a_k = 0$). If no such maximum number exists, we say that $p^r(M) = \infty$. Now a description of a "hole" with a 2-dimensional boundary can be given by saying that the 2-dimensional Betti number is one. To say that holes cannot be closed is included in the following theorem.

4.2 THEOREM. *If* $(A_i) \to A$ *r-regularly and* $p^r(A_i) \leqq n$ *for all* i, *then* $p^r(A) \leqq n$.

It is also true that holes cannot be formed under regular convergence according to the next theorem.

4.3 THEOREM. *If* $(A_i) \to A$ *r-regularly and* $p^r(A) \leqq n$, *then* $p^r(A_i) \leqq n$ *for all but a finite* i.

These theorems are due to H. A. Arnold [6].

In the case of 0-regular convergence, we saw that the limit set is always locally connected. The following theorem states the general result due to G. T. Whyburn [7].

4.4 THEOREM. *If* $(A_i) \to A$ *r-regularly, and each* A_i *is* lc^r (*locally-j-connected for all* $j \leqq r$), *then* A *is* lc^r.

In case the coefficient group is an arbitrary ring with a unit, E. G. Begle has shown [8] that the above theorem is not always true; but if each A_i is also required to be lc^n, then the theorem is true.

5. **Manifolds.** We have already seen that under 0-regular convergence a sequence of the simplest manifolds, i.e. simple closed curves or arcs, has for its limit a manifold of the same kind (if nondegenerate). It is natural to consider the behavior of higher-dimensional manifolds under the higher-dimensional regular convergence. However it is well known that no topological characterizations exist for the manifolds of dimension higher than 2; therefore, it is necessary

to consider generalized manifolds instead of the classical ones. A generalized manifold is a space which has many of the same homology characteristics as a classical manifold of the same dimension and reduces to the classical case for dimensions $\leqq 2$. The definition we shall use is due to R. L. Wilder [9].

5.1 Definition. The compact n-dimensional (Menger-Urysohn dimension—[10]) metric space M is called a *closed* (*orientable*) *n-dimensional manifold* if:

(1) $p^n(M)=1$, but $p^n(F)=0$ for every proper closed subset F of M.

(2) If $p\in M$, there exists an $\epsilon>0$ such that if V^i $(1\leqq i\leqq n-1)$ is an i-cycle of the sphere $S(p, \epsilon)$, then $V^i\sim 0$ in M.

(3) If $p\in M$, and $\epsilon>0$ are arbitrary, then there exist positive numbers δ and η, $\epsilon>\delta>\eta$, such that if V^i $(0\leqq i\leqq n-2)$ is a cycle of the boundary of the sphere $S(p, \delta)$, then $V^i\sim 0$ in $S(p, \epsilon)-S(p, \eta)$; if V^{n-1} is a cycle of this boundary, then $V^{n-1}\sim 0$ in $M-S(p, \eta)$.

By using cohomology theory E. G. Begle has given a definition [11] that is essentially equivalent. It is easily seen that the simple closed curve and 2-sphere satisfy Definition 5.1 for dimensions 1 and 2, respectively. Conversely, Wilder has shown that a 1- or 2-dimensional generalized manifold must be a classical one.

Begle has proved the following theorem [8].

5.2 Theorem. *If $M_i\to M$ $(r-1)$-regularly, where each M_i is a closed r-dimensional orientable generalized manifold and M is r-dimensional, then M is a closed r-dimensional orientable generalized manifold.*

This theorem was proved in the case where the coefficient group for cycles can be any commutative ring with a unit.

In the 2-dimensional cases the following theorems can be stated. The first was announced by H. E. Vaughan [12] and also proved by Begle [8].

5.3 Theorem. *Let (M_i) be a sequence of closed, orientable 2-dimensional manifolds and let (M_i) converge 1-regularly to a nondegenerate set M. Then M is a closed, orientable 2-dimensional manifold, and for all sufficiently large i, M and M_i are homeomorphic.*

The author has also defined a generalized manifold with boundary [29] and the following theorem is proved [30].

5.4 Theorem. *If S is an orientable n-dimensional generalized closed manifold and $K\subset S$ is an $(n-1)$-dimensional generalized closed manifold such that $S-K=A\cup B$ separate, and K is the common boundary of A and B, then $\bar{A}$ and $\bar{B}$ are n-dimensional generalized closed manifolds with boundary K.*

For our purposes the result of this theorem may be used as the definition of a manifold with boundary, and the author has proved the following result [31].

5.5 THEOREM. *If (M_i) is a sequence of orientable generalized n-manifolds with boundaries (K_i), such that $M_i \to M$ $(n-1)$-regularly and $K_i \to K$ $(n-2)$-regularly, then M is an orientable generalized n-manifold with boundary K.*

In the 2-dimensional case it is proved as in 5.3 that all but a finite number of the M_i are homeomorphic with M. Whyburn first proved the case where each M_i is a 2-cell, but used the following theorem to do it.

5.6 THEOREM. *If a sequence of 2-cells converge 1-regularly to a limit set, then the limit set is a base set.*

6. **An alternate definition of 0-regular convergence and its generalization.** Whyburn has given the following interesting characterization of 0-regular convergence [3].

6.1 THEOREM. *Let the sequence of closed sets (M_n) converge to the limiting set M. Then in order that (M_n) converge 0-regularly to M it is necessary and sufficient that for each sequence of decompositions $M_{n_i} = A_{n_i} + B_{n_i}$ into closed sets such that $(A_{n_i}) \to A$, $(B_{n_i}) \to B$, and $(A_{n_i} \cdot B_{n_i}) = (X_i) \to X$, we have $A \cdot B = X$.*

No similar characterization of the higher-dimensional cases is known, but the following similar theorems were announced by H. A. Arnold [6].

6.2 THEOREM. *If $(M_n) \to M$ r-regularly, then for every sequence of decompositions $M_{n_i} = A_{n_i} + B_{n_i}$ into closed sets, such that the sequence $(X_i) = (A_{n_i} \cdot B_{n_i}) \to X$ and the sequences (B_{n_i}), $(A_{n_i}) \to (r-1)$-regularly to B and A, respectively, then $(X_i) \to X$ $(r-1)$-regularly.*

6.3 THEOREM. *If $(M_n) \to M$ r-regularly, and $(A_n \cdot B_n) \to A \cdot B$ r-regularly, then $(A_n) \to A$ and $(B_n) \to B$, both r-regularly.*

7. **Regular convergence in terms of Čech cycles.** The use of Vietoris cycles ties the work to a metric space. If one wishes to break away from this restriction, the Čech theory of cycles in terms of coverings of the space by open sets can be used. The definition of local-r-connectivity, r-dimensional Betti number, generalized manifold, etc., can all be rephrased in terms of this theory with only the assumption of bicompactness (every open covering can be reduced to a finite sub-covering). These definitions are all given in R. L. Wilder's Col-

loquium publication [13]. Here the coefficient group is assumed to be an arbitrary field.

The author has proved [14] that the theorems regarding the Betti numbers, local connectivity, and generalized manifolds, 4.2, 4.3, 4.4, and 5.2, still hold in this case.

In proving the above results the following theorem concerning the existence of a normal refinement was proved.

7.1 THEOREM. *If* $(A_i)\rightarrow A$ *r-regularly, then corresponding to any covering of the space there is a covering which is a normal refinement of it relative to r-dimensional Čech cycles of* A_i *for any* i.

Wilder defines and proves the existence of such a covering for one set [13], but to the author's knowledge this is the only known theorem on the existence of such a refinement for an infinite collection of sets.

8. **Regular convergence spaces.** The collection of all closed lc^r subsets of a compact metric space M can be made into a hyperspace K^r by definining the notion of convergence by means of regular convergence. Thus we shall say that the sequence of points (a_i) of the hyperspace converges to the limit point a if their corresponding subsets (A_i) converge r-regularly to the set A corresponding to a. It is clear from Theorem 4.4 that the space K^r is closed relative to the similarly constructed hyperspace of all closed subsets of M. Also by 4.2 and 4.3 the set K^r_n consisting of all points of K^r corresponding to sets whose r-dimensional Betti number is n is closed. It is shown by the author [15] that

(8.1) K^r *is metrizable and separable.*

Begle also obtained this result independently [8] and showed that

(8.2) K^r *is topologically complete.*

Many point set properties can be related to special subsets of K^r as stated in the following theorems:

8.3 *If* $r>0$, *then a necessary and sufficient condition that* M *contain no simple closed curve is that* K_1 (*the subspace of* K^r *corresponding to locally connected continua*) *be connected. In case* M *is locally connected this is a necessary and sufficient condition for a dendrite* [4].

8.4 *A necessary and sufficient condition that every convergent sequence of arcs in* M *converge* 0*-regularly is that* K_1 *be compact.*

8.5 *If* M *is compact, then a necessary and sufficient condition that every convergent sequence of arcs converge* 0*-regularly is that corresponding to every* $\epsilon>0$ *there exists a* $\delta>0$ *such that if* p *and* q *are* 2 *points of* M *whose distance apart is* $<\delta$, *then every arc joining* p *and* q *has diameter* $<\epsilon$.

8.6 *If M is a Euclidean cube of dimension $\geqq r+2$ and of diameter 1, then the subset N of K^r consisting of those subsets of M which are lc^{r+1} is of the 1st Baire category in K^r.*

9. Relation to monotone transformations.

9.1 DEFINITION. If A is compact metric and the transformation $T(A)=B$ is continuous, then T is said to be *r-monotone* if for each $b \in B$, $p^i(T^{-1}(b))=0$, $i \leqq r$. In the 0-dimensional case this says that the inverse of every point is connected and reduces to the usual definition of a monotone transformation. It has been shown [4] that the monotone image of

(1) an arc is an arc,

(2) a simple closed curve is a simple closed curve (or a point),

(3) a topological sphere is a cactoid,

(4) a 2-cell with boundary J is a hemicactoid with boundary curve $f(J)$.

Also Wilder has shown [22] that

(5) the r-monotone image of a closed $(r+1)$-dimensional orientable generalized manifold is also a closed $(r+1)$-dimensional orientable generalized manifold.

Notice that if $(A_i) \rightarrow A$ 0-regularly and if each A_i is a set of type 1, 2, 3, or 4, we have already stated that A will also be a set of the same type. Also if the convergence is r-regular and each A_i is of type 5, then A is also of that type. This suggests that there may be a relationship between monotone transformations and regular convergence. The following theorem due to Whyburn [7] demonstrates this relationship.

9.2 THEOREM. *Let the sequence of r-monotone transformations $T_i(A)=B_i$ converge uniformly to the limit transformation $T(A)=B$. In order that the sequence $(B_i) \rightarrow$ to B r-regularly, it is necessary and sufficient that T be s-monotone, $s \leqq r$, and B be an lc^r.*

10. Regular transformations. Suppose A and B are compact metric spaces and the continuous mapping T carries A onto B. T is called *interior* [16] if it carries open sets into open sets. Eilenberg showed [17] that T is interior in this case if and only if for each sequence $(b_i) \rightarrow b$ in B, $(T^{-1}(b_i)) \rightarrow T^{-1}(b)$. A natural generalization is to require the convergence to be r-regular in which case the transformation is called *r-regular*. The 0-dimensional case was first studied by Wallace [18] and the r-dimensional case by Puckett [19] and the author [20]. Puckett showed that if T is $(n-1)$-regular, then for any two points b and b' of B the Betti groups of dimensions $\leqq n$ of $T^{-1}(b)$ and $T^{-1}(b')$ are isomorphic. The author has shown (a) that under an $(n-1)$-

regular transformation every small Vietoris cycle in B of dimension $\leqq n$ is the image of a small cycle in A. Also (b) that if the transform of a cycle of dimension $\leqq (n-1)$ in A is ~ 0 in a small subset of B, then the original cycle is ~ 0 in a small subset of A. From proposition (a) the following results can be proved. In each case assume $T(A)=B$ is $(n-1)$-regular.

10.1 THEOREM. *If the sequence of closed sets* $(Y_i) \rightarrow Y$ *in* B, *then* $(T^{-1}(Y_i)) \rightarrow T^{-1}(Y)$ *and if the latter convergence is n-regular so is the former.*

10.2 THEOREM. *If* T *is factored,* $T=T_2T_1$, *so that* $T_1(A)=C$ *is* $(n-2)$*-regular, then* $T_2(C)=B$ *is* $(n-1)$*-regular.*

10.3 THEOREM. *If* A *is* lc^n, *then* B *is* lc^n.

10.4 THEOREM. *If* A *is a continuum, then* T *can be factored,* $T=T_2T_1$, *so that* $T_1(A)=C$ *is monotone and* $(n-1)$*-regular and* $T_2(C)=B$ *is of constant multiplicity and locally topological.*

From proposition (b) the following results are obtained:

10.5 THEOREM. *In order that* $T(A)=B$ *be n-regular it is necessary and sufficient that for any sequence of closed sets* $(Y_i) \rightarrow Y$ *n-regularly in* B, *we have* $(T^{-1}(Y_i)) \rightarrow T^{-1}(Y)$ *n-regularly.*

10.6 THEOREM. *If* $T(A)=B$ *is n-regular and* $Y \subset B$ *is* lc^n, *then* $T^{-1}(Y)$ *is* lc^n.

10.7 THEOREM. *If* $T_1(A)=C$ *and* $T_2(C)=B$ *are n-regular, then so also is* $T=T_2T_1$.

Puckett proved the following interesting theorems [19; 21] in the 0-regular case, where in each case we assume $T(A)=B$ is 0-regular and monotone (i.e. the first factor guaranteed by 10.4 above).

10.8 THEOREM. *If* A *is a continuum, then its* 1*-dimensional Betti group is the direct sum of* 2 *groups* U *and* V, *where* U *is isomorphic to the* 1*-dimensional Betti group of* B *and* V *is isomorphic to the Betti group of each set* $T^{-1}(b)$ (b *is any point of* B).

10.9 THEOREM. *If* A *is a* 2*-dimensional pseudo-manifold (i.e. a classical* 2*-dimensional manifold with or without boundary among* q *points of which identifications have been performed to produce* r *local separating points of* A), *then the following situations can occur and only these:*

(1) T *is topological or* B *is a single point;*

$$(2)\ A = \begin{Bmatrix} \textit{sphere} \\ \textit{2-cell} \\ \textit{circular ring} \end{Bmatrix}, \ B = \textit{arc};$$

$$(3)\ A = \begin{Bmatrix} \textit{torus} \\ \textit{Klein bottle} \\ \textit{circular ring} \\ \textit{Möbius band} \\ \textit{pinched sphere} \\ \textit{2-cell with 2 boundary} \\ \quad \textit{points identified} \end{Bmatrix}, \ B = \textit{simple closed curve.}$$

Furthermore in the above theorem the topological behavior of each transformation can be precisely described. In (2) of the theorem if A is the sphere $x^2+y^2+z^2=1$, then T is essentially the mapping that carries the circle cut by $z=k$ $(-1 \leqq k \leqq 1)$ into the point $(0, 0, k)$ on the z-axis. If A is the 2-cell $x^2+y^2 \leqq 1$, then T is essentially the mapping that carries the segment cut by $y=k$ $(-1 \leqq k \leqq 1)$ into the point $(0, k)$ on the y-axis. If A is the circular ring $1 \leqq \rho \leqq 2$ (polar coordinates), then T is essentially the mapping that carries the circle $\rho=k$, $1 \leqq k \leqq 2$, into the point $\rho=k$, $\theta=0$. In (3) if A is the torus obtained from the cylinder $x^2+y^2=1$, $-1 \leqq z \leqq 1$, by identifying the circles corresponding to $z=-1$ and $z=1$, then T is essentially the mapping that carries the circle corresponding to $z=k$ into the point $(0, 0, k)$ of the simple closed curve obtained from the line segment $x=0$, $y=0$, $-1 \leqq z \leqq 1$, by identifying $(0, 0, 1)$ and $(0, 0, -1)$. If A is the Klein bottle obtained from the same cylinder by identifying the ends with opposite orientations, then the same transformation as before can be used. If A is the circular ring, $1 \leqq \rho \leqq 2$ (polar coordinates), then T is essentially the mapping that carries the segment $\theta=k$, $1 \leqq \rho \leqq 2$, into the point $\theta=k$, $\rho=1$. If A is the Möbius band obtained by cutting the above ring along $\theta=0$ and applying a twist before sewing it back together, then the same T as before can be used. If A is the pinched sphere obtained by rotating the circle $(x-1)^2+z^2=1$ about the z-axis, then T is essentially the mapping that carries the circle generated by the point $(x, \pm(1-(x-1)^2)^{1/2})$ into the point $(x, \pm(1-(x-1)^2)^{1/2})$. Finally if A is the 2-cell with 2 boundary points identified obtained by rotating the circle $(x-1)^2+z^2=1$ one-half a revolution about the z-axis, then T is the same as in the preceding case with half circles going into points.

11. Other types of regular convergence. We pointed out in §2 that regular convergence was derived from the definition of local connectedness by stringing the property out over the sets of a convergent sequence. This same technique can be used on other local properties such as (a) r-dimensional semi-local connectedness [23], (b) complete r-avoidability [24], and (c) homotopy local connectedness for dimensions $\leqq r$ (LC^r) [25]. The convergence obtained by using property (a) is called r-dimensional coregular convergence [26], that obtained from property (b) is called r-c.a. regular convergence [26], and that obtained from (c) homotopy-r-regular convergence [27]. It is shown [26] that if the convergence of a sequence of sets is $(r-1)$-regular and r-coregular then the limit set contains no r-cut points [24]. If the convergence is n-regular, i-c.a. regular for $i \leqq n-2$, and $(n-1)$-coregular, where the members of the sequence are each n-dimensional closed Cantorian manifolds [28], then the limit set is a closed (orientable) n-dimensional generalized manifold (Definition 5.1). It is shown [26] that in the 0-dimensional case, 0-c.a. convergence implies both 0-regular and 0-coregular convergence for continua. Also a sequence of 2-dimensional compact classical manifolds converge 0-c.a. regularly to another such manifold (or a point) and if each member of the sequence has n disjoint simple closed curves as boundary, then the limit set has $m \leqq n$ simple closed curves as its boundary—if each member is a sphere with n handles, then the limit set is a sphere with $m \leqq n$ handles. If the sequence converges homotopy-p-regularly where all members of the sequence and the limit set are LC^p (dimension $\leqq p$), then it is shown [27] that infinitely many members of the sequence have the same homotopy type as the limit set. Notice that it was not asserted that LC^p for the members of the sequence implies the same property for the limit sets, however a similar result can be shown [27].

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THE CONVOLUTION TRANSFORM

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Introduction. The material I am reporting on here was prepared in collaboration with I. I. Hirschman. It will presently appear in book form in the Princeton Mathematical Series. I wish also to refer at once to the researches of I. J. Schoenberg and his students. Their work has been closely related to ours and has supplemented it in certain respects. Let me call attention especially to an article of Schoenberg [5, p. 199] in this Bulletin where the whole field is outlined and the historical development is traced. In view of the existence of this paper I shall try to avoid any parallel development here. Let me take rather a heuristic point of view and concentrate chiefly on trying to entertain you with what seems to me a fascinating subject.

THE CONVOLUTION TRANSFORM

1. **Convolutions.** Perhaps the most familiar use of the operation of convolution occurs in its application to one-sided sequences $\{a_n\}_0^\infty$, $\{b_n\}_0^\infty$. The convolution (Faltung) of these two sequences is defined as the new sequence $\{c_n\}_0^\infty$,

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} a_{n-k} b_k. \tag{1.1}$$

The operation arises when power series are multiplied together:

$$\sum_{k=0}^{\infty} a_k s^k \sum_{k=0}^{\infty} b_k s^k = \sum_{k=0}^{\infty} c_k s^k.$$

The convolution of two-sided sequences,

$$c_n = \sum_{k=-\infty}^{\infty} a_k b_{n-k} = \sum_{k=-\infty}^{\infty} a_{n-k} b_k, \tag{1.2}$$

presents itself when two Laurent series are multiplied.

Hardly less familiar is the continuous analogue of (1.2),

$$c(x) = \int_{-\infty}^{\infty} a(x - y)b(y)dy = \int_{-\infty}^{\infty} a(y)b(x - y)dy, \tag{1.3}$$

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which arises when two Fourier integrals or two bilateral Laplace integrals are multiplied together [6, p. 258]. It is customary to abbreviate the convolution operation by the symbol $*$, so that (1.3) becomes $c(x) = a(x) * b(x)$. If $a(x)$ and $b(x)$ both vanish on $(-\infty, 0)$, (1.3) reduces to

$$c(x) = \int_0^x a(x-y)b(y)dy, \tag{1.4}$$

the analogue of (1.1).

2. **The convolution transform.** We may interpret (1.3) as an integral transform, designating one of the functions $a(x)$ or $b(x)$ as the kernel, the other being then *transformed* into $c(x)$. It is small exaggeration to say that nearly all the integral transforms in mathematical literature are either in this form or can be put into it by change of variable. We give below a number of examples. In subsequent work we shall denote the kernel by $G(x)$ and suppose that $f(x)$ is the transform of $\phi(x)$:

$$f(x) = G(x) * \phi(x) = \int_{-\infty}^{\infty} G(x-y)\phi(y)dy. \tag{2.1}$$

Example A. *A basic exponential transform.*

$$G(x) = g(x) = \begin{cases} e^x & (-\infty, 0), \\ 0 & (0, \infty), \end{cases} \tag{2.2}$$

$$f(x) = g(x) * \phi(x) = e^x \int_x^{\infty} e^{-y}\phi(y)dy. \tag{2.3}$$

Example B. *The Laplace transform (unilateral).*

$$F(x) = \int_0^{\infty} e^{-xy}\Phi(y)dy. \tag{2.4}$$

Replace x by e^x, y by e^{-y}. Then (2.4) becomes

$$F(e^x)e^x = \int_{-\infty}^{\infty} e^{-e^{x-y}} e^{x-y}\Phi(e^{-y})dy,$$

and this is equation (2.1) if

$$f(x) = F(e^x)e^x, \qquad G(x) = e^{-e^x}e^x, \qquad \phi(x) = \Phi(e^{-x}).$$

Example C. *The Stieltjes transform.*

$$F(x) = \int_0^{\infty} \frac{\Phi(y)}{x+y}dy.$$

This is in fact the first iterate of the Laplace transform (2.4). It takes the form (2.1) when

$$f(x) = F(e^x)e^{x/2}, \qquad G(x) = (1/2) \operatorname{sech} (x/2), \qquad \phi(x) = \Phi(e^x)e^{x/2}.$$

EXAMPLE D. *The Weierstrass (or Gauss) transform.*

$$f(x) = (4\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-(x-y)^2/4}\phi(y)dy. \tag{2.5}$$

This is already in convolution form with kernel $G(x)$ equal to the Gauss frequency function $(4\pi)^{-1/2}e^{-x^2/4}$.

EXAMPLE E. *A general transform.*

$$F(x) = \int_0^{\infty} K(xy)\Phi(y)dy. \tag{2.6}$$

This is clearly equivalent to (2.1) after exponential change of variables. Many of the classical transforms appear in this form. Examples are: Laplace, Fourier-sine, Fourier cosine, Hankel, Meier. The five corresponding kernels are

$$K(x) = e^{-x}, \sin x, \cos x, J_0(x), x^{1/2}K_0(x),$$

where $J_0(x)$ is the Bessel function, $K_0(x)$ the modified Bessel function:

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos (x \cos y)dy,$$

$$K_0(x) = \int_0^{\infty} e^{-x \cosh y}dy.$$

We observe that the general Fourier, the bilateral Laplace, and the Mellin transforms can be expressed as the sum of two integrals (2.6).

In view of these examples the importance of the convolution transform as a unifying influence can scarcely be doubted. The two basic problems for any transform are (a) inversion, (b) representation. In (a) we seek to recover $\phi(x)$ from $f(x)$, the kernel $G(x)$ being known; in (b) we inquire what functions $f(x)$ can be written as convolutions (2.1) for a given kernel $G(x)$. In the present paper we restrict attention to the former problem, referring to [3] or [4] for the latter.

3. **Operational calculus.** A very useful guide to the study of the convolution transform is the operational calculus. Its practical importance was brought forcefully to public attention by Heaviside when he used it so advantageously in the study of electric circuits. In brief, the technique consists in treating some operational symbol,

such as D for differentiation, as if it were a number throughout some calculation and finally in restoring to it its original operational meaning. Of course the success of the method depends upon the existence of a correspondence between the operational laws of combination on the one hand, and the algebraic ones on the other, but there is obviously no compulsion to investigate this correspondence if one can check results directly (as in the case of Heaviside's differential equations).

Let us illustrate. Denote by D the operation of differentiation with respect to x and by a an arbitrary constant. From the symbolic expansion

$$e^{aD} = \sum_{k=0}^{\infty} \frac{a^k D^k}{k!}$$

we obtain the Maclaurin series

$$e^{aD}\phi(x) = \sum_{k=0}^{\infty} \frac{a^k \phi^{(k)}(x)}{k!} = \phi(x + a). \tag{3.1}$$

We now *define* $e^{aD}\phi(x)$ as $\phi(x+a)$ even if $\phi(x)$ is not differentiable. Suppose that we seek a solution of the differential equation

$$f(x) - f'(x) = \phi(x)$$

by the operational method. Using the symbol D, we have

$$(3.2)\qquad \begin{aligned} (1 - D)f(x) &= \phi(x), \\ f(x) &= \frac{1}{1 - D}\,\phi(x), \end{aligned}$$

but it remains to interpret the operator $1/(1-D)$. Now

$$\frac{1}{1 - x} = \int_{-\infty}^{\infty} e^{-xy} g(y)\,dy, \qquad -\infty < x < 1, \tag{3.3}$$

where $g(y)$ is the function of Example A. This is easily verified by direct integration. Hence

$$\frac{1}{1 - D}\,\phi(x) = \int_{-\infty}^{\infty} e^{-yD}\phi(x) g(y)\,dy.$$

By (3.1) we thus obtain for the solution of (3.2)

$$f(x) = \int_{-\infty}^{\infty} \phi(x - y) g(y)\,dy = g(x) * \phi(x).$$

It is now an easy matter to verify that this is in fact a solution, at least for a large class of functions $\phi(x)$, by substituting the integral (2.3) in (3.2):

$$(1 - D)g(x) * \phi(x) = \phi(x).$$

Note that we have found an inversion operator, $1-D$, for the convolution transform (2.1) when the kernel $G(x)$ is the special function $g(x)$. Observe how it was obtained from equation (3.3).

More generally, this same operational method would lead us to expect that $E(D)$ would invert (2.1) if

$$\frac{1}{E(x)} = \int_{-\infty}^{\infty} e^{-xt}G(t)dt.$$

In other words, *the inversion function is the reciprocal of the Laplace transform of the kernel.* But how is $E(D)$ to be interpreted in the general case? If $E(x)$ is a polynomial there can be little doubt, but what if $E(x) = \cos x$ or $1/\Gamma(1-x)$, for example?

4. **The Laguerre-Pólya class.** We can make an effective interpretation of $E(D)$ if $E(x)$ belongs to a large class of functions originally studied by E. Laguerre (for references see Schoenberg [5]). We "normalize" the class, $E(0)=1$, in accordance with the following definition.

DEFINITION 4. *$E(x)$ belongs to class E if and only if*

$$E(x) = e^{-cx^2+bx}\prod_{k=1}^{\infty}\left(1 - \frac{x}{a_k}\right)e^{x/a_k}, \tag{4.1}$$

where a_k, b, c are real, $c \geqq 0$, and

$$\sum_{k=1}^{\infty} \frac{1}{a_k^2} < \infty.$$

For example, $1-x$, e^x, $\cos x$, $1/\Gamma(1-x)$, e^{-x^2} all belong to the class. Laguerre showed, Pólya introducing a refinement, that a function belongs to the class E if and only if it is the uniform limit of polynomials, each of which has real roots only and is equal to 1 at $x=0$. For example,

$$e^{-x^2} = \lim_{n\to\infty}\left(1 - \frac{x^2}{n}\right)^n.$$

Note that the corresponding equation for e^{x^2} would introduce polynomials with imaginary roots, and that $e^{x^2} \notin E$.

We mention this beautiful characterization of the class E only in passing, for we are not directly concerned with it here.[1] What does concern us is that the reciprocal of every function of E, except e^{bx}, is a Laplace transform, just as for $E(x)=1-x$ in equation (3.3). We state the result.

THEOREM 4.1. *If* $E(x)\in E$ *and* $E(x)\neq e^{bx}$, *then*

$$\frac{1}{E(x)} = \int_{-\infty}^{\infty} e^{-xy}G(y)dy, \tag{4.2}$$

the integral converging in the largest neighborhood of the origin which is free of zeros of $E(x)$.

It is not our purpose to give complete proofs of theorems here, but rather to outline methods. Let us first replace x by the complex variable $s=\sigma+i\tau$ and continue the function $E(x)$ analytically into the complex plane. We then show easily from (4.1) that $|E(\sigma+i\tau)| \geq |E(\sigma)|$. Next, the product relation (4.1) may be used to show that

$$|E(\sigma+i\tau)|^{-1} = O(|\tau|^{-p}), \qquad |\tau|\to\infty,$$

uniformly in any vertical strip $|\sigma|\leq R$. This is for any constant p, however large, if $E(s)$ is not the product of e^{bs} by a polynomial (in which case it is the degree of the polynomial). We can now appeal to a familiar theorem from the general theory of Laplace integrals [1, p. 126] which insures a representation (4.2) for any function $1/E(s)$ which is analytic in and uniformly small at the two ends of a vertical strip. Since $1/E(s)$ is analytic in the largest vertical strip containing the origin and free of zeros of $E(s)$ the representation (4.2) is valid there.

We need also some properties of the function $G(y)$ of (4.2).

THEOREM 4.2. *The function* $G(y)$ *of Theorem* 4.1 *has the properties*:

A. $$G(y)\geq 0, \qquad -\infty<y<\infty,$$

B. $$\int_{-\infty}^{\infty} G(y)dy = 1,$$

C. $$\int_{-\infty}^{\infty} yG(y)dy = b,$$

D. $$\int_{-\infty}^{\infty} (y-b)^2G(y)dy = c + \sum_{k=1}^{\infty} a_k^{-2}.$$

---

[1] In [4] we use it to show that the kernels $G(x)$ under discussion form a semi-group and hence in some sense exhaust *all* kernels amenable to our methods.

In the language of statistics, $G(y)$ is a frequency function of mean b and variance $c+\sum_{k=1}^{\infty} a_k^{-2}$. With the exception of A these properties are all corollaries of Theorem 4.1. Conclusion B follows from the normalization $E(0)=1$ when we set $x=0$ in (4.2); C follows by computing the first derivative of the integral (4.2) at $x=0$ and the computation is best done logarithmically; D follows by computing the second derivative of $e^{bx}/E(x)$ at $x=0$.

Property A is a little harder to prove, but the essential reason for its truth is almost intuitive. The reciprocal of each factor in (4.1) (except e^{bx}) is the Laplace transform of a non-negative function:

$$
\begin{gathered}
e^{cx^2} = (4\pi c)^{-1/2}\int_{-\infty}^{\infty} e^{-xy}e^{-y^2/(4c)}\,dy, \\
\left(1-\frac{x}{a_k}\right)^{-1} e^{-x/a_k} = |a_k| \int_{-\infty}^{\infty} e^{-xy} g(a_k y - 1)\,dy,
\end{gathered}
\tag{4.3}
$$

where $g(x)$ is the non-negative function of Example A. By the fundamental "product theorem" for Laplace transforms referred to in §1 we should expect that

$$
G(y) = (4\pi c)^{-1/2} e^{-(y-b)^2/4c} * |a_1|\, g(a_1 y - 1) * |a_2|\, g(a_2 y - 1) * \cdots
$$

if we are optimistic about matters of convergence. But then A is obvious since the convolution of positive functions is again positive.

In the more leisurely development available in a book [4] we have actually followed a different course in the proofs of both the theorems of the present section.

5. **Inversion.** Let us now show that the operator $E(D)$, if properly interpreted, is indeed effective for the inversion of (2.1). We treat first the case $c=0$, the factor e^{-cx^2} in (4.1) being then missing. For the purposes of presentation here we assume the simplest of conditions on $\phi(x)$. It should be emphasized, however, that the ultimate in generality has been achieved in this direction. We have shown that if $\phi(x)$ is any function for which (2.1) converges, then

$$
E(D)G(x) * \phi(x) = \phi(x)
$$

almost everywhere. This result should be contrasted with Jordan's theorem for Fourier series or integrals where some such local condition as bounded variation is needed.

THEOREM 5. *If* $E(x)\in E$ *with* $b=c=0$, *if* $G(x)$ *is the function of Theorem* 4.1, *and if* $\phi(x)$ *is bounded and continuous on* $(-\infty, \infty)$, *then*

$$
E(D)G(x) * \phi(x) = \phi(x), \qquad -\infty < x < \infty, \tag{5.1}
$$

where

$$E(D) = \lim_{n\to\infty} \prod_{k=1}^{n}\left(1 - \frac{D}{a_k}\right)e^{D/a_k}. \tag{5.2}$$

We sketch the proof. Set

$$P_n(D) = \prod_{k=1}^{n}\left(1 - \frac{D}{a_k}\right)e^{D/a_k}, \tag{5.3}$$

so that (5.1) is equivalent to

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} G_n(x - y)\phi(y)dy = \phi(x),$$

where

$$G_n(x) = P_n(D)G(x).$$

We have applied the operator (5.3) under the integral sign (2.1), a step easily justified with present hypotheses. By the classical inversion of (4.2), see [6, p. 241],

$$G(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{sx}}{E(s)}\,ds.$$

Again applying (5.3),

$$G_n(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{sx}P_n(s)}{E(s)}\,ds.$$

But the function

$$E_n(s) = \frac{E(s)}{P_n(s)}$$

itself belongs to E, so that Theorem 4.2 may be used to obtain the properties of $G_n(x)$. Set

$$\begin{aligned} I_n(x) &= \int_{-\infty}^{\infty} G_n(x - y)\phi(y)dy - \phi(x) \\ &= \int_{-\infty}^{\infty} G_n(y)\,[\phi(x - y) - \phi(x)]dy. \end{aligned}$$

For a fixed x and $\delta > 0$ write the integral $I_n(x)$ as the sum of two others $I_n'(x)$ and $I_n''(x)$ corresponding to the ranges of integration $|y| \leq \delta$ and $|y| > \delta$, respectively. Then

$$| I_n'(x) | \leq \max_{|y|\leq\delta} | \phi(x-y) - \phi(x) |$$

by properties A and B of Theorem (4.2). Also

$$\begin{aligned} | I_n''(x) | &\leq 2 \sup_{-\infty<y<\infty} | \phi(y) | \int_{|t|>\delta} \frac{y^2}{\delta^2} G_n(y)dy \\ &\leq \frac{2}{\delta^2} \sup_{-\infty<y<\infty} | \phi(y) | \sum_{k=n+1}^{\infty} a_k^{-2}, \end{aligned}$$

by property D of Theorem 4.2. It is thus clear that $I_n'(x)$ can be made small by choice of δ, $I_n''(x)$ by choice of n, so that $I_n(x)\to 0$ when $n\to\infty$, as desired.

The choice $b=0$ was inconsequential, amounting essentially to a choice of origin. A case in which $c\neq 0$ will be treated in §7.

6. **Application to the Laplace transform.** We now apply the foregoing theory to Example B of §2. But let us first recall the following real inversion [6, p. 288] of the Laplace integral (2.4),

$$\lim_{n\to\infty} \frac{(-1)^n}{n!} F^{(n)}\left(\frac{n}{y}\right)\left(\frac{n}{y}\right)^{n+1} = \Phi(y). \tag{6.1}$$

It is valid, whenever (2.4) converges, for almost all x and certainly at points where $\Phi(x)$ is continuous. For example, for the pair $\Phi(x) = e^{-x}$, $F(x) = (x+1)^{-1}$ it becomes $\lim_{n\to\infty} [1 + (y/n)]^{-n-1} = e^{-y}$. We shall show that (5.1) reduces to (6.1) in Example B. The function $E(x)$ in the present case is $1/\Gamma(1-x)$. For, the Laplace transform of the kernel is

$$\int_{-\infty}^{\infty} e^{-e^y} e^y e^{-xy} dy = \int_0^{\infty} e^{-y} y^{-x} dy = \Gamma(1-x), \qquad -\infty < x < 1.$$

Since $1/\Gamma(1-x)$ is a function of class E with $c=0$, Theorem 5 is applicable.[3] We need the familiar product expansion of the gamma function,

$$\frac{1}{\Gamma(1-x)} = e^{-\gamma x} \prod_{k=1}^{\infty} \left(1 - \frac{x}{k}\right) e^{x/k},$$

$$\gamma \sim \sum_{k=1}^{n} \frac{1}{k} - \log n, \qquad n \to \infty.$$

Set

---

[3] We really need a slight modification of Theorem 5 since $b=-\gamma$ in the present example.

$$P_n(x) = n^x \prod_{k=1}^{n} \left(1 - \frac{x}{k}\right),$$

so that (5.1) is equivalent to

$$\lim_{n\to\infty} P_n(D) e^x F(e^x) = \Phi(e^{-x}), \tag{6.2}$$

in the present case. But the left-hand side of (6.2) takes on an especially simple form arising from the equation

$$(1 - D) e^x F(e^x) = - e^{2x} F'(e^x).$$

By induction

$$\prod_{k=1}^{n} \left(1 - \frac{D}{k}\right) e^x F(e^x) = \frac{(-1)^n}{n!} e^{(n+1)x} F^{(n)}(e^x).$$

Of course n^D or $e^{(\log n)D}$ means translation of x through $\log n$ or multiplication of e^x by n. Hence

$$P_n(D) e^x F(e^x) = \frac{(-1)^n}{n!} n^{(n+1)x} F^{(n)}(n e^x),$$

and if $e^{-x} = y$, (6.2) becomes (6.1), as predicted.

7. **The Weierstrass transform.** If we denote the "source solution" of the heat equation by $k(x, t)$,

$$k(x, t) = (4\pi t)^{-1/2} e^{-x^2/4t}, \qquad -\infty < x < \infty,\ 0 < t < \infty,$$

then equation (2.5) becomes

$$f(x) = k(x, 1) * \phi(x). \tag{7.1}$$

Recalling that the inversion function predicted by the operational calculus is the reciprocal of the Laplace transform of the kernel and from (4.3) that

$$e^{s^2} = \int_{-\infty}^{\infty} e^{-sy} k(y, 1)\, dy, \tag{7.2}$$

we expect that

$$e^{-D^2} f(x) = \phi(x).$$

This operational equation was already observed by A. Eddington [2], who replaced e^{-D^2} by its Taylor development. However, this interpretation is usually ineffective because the resulting series diverges for most functions $f(x)$. We employ a different method.

Equation (4.3) is valid for all complex s, so that if we replace x by $-iD$ therein we obtain

$$e^{-tD^2} = (4\pi t)^{-1/2}\int_{-\infty}^{\infty} e^{iyD}e^{-y^2/4t}dy.$$

Again using (3.1) with $a = ix$,

$$e^{-tD^2}f(x) = (4\pi t)^{-1/2}\int_{-\infty}^{\infty} f(x + iy)e^{-y^2/4t}dy.$$

This integral is clearly a complex integral evaluated along a vertical line in the complex plane. If we set $x+iy=s$ it becomes

$$(7.3) \qquad e^{-tD^2}f(x) = -i(4\pi t)^{-1/2}\int_{x-i\infty}^{x+i\infty} f(s)e^{(s-x)^2/4t}ds.$$

For all functions $f(x)$ arising from equation (7.1) the integral (7.3) will be independent of the path of integration, and we accordingly make our definition more flexible by replacing the path of (7.3) by an arbitrary one.

DEFINITION 7. *The operator* e^{-D^2} *is defined as*

$$(7.4) \qquad \lim_{t\to 1-} e^{-tD^2}f(x) = \lim_{t\to 1-} -i(4\pi t)^{-1/2}\int_{c-i\infty}^{c+i\infty} f(s)e^{(s-x)^2/4t}ds,$$

where c is a suitable constant.

This definition should be compared and contrasted with (5.2). The continuous parameter t in (7.4) corresponds to the discrete one n in (5.2). The use of the parameter $t<1$ in (7.4) amounts to the use of Abel summability on an integral that would diverge for certain functions $f(x)$ when $t=1$.

We can now state our inversion result in terms of this operator. As in §5 we do so with simplifying but unnecessary restrictions on $\phi(x)$.

THEOREM 7. *If $\phi(x)$ is bounded and continuous on* $(-\infty, \infty)$ *and if $f(x)$ is its Weierstrass transform, given by* (7.1), *then*

$$(7.5) \qquad e^{-D^2}f(x) = \phi(x), \qquad -\infty < x < \infty.$$

We sketch the proof. Inverting equation (7.2), we have

$$k(x, 1) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{sx+s^2}ds.$$

If we compute the integral (7.4), $c=0$ when $f(x)$ is replaced by $k(x, 1)$, we have

$$e^{-tD^2}k(x, 1) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{sx+s^2}e^{-ts^2}ds,$$

and this is $k(x, 1-t)$ as we see by inverting (4.3). Hence after validating the application of e^{-tD^2} under the integral sign (Fubini's theorem) one obtains

$$(7.6) \qquad e^{-tD^2}f(x) = \int_{-\infty}^{\infty} k(x-y, 1-t)\phi(y)dy = k(x, 1-t)*\phi(x).$$

But this is the Weierstrass singular integral so familiar in the theory of heat conduction. It tends to $\phi(x)$ as $t \to 1-$, and (7.5) follows.

From the point of view of heat conduction $e^{tD^2}\phi(x)$ or $k(x, t)*\phi(x)$ is the temperature of an infinite rod t seconds after it was $\phi(x)$ and the Weierstrass transform of $\phi(x)$ is the temperature one second after it was $\phi(x)$. This makes (7.6) intuitive: $e^{-tD^2}f(x)$ is the temperature t seconds *before* it was $f(x)$ or $1-t$ seconds after it was $\phi(x)$.

8. Table of inversion operators. We conclude with a summary of the inversion operators mentioned in §2. Many others are treated in [4].

	Kernel $G(x)$	Inversion operator $E(D)$
A.	$g(x)$	$1-D$
B.	$e^{-e^x}e^x$	$1/\Gamma(1-D)$
C.	$(2\pi)^{-1}\operatorname{sech}(x/2)$	$\cos \pi D$
D.	$(4\pi)^{-1/2}e^{-x^2/4}$	e^{-D^2}
E.	$(2/\pi)K_0(e^x)e^x$	$\pi 2^D\Gamma^{-2}[(1-D)/2]$

In Example C, the fact that the Stieltjes transform is the iterated Laplace transform expressed in convolution form becomes

$$e^{x/2}F(e^x) = e^{-e^x}e^{x/2} * e^{-e^x}e^{-x/2} * e^{x/2}\Phi(e^x).$$

The corresponding relation for the inversion functions $E(s)$ is the classical equation

$$\Gamma\left(\frac{1}{2}-s\right)\Gamma\left(\frac{1}{2}+s\right) = \frac{\pi}{\cos \pi s}.$$

Example E is the Meier transform.

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HARVARD UNIVERSITY

THE FIRST SUMMER MATHEMATICAL INSTITUTE

The First Summer Mathematical Institute, devoted to Lie algebras and Lie groups, was held from June 20 to July 31, 1953 at Colby College, Waterville, Maine. It was supported by a grant from the National Science Foundation to the American Mathematical Society. The Organization Committee consisted of C. C. Chevalley, A. M. Gleason, and Nathan Jacobson (chairman).

The following twenty mathematicians attended the Institute by invitation of the Organization Committee: Armand Borel, C. C. Chevalley, W. L. Chow, A. M. Gleason, Morikumi Goto, G. P. Hochschild, Kenkichi Iwasawa, Nathan Jacobson, Irving Kaplansky,[1] E. R. Kolchin, W. G. Lister, Deane Montgomery,[1] G. D. Mostow, Hans Samelson, R. D. Schafer, E. V. Schenkman, H. C. Wang, Hidehiko Yamabe, Hans Zassenhaus, and Leo Zippin.[1]

The Institute was open to all interested mathematicians and the following nine attended: S. A. Amitsur, S. G. Bourne, J. L. Brenner, H. E. Campbell, C. W. Curtis, E. C. Paige, Jr., A. J. Penico, G. B. Seligman, and M. L. Tomber.

The formal program of the Institute consisted of a seminar on simple Lie algebras and the following four series of lectures:

Armand Borel, *The cohomology of compact connected Lie groups and their coset spaces*;
C. C. Chevalley, *Cartan subalgebras and Cartan subgroups*;
Hidehiko Yamabe, *Structure of locally compact groups*;
Hans Zassenhaus, *Representation theory of Lie algebras of characteristic p*.

These are summarized below.

In addition there was a series of single talks on various mathematical topics as follows:

Kenkichi Iwasawa, *On some totally disconnected compact groups.*
H. C. Wang, *Closed manifolds with homogeneous complex analytic structure.*
Morikumi Goto, *Dense imbeddings of locally compact connected groups.*
E. R. Kolchin, *Galois theory of differential fields.*
W. L. Chow, *The Albanese variety of an algebraic variety.*
A. M. Gleason, *Lattices of topologies.*
Hans Zassenhaus, *Trace functions of characteristic p.*
G. D. Mostow, *Uniform subgroups of solvable groups.*

Received by the editors June 7, 1954.

[1] Attended for the first three weeks only.

C. C. Chevalley, *Galois theory of semi-simple Lie algebras.*

Colby College provided office space and dormitory accommodations for the members of the Institute and their families and made available their recreation area on Great Pond where a picnic was held on the fourth of July.

Beyond the lectures themselves, the informal contacts that the Institute afforded were tremendously valuable. Several papers, many of them collaborations, will soon appear in a Memoirs volume enentitled *Proceedings of the First Summer Mathematical Institute*, and it is expected that many other results will evolve from the stirring of ideas effected by the six weeks of study and conversation.

A. M. GLEASON

THE COHOMOLOGY OF COMPACT CONNECTED LIE GROUPS AND THEIR COSET SPACES

Lectures by Armand Borel

This seminar was devoted to a survey of the main results obtained so far in the study of the cohomology ring of compact connected Lie groups and their homogeneous spaces. One may roughly divide these twelve talks into four groups.

In the first part were discussed general properties of groups, i.e., properties which derive solely from the existence of a product (say associative with unit); that such an approach is fruitful was shown by H. Hopf, who proved that the cohomology ring $H^*(X, K_0)$ of such a compact manifold, with respect to a field of coefficients K_0 of characteristic zero, is a Grassmann algebra generated by elements of odd degrees, or in other words that it is isomorphic to the cohomology ring of a product of odd-dimensional spheres. A generalization of this theorem, valid for coefficient fields of any characteristic, was established. The product in X also allows one to define a product (introduced by Pontrjagin) in the homology group of X, which thus becomes a ring; over a field of characteristic zero, it is again an exterior algebra (Samelson's theorem), but this result has been generalized only in some special cases; the chief new difficulty arising here is the fact that the Pontrjagin product does not have the familiar anticommutativity property of the cup-product in general.

The second part was devoted to relations between the cohomology ring $H^*(G, K_p)$ (K_p field of characteristic p) of a compact connected Lie group G and the cohomology ring $H^*(B_G, K_p)$ of a classifying space B_G for G, a point which is at the source of most of the progress made recently in these questions. We recall that a fiber bundle E, with base B and typical fiber F, is said to be a principal bundle with structural group G if G operates on E, leaving each fiber invariant, and being simply transitive on them (thus F is homeomorphic to G); it is a universal bundle for G and for n if moreover its n first homotopy groups vanish. The existence of universal bundles for any compact Lie group and any n is well known and important. The base space of a universal bundle is called a classifying space for G (and for n), and denoted B_G; it owes its name to the fact that the different structures of principal bundles with structural group G over a given locally finite polyhedron B (of dimension $\leqq n$) are in 1-1 correspondence with the homotopy classes of continuous maps of B into B_G. By use of Leray's spectral sequence of fiber bundles one can establish relations between $H^*(G, K_p)$ and $H^*(B_G, K_p)$, or more precisely one can see how certain assumptions on the structure of one of them affect the other; that study has not yet been performed in the most general case; however, partial results (with a rather wide range of applications) have been obtained and were discussed. The main one says that if $H^*(G, K_p)$ is an exterior algebra generated by r elements of odd degrees r_i $(1 \leqq i \leqq m)$, then $H^*(B_G, K_p)$ is a ring of polynomials over r variables of degrees r_i+1, and conversely.

When U is a closed subgroup of G, there exists a natural map of B_U into B_G, denoted $\rho(U, G)$, which is particularly interesting when $U=T^r$ is a maximal torus of G; in that case $H^*(B_U, K_p)$ is a ring of polynomials in r variables of degree two, on which the Weyl group of G acts in a natural fashion (e.g., over the real field, $H^*(B_{T^r}, R)$ may be identified with the ring of polynomials with real coefficients on the Lie algebra of T^r). It was shown that under suitable assumptions, always fulfilled in characteristic zero (mainly $H^*(G, Z)$ has no p-torsion, i.e., has no torsion coefficient divisible by p), the homomorphism $\rho^*(T^r, G)$ induced by $\rho(T^r, G)$ maps $H^*(B_G, K_p)$ isomorphically into $H^*(B_{T^r}, K_p)$, essentially on the ring of invariants of the Weyl group. Thus one gets a group-theoretical interpretation of $H^*(B_G, K_p)$, which, via the theorem stated at the end of the preceding paragraph, also gives relations between $H^*(G, K_p)$ and the Weyl group (obtained first in characteristic zero by C. Chevalley). Whether one can tie up so closely $H^*(B_G, K_p)$ with group theoretical properties of G in the general case where G has p-torsion is unknown; modulo 2, however, a special case has been worked out, that of the orthogonal group, where replacing the maximal torus by the subgroup of diagonal matrices leads to results bearing a close and suggestive analogy to the preceding ones.

The methods and results expounded in the first two parts do not allow one in general to determine $H^*(G, K_p)$ for a given G having p-torsion, and for that purpose one has to use special devices, taking advantage of some particular properties of the group in consideration. The third group of talks described how the above general theorems combined with the study of spectral sequences of special fiberings give the cohomology mod p of the classical groups, the spinor groups, and the first two exceptional groups G_2, F_4.

The last part of the seminar was devoted to homogeneous spaces. Here the problem is to find relations between the cohomology rings of a group G, a subgroup U, and the coset space G/U; to be more specific, one wants mainly to determine the last one by means of the two others and of some information describing the position of U in G. Rather complete results in that direction have been obtained for the cohomology over the real numbers, by use of differential-geometric and algebraic methods, the most comprehensive one being a theorem of H. Cartan; it shows how to calculate (at least theoretically) $H^*(G/U, R)$, knowing $H^*(G, R)$, and certain data which turn out to be essentially $H^*(B_U, R)$ and $\rho^*(U, G)$, and there was given here a topological proof, in the framework of fiber bundle theory.

Mod p, the situation is less satisfactory; though the methods and results known so far are sufficient to deal with many classical homogeneous spaces, they nevertheless pertain only to special cases. The discussion was restricted mainly to one of these, that of equality of ranks (i.e., G and U have a maximal torus in common); also, the analogy mentioned previously between the role of maximal tori in real cohomology and of the diagonal matrices for the cohomology mod 2 of the orthogonal group $O(n)$ was pursued further and allowed us to determine $H^*(O(n)/U, K_2)$ when U has the same "rank mod 2" as $O(n)$, i.e., when it contains an abelian subgroup of type $(2, \cdots, 2)$ maximal in $O(n)$.

ARMAND BOREL

CARTAN SUBALGEBRAS AND CARTAN SUBGROUPS

Lectures by C. C. Chevalley

We consider a field K of characteristic zero and a vector space V of finite dimension n over K. The set $L(V)$ of all K-endomorphisms of V is a vector space of dimension n^2 over K. $L(V)$ becomes a Lie algebra if it is endowed with the bracket operation $[X, Y] = XY - YX$; we denote this Lie algebra by $\mathfrak{gl}(V)$. If the intersection $\mathfrak{G}$ of some algebraic variety in $L(V)$ with the group of nonsingular endomorphisms of V is a group, then $\mathfrak{G}$ is called an algebraic group of automorphisms of V. Forming the differentials (formal) at the unit element of the polynomials defining $\mathfrak{G}$, we get a family of linear functions on $\mathfrak{gl}(V)$ which annihilate a subalgebra of $\mathfrak{gl}(V)$ called the Lie algebra of $\mathfrak{G}$. Not every subalgebra of $\mathfrak{gl}(V)$ arises in this way; those that do are called algebraic Lie algebras. Algebraic groups and their Lie algebras can now be studied by the methods of algebraic geometry.

A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is said to be nilpotent if, for each $X\in\mathfrak{h}$, the endomorphism $Y\rightarrow[X, Y]$ of $\mathfrak{g}$ is nilpotent. The normalizer of a subalgebra $\mathfrak{h}$ is the largest subalgebra of $\mathfrak{g}$ in which $\mathfrak{h}$ is an ideal. We define a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ as a nilpotent subalgebra of $\mathfrak{g}$ which is its own normalizer.

A group is nilpotent if the ascending central series terminates with the whole group. The subgroup $\mathfrak{H}$ of $\mathfrak{G}$ is called a Cartan subgroup if $\mathfrak{H}$ is a maximal nilpotent subgroup of $\mathfrak{G}$ and if every normal subgroup $\mathfrak{K}$ of finite index in $\mathfrak{H}$ is of finite index in the normalizer of $\mathfrak{K}$.

The main objective of the lectures was the proof of the following theorem.

Let $\mathfrak{G}$ be an irreducible algebraic group and let $\mathfrak{g}$ be the Lie algebra of $\mathfrak{G}$. If $\mathfrak{H}$ is a Cartan subgroup of $\mathfrak{G}$, then $\mathfrak{H}$ is algebraic and its Lie algebra is a Cartan subalgebra of $\mathfrak{g}$. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then it is the Lie algebra of some Cartan subgroup of $\mathfrak{G}$.

A. M. GLEASON

THE STRUCTURE OF LOCALLY COMPACT GROUPS

Lectures by Hidehiko Yamabe

The main objective in the theory of locally compact groups is to relate them to Lie groups. The structure theorem can be stated as follows:

THEOREM. *Let G be a connected locally compact group. In every neighborhood of the identity there is a compact normal subgroup N such that G/N is a Lie group.*

While this result is not valid for disconnected locally compact groups, every locally compact group contains an open subgroup for which the above conclusion holds. This fact is easily deduced from the theorem as stated.

The theorem was first proved for groups of finite topological dimension by Montgomery and Zippin [3]. Yamabe [5; 6] has proved the general case using a refinement of a technique introduced by Gleason

[1] in proving that finite-dimensional locally compact groups without small subgroups are Lie groups.

The reasoning given by von Neumann [4] in his solution of Hilbert's fifth problem for compact groups in conjunction with this theorem readily proves the affirmative answer to the customary form of Hilbert's fifth problem: Every locally Euclidean group is a Lie group.

Yamabe begins with a connected locally compact group G and proves that in any neighborhood of the identity there is a compact normal subgroup N for which G/N has no small subgroups; i.e., there is a neighborhood V of the identity in G/N which contains no entire nontrivial subgroup. The remainder of the proof consists in showing that G/N is a Lie group.

Starting once again with a connected locally compact group H having no small subgroups, he constructs a linear space T of "tangent vectors to H" in $L^2(H)$, the Hilbert space of square integrable functions with respect to the Haar measure of H, and a homeomorphism of a neighborhood of the origin in T onto a compact subset of G. (This turns out to be the exponential mapping when T is identified with the Lie algebra of G.) The existence of such a homeomorphism proves that T has finite dimension. The inner automorphisms of H induce linear transformations on T; this gives a finite-dimensional representation of H whose kernel is just the center C. (This is the adjoint representation in the classical theory of Lie groups.) By von Neumann's theorem H/C is a Lie group. Since C is a locally compact abelian group without small subgroups, it is already known to be a Lie group from the work of Pontrjagin. At this point one can apply the theorem of Kuranishi [2] and deduce that H is a Lie group, but, since Kuranishi's proof depends on long arguments with differential equations, it is more satisfying to prove that H is a Lie group directly. This Yamabe does by explicitly calculating the group multiplication in terms of the canonical coordinates induced in H by the exponential mapping of the tangent vectors.

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A. M. GLEASON

REPRESENTATION THEORY OF LIE ALGEBRAS OF CHARACTERISTIC p

Lectures by Hans Zassenhaus

1. **Introduction.** The first serious investigations of the linear representations of Lie algebras of prime characteristic were carried out by Zassenhaus [10], who made an exhaustive study of the irreducible representations of nilpotent Lie algebras, and by Ho-Jui Chang [3], who determined the irreducible representations of the Witt algebras. At about the same time, Jacobson began his work on restricted Lie algebras of characteristic $p>0$, and proved a number of important theorems concerning their representations. These results give many indications of the shape a general theory must assume, but such a theory has been slow to materialize. Nevertheless, important results of a general nature have been obtained recently by Jacobson and Zassenhaus.

Jacobson's contribution, to be found in [7], included the observation that the universal associative algebra $\mathfrak{A}$ of a Lie algebra $\mathfrak{L}$ over a field of characteristic $p>0$ is a finitely generated module over its center. This result led to a new and simple proof of Iwasawa's theorem that $\mathfrak{L}$ has a faithful finite-dimensional representation, a proof of a conjecture of Chevalley which asserts that every Lie algebra of prime characteristic has a representation which is not completely reducible, and a proof of the fact that $\mathfrak{A}$ can be imbedded in a division algebra, finite-dimensional over its center.

By exploiting the fact that $\mathfrak{A}$ can be imbedded in a division ring, Zassenhaus exhibited a bound on the degree of the absolutely irreducible representation (a.i.r.) of $\mathfrak{L}$, and proved that the maximum degree was assumed except for those a.i.r. which annihilate a certain discriminant ideal associated with $\mathfrak{A}$. His methods also lead to the theorem that $\mathfrak{L}$ has indecomposable representations of arbitrarily high degree.

This report contains a more detailed description of these results, together with some comments on the proofs. A few unsolved problems have also been listed.

2. **The universal associative algebra.** We begin with the result of Birkhoff [1] and Witt [9], which asserts that a Lie algebra $\mathfrak{L}$ over an arbitrary field K has a faithful imbedding η in an associative algebra

$\mathfrak{A}^*$, which is universal in the sense that if σ is any other imbedding of $\mathfrak{L}$, then there exists an associative homomorphism of $\mathfrak{A}^*$ onto the enveloping algebra of σ. A universal imbedding of a finite-dimensional Lie algebra $\mathfrak{L}$ with basis $x_1, \cdots, x_n$ may be constructed as follows. Form the free associative algebra $\mathfrak{F}$ over K on the elements $x_1, \cdots, x_n$, and consider the difference algebra $\mathfrak{F}/\mathfrak{B}$, where $\mathfrak{B}$ is the ideal generated by the elements $x_ix_j - x_jx_i - \sum \gamma_{ijk}x_k$, and $[x_ix_j] = \sum \gamma_{ijk}x_k$ is the multiplication table in $\mathfrak{L}$. Then the mapping η: $\sum \alpha_i x_i \to \sum \alpha_i \cdot (x_i + \mathfrak{B})$ of $\mathfrak{L}$ into $\mathfrak{A}^*$ is a universal imbedding, and $\mathfrak{A}^*$ is the enveloping algebra of η. If we denote the cosets of the x_i by $\mathfrak{x}_i$, then it can be proved that the standard monomials $\mathfrak{x}_1^{e_1} \cdots \mathfrak{x}_n^{e_n}$, $(e_1, \cdots, e_n) \neq 0$, form a basis of $\mathfrak{A}^*$ over K, and that, since the $\mathfrak{x}_i$ are linearly independent, the imbedding η is faithful. We shall identify $\mathfrak{L}$ with the linear part of $\mathfrak{A}^*$, and in particular, write $x_1, \cdots, x_n$ for the $\mathfrak{x}_i$.

The importance of the universal enveloping algebra $\mathfrak{A}^*$ for the theory of representations stems from the fact that there is a natural one-to-one correspondence between the representations of $\mathfrak{L}$ and the representations of $\mathfrak{A}^*$, which can be described as follows. If σ is a representation[3] of $\mathfrak{L}$, then the unique associative homomorphism of $\mathfrak{A}^*$ onto the enveloping algebra of σ defines a representation of $\mathfrak{A}^*$, and conversely, if τ is a representation of $\mathfrak{A}^*$. and if η is the universal imbedding, then $\sigma = \eta\tau$ is a representation of $\mathfrak{L}$.

For various technical reasons it is convenient to adjoin an identity element $1 \equiv x_1^0 x_2^0 \cdots x_n^0$ to $\mathfrak{A}^*$, and the algebra $\mathfrak{A}$ obtained, while it is not, strictly speaking, a universal enveloping algebra of $\mathfrak{L}$, will be called nevertheless the *universal associative algebra* of $\mathfrak{L}$. The one-to-one correspondence between the representations of $\mathfrak{L}$ and those of $\mathfrak{A}$ persists, for we can extend the representation of $\mathfrak{A}^*$ corresponding to a given representation of $\mathfrak{L}$ to one of $\mathfrak{A}$ by mapping the identity element of $\mathfrak{A}$ onto the identity linear transformation.

We now introduce an increasing *filtration* of $\mathfrak{A}$, that is, a family $\{\mathfrak{A}_i, -\infty < i < \infty\}$ of subspaces of $\mathfrak{A}$ such that $\mathfrak{A}_i \subseteq \mathfrak{A}_{i+1}$, $\bigcup \mathfrak{A}_i = \mathfrak{A}$, and $\mathfrak{A}_i\mathfrak{A}_j \subseteq \mathfrak{A}_{i+j}$. In fact, we set $\mathfrak{A}_{-1} = \mathfrak{A}_{-2} = \cdots = (0)$; $\mathfrak{A}_0 = K \cdot 1$, $\mathfrak{A}_1 = \mathfrak{L}$, and we define $\mathfrak{A}_k$, $k > 1$, inductively to be the subspace $\mathfrak{A}_{k-1} + \mathfrak{L}^k$, where $\mathfrak{L}^k$ consists of all finite sums of products $\alpha u_1 u_2 \cdots u_k$, $\alpha \in K$, $u_i \in \mathfrak{L}$. Then we define the *degree* $d(x)$ of an element x by the formula $d(x) = \min_{x \in \mathfrak{A}_i} i$; then d is a function on $\mathfrak{A}$ to the non-negative integers and $-\infty$ such that (i) $d(\alpha x) \leqq d(x)$, $\alpha \in K$, (ii) $d(x+y) \leqq \max (d(x), d(y))$, and (iii) $d(xy) \leqq d(x) + d(y)$. This definition is independent of the choice of a basis in $\mathfrak{A}$, but it can be shown easily that $d(x)$ coincides with the notion of degree one is led to by writing out x as a linear combination of the standard monomials.

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[3] It is not assumed that the representation space is finite-dimensional.

Corresponding to the algebra $\mathfrak{A}$, together with the filtration $\{\mathfrak{A}_i\}$, we can associate a graded algebra $\mathfrak{M}$ which is, as a vector space, the direct sum of the spaces $\mathfrak{A}_i/\mathfrak{A}_{i-1}$, and where the product of two cosets $\mathfrak{s}_i \in \mathfrak{A}_i/\mathfrak{A}_{i-1}$ and $\mathfrak{s}_j \in \mathfrak{A}_j/\mathfrak{A}_{j-1}$ is defined to be the coset in $\mathfrak{A}_{i+j}/\mathfrak{A}_{i+j-1}$ of $s_i s_j$, where $s_i \in \mathfrak{s}_i$, and $s_j \in \mathfrak{s}_j$. The following result is the key to many of the arithmetic properties of the algebra $\mathfrak{A}$.

THEOREM 1. *The graded algebra $\mathfrak{M}$ associated with the universal associative algebra $\mathfrak{A}$ of a Lie algebra $\mathfrak{L}$, together with its filtration by degrees, is isomorphic to the polynomial ring $K[X_1, \cdots, X_n]$, where n is the dimension of $\mathfrak{L}$.*

This result seems to provide the natural setting for the following properties of the algebra $\mathfrak{A}$, which can also be proved directly. The first result of interest states that the function d satisfies a sharper form of the relation (iii) above, namely $d(xy) = d(x) + d(y)$. This formula implies on the one hand that $\mathfrak{A}$ has no divisors of zero, and on the other hand, that the units of $\mathfrak{A}$ coincide with the nonzero elements of $K \cdot 1$, so that $\mathfrak{A}$ is semi-simple in the sense of Jacobson.

A second property of $\mathfrak{A}$, first established by Birkhoff and Whitman [2], states that $\mathfrak{A}$ satisfies the maximum condition for left ideals.

We assume now, and throughout the remainder of the article, that the base field K has characteristic $p > 0$.

N. Jacobson proved in [7] that for each generator x_i of $\mathfrak{A}$ there exists a nonzero polynomial $f_i \in K[\lambda]$ such that $y_i = f_i(x_i)$ is in the center $\mathfrak{C}$ of $\mathfrak{A}$. Let $\mathfrak{R} = K[y_1, \cdots, y_n]$. Then $\mathfrak{R}$ has two properties of importance: the commuting elements y_i are algebraically independent over K, so that $\mathfrak{R}$ is isomorphic to the polynomial ring in n variables; and it is possible to find a finite $\mathfrak{R}$-linearly independent set of p^N generators for the $\mathfrak{R}$-module $\mathfrak{A}$, where p^N is the product of the degrees of the polynomials y_i. Now let R be the quotient field of $\mathfrak{R}$, and form the tensor product $D = \mathfrak{A} \otimes R$ with respect to $\mathfrak{R}$. Since $\mathfrak{A}$ is a free $\mathfrak{R}$-module, the natural mapping $x \rightarrow x \otimes 1$ of $\mathfrak{A}$ into $\mathfrak{A} \otimes R$ is an isomorphism of $\mathfrak{A}$ into D, and we shall identify $\mathfrak{A}$ with its image in D. Then D has no divisors of zero, and since D is a finite-dimensional algebra over R, D is a division algebra. The center of D is the quotient field C of $\mathfrak{C}$, the center of $\mathfrak{A}$, and the dimension of D over C is an even power of p, which we shall denote by p^{2m}. Finally it can be proved that $\mathfrak{A}$ is a bounded maximal order[3] in D, that $\mathfrak{C}$ is integrally closed in C, and that $\mathfrak{C}$ is integrally dependent on $\mathfrak{R}$ so that $\mathfrak{C}$ has degree of transcendence n over K, where n is the dimension of $\mathfrak{L}$.

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[3] Cf. N. Jacobson, *The theory of rings*, New York, 1943, Chap. 6.

3. **General properties of the representations.** We consider first an arbitrary irreducible representation σ of $\mathfrak{A}$ with kernel $\mathfrak{U}$. It is proved in [4] that $\mathfrak{A}/\mathfrak{U}$ is a finite-dimensional algebra over K, and this implies that every irreducible representation of $\mathfrak{A}$ (and hence of $\mathfrak{L}$) has finite degree. If we combine this fact with the result that $\mathfrak{A}$ is a semi-simple algebra, so that the intersection of the kernels of the irreducible representations of $\mathfrak{A}$ is the zero ideal, then we can prove that $\mathfrak{A}$ has sufficiently many (finite-dimensional) irreducible representations. A consequence of this result is the following theorem (cf. [4]), which is actually a sharpened form of a result proved by Iwasawa [5] and Jacobson [7].

THEOREM 2. *Every finite-dimensional Lie algebra over a field of characteristic $p>0$ has a faithful finite-dimensional completely reducible representation.*

The rift between characteristic p and characteristic 0 widens when one observes that there is no class of algebras of characteristic p every one of whose representations is completely reducible. The precise result, which was conjectured by Chevalley and proved by Jacobson [7], is the following.

THEOREM 3. *Every finite-dimensional Lie algebra over a field of prime characteristic has a finite-dimensional representation which is not completely reducible.*

To gain a deeper insight towards the behavior of the irreducible representations, we adopt the approach of H. Zassenhaus, and consider the relations among the representations of the center $\mathfrak{C}$ of $\mathfrak{A}$ and the representations of $\mathfrak{A}$ itself. A homomorphism Φ of $\mathfrak{A}$ is said to be an *extension* of a homomorphism ϕ of $\mathfrak{C}$ if ϕ is the restriction of Φ to $\mathfrak{C}$. We note that if Φ extends ϕ, then the image $\mathfrak{A}\Phi$ of $\mathfrak{A}$ is a $\mathfrak{C}\phi$-module. Then we can state a theorem due to Zassenhaus, which asserts that if ϕ is a homomorphism of $\mathfrak{C}$ onto the base field K, then there exists a *generic extension* Φ of ϕ with the property that if Ψ is any other extension of ϕ, then there exists a $\mathfrak{C}\phi$-module homomorphism τ of $\mathfrak{A}\Phi$ onto $\mathfrak{A}\Psi$. The proof of this result depends only upon the structure of $\mathfrak{A}$ as a finitely generated $\mathfrak{C}$-module, and is based on the theory of elementary ideals due to Steinitz and Zassenhaus.[4]

The main results on the absolutely irreducible representations which have been obtained to date, and which are due to Zassenhaus, can be derived in two ways: either by making use of the minimum polynomial of $\mathfrak{A}$ as Zassenhaus has done, or by applying certain results on minimal identities for matrix algebras due to Levitzki and Amitsur [8]. We shall indicate both approaches to the subject, because the relationship between them is a problem still to be worked out.

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[4] Cf. H. Zassenhaus, *Lehrbuch der Gruppentheorie*, Leipzig, 1937.

First we consider Zassenhaus' approach. We select a module basis $a_1, \cdots, a_r$ for $\mathfrak{A}$ over $\mathfrak{C}$, and define the *general element* a of $\mathfrak{A}$ to be $\sum_{i=1}^{r} X_i a_i$, an element of the ring $\mathfrak{A}_{\mathfrak{P}}$ obtained by extending the coefficient ring of $\mathfrak{A}$ from $\mathfrak{C}$ to $\mathfrak{P} = \mathfrak{C}[X_1, \cdots, X_r]$, where the X_i are algebraically independent over $\mathfrak{C}$. Then $A_{\mathfrak{P}}$ is a subring of the algebra D_Q obtained by extending the coefficient field of D from the quotient field of $\mathfrak{C}$ to Q, the quotient field of $\mathfrak{P}$, and a is an element of the finite-dimensional algebra D_Q. Then a has a minimum polynomial $f(\lambda)$, whose coefficients are in $\mathfrak{P}$ since $\mathfrak{C}$ is integrally closed.

By comparing the degree of $f(\lambda)$ with the degree of the minimum polynomial of the general element of $A\Psi$, where Ψ is an a.i.r. of A, the following result is obtained.

THEOREM 4. *The degree of every absolutely irreducible representation of $\mathfrak{A}$ is less than or equal to p^m, where p^{2m} is the dimension of D over its center.*

The trace of the general element a of $\mathfrak{A}$, which is the negative of the second coefficient of $f(\lambda)$, has the form $\sum_{i=1}^{r} \lambda_i X_i$, $\lambda_i \in \mathfrak{C}$. We then define the *reduced trace* $\mathrm{tr}(u)$ of an arbitrary element $u \in \mathfrak{A}$, $u = \sum \xi_i a_i$, $\xi_i \in \mathfrak{C}$, by the formula $\mathrm{tr}(u) = \sum \lambda_i \xi_i$. Corresponding to the trace function, there is a bilinear form g on $\mathfrak{A} \times \mathfrak{A} \rightarrow \mathfrak{C}$, namely $g(x, y) = \mathrm{tr}(xy)$. We define the *discriminant* $\mathfrak{d}$ of $\mathfrak{A}$ over $\mathfrak{C}$ to be the ideal in $\mathfrak{C}$ generated by the determinants $|g(u_i, v_k)|$, where $u_1, \cdots, u_{p^{2m}}$ and $v_1, \cdots, v_{p^{2m}}$ are arbitrary elements of $\mathfrak{A}$. Then we can state

THEOREM 5. *Let K be an algebraically closed field. Let ψ be a homomorphism of $\mathfrak{C}$ onto K, and let Ψ be a generic extension of ψ. Then the following statements are equivalent.*

(i) $\mathfrak{A}\Psi$ *is a central simple algebra over* K;

(ii) $\mathfrak{A}\Psi$ *has an irreducible representation of degree p^m, and hence every irreducible representation extending ψ has degree p^m*;

(iii) $\mathfrak{d}\psi \neq (0)$.

The bound on the degrees of the a.i.r. of $\mathfrak{A}$ can also be obtained by applying the result of Levitzki and Amitsur, which states that the standard identity $S_{2n}(x)$ in $2n$ variables vanishes identically on the full matrix algebra of order n, but not on the full matrix ring of order s for $s > n$. Moreover, if we consider the ideal $\tilde{\mathfrak{d}}$ in $\mathfrak{A}$ generated by the elements $S_{2(p^m-1)}(a_1, \cdots, a_{2(p^m-1)})$ where the a_i are arbitrary in $\mathfrak{A}$, then a result analogous to Theorem 5 can be proved, specifically that the degree of an a.i.r. Ψ is p^m if and only if $\tilde{\mathfrak{b}}\Psi \neq (0)$.

The following results have been obtained by Zassenhaus concerning the indecomposable representations.

THEOREM 6. *Let K be algebraically closed, and let Ψ be an indecomposable representation of $\mathfrak{U}$. Then any two irreducible constituents of Ψ have the same restrictions to $\mathfrak{C}$.*

There are examples, however, which show that an indecomposable representation may have inequivalent irreducible constituents.

THEOREM 7. *Let K be algebraically closed, and let ψ be an arbitrary homomorphism of $\mathfrak{C}$ onto K. Then there exist indecomposable representations of arbitrarily high degree, each of which has an irreducible constituent extending ψ.*

4. **Special results and problems.** The representation theory of nilpotent Lie algebras over an algebraically closed field of characteristic $p>0$ has been studied fully by Zassenhaus in [10] and [11]. We summarize his main results, some of which (in particular Theorem 8) have been extended to nilpotent Lie algebras over an infinite field by N. Jacobson and the writer.

THEOREM 8. *If Ψ is an indecomposable representation of a nilpotent Lie algebra $\mathfrak{L}$ over an algebraically closed field, then any two irreducible constituents of Ψ are equivalent.*

THEOREM 9. *Let $\mathfrak{L}$ be a nilpotent Lie algebra over an algebraically closed field K. Then the degree of any irreducible representation of $\mathfrak{L}$ is a power of p. If we select in $\mathfrak{L}$ a basis $(a_1, \cdots, a_n)$ such that, whenever $i<j$, $[a_ia_j] \in (a_{i+1}, \cdots, a_n)$, then for each ordered set $(\lambda_1, \cdots, \lambda_n)$ of elements of K, there exists one and only one irreducible representation $\Gamma: x \rightarrow x\Gamma$ of $\mathfrak{L}$ such that λ_i is the unique eigenvalue of $a_i\Gamma$, $i=1, \cdots, n$.*

A determination of the irreducible representations of the Witt algebras has been given by Ho-Jui Chang in [3]. By making use of graded algebra techniques, Zassenhaus has given new proofs of a number of these results [12]; in particular he has given a direct proof of the theorem of Chang which states that the integer m for the p-dimensional Witt algebra $\mathfrak{L}$ is exactly $(p-1)/2$, and thus the bound on the degree of the absolutely irreducible representations of $\mathfrak{L}$ is given explicitly in this case. The values of m for simple algebras other than the Witt algebras are unknown at this time.

We know that in general the center $\mathfrak{C}$ of $\mathfrak{U}$ does not coincide with the ring $\mathfrak{R} = K[y_1, \cdots, y_n]$ described above. It seems to be an interesting problem to determine the structure of the field extension of the quotient field of $\mathfrak{C}$ over the quotient field of $\mathfrak{N}$. It is known that for nilpotent Lie algebras the extension is purely inseparable.

The existence of an extension to $\mathfrak{A}$ of a given homomorphism ψ of $\mathfrak{C}$ is known only when ψ maps $\mathfrak{C}$ onto a field. For the study of the indecomposable representations it is of importance to know whether ψ can be extended if $\mathfrak{C}\psi$ is a primary algebra. Progress in this direction may wait upon a knowledge of the ideal theoretic properties of $\mathfrak{A}$ considered as a maximal order in D.

Jacobson has proved in [6] that a restricted Lie algebra $\mathfrak{L}$ has only a finite number of inequivalent irreducible representations. The representations of $\mathfrak{L}$ are in 1-1 correspondence with the representations of the $\mathfrak{U}$-algebra of $\mathfrak{L}$, which is a finite-dimensional algebra. The problem of the determination of the representations of $\mathfrak{L}$ involves first of all detailed information concerning the radical of the $\mathfrak{U}$-algebra of $\mathfrak{L}$.

Another problem concerns the construction of representations of a Lie algebra $\mathfrak{L}$ having prescribed constituents. For example, if Ψ_1 and Ψ_2 are irreducible representations of $\mathfrak{L}$, does there exist an indecomposable representation of $\mathfrak{L}$ having Ψ_1 and Ψ_2 as constituents? This problem can be formulated in terms of a cohomology theory for representation spaces.

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C. W. CURTIS

SEMINAR ON SIMPLE LIE ALGEBRAS

The heart of the structure theory for Lie algebras of characteristic 0 is given by the following two statements: (1) An algebra with no solvable ideals is a direct sum of simple algebras, (2) The simple algebras are given by an explicit list, comprising four infinite families and five exceptional algebras. Statement (1) becomes false for characteristic p, and finding a substitute for it is a problem that awaits solution. It seems to be timely to attack the classification of simple Lie algebras of characteristic p. Very likely a complete solution of this second problem would carry with it valuable hints for the solution of the first. It is our purpose to give a list (believed to be essentially complete) of the known simple Lie algebras over an algebraically closed field. The list naturally falls into three parts: the classical algebras, the exceptional algebras, and the Witt algebra and its generalizations.

Classical algebras. Type A. Let T be the Lie algebra of all n by n $(n \geqq 2)$ matrices of trace 0 over a field F. If the characteristic of F is 0 or prime to n, T is a simple algebra (of dimension n^2-1). If the characteristic does divide n, T still has a one-dimensional center Z consisting of all scalar matrices. Except when the characteristic is two and $n=2$, T/Z is simple and has dimension n^2-2.

Types B, D. A matrix (a_{ij}) is *alternate* if $a_{ii}=0$, $a_{ij}=-a_{ji}$ for all i and j. Let S be the Lie algebra of all alternate n by n matrices over a field. For $n=3$ and $n \geqq 5$, S is a simple algebra of dimension $n(n-1)/2$.

Type C. On a two-by-two matrix algebra we may define an involution $*$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

A matrix of even order, split into two by two blocks as (A_{ij}), admits an extension of this involution: $(A_{ij})^* = (A_{ji}^*)$. Let S denote the Lie algebra of all matrices of order $2m$, skew with respect to this involution; S has dimension $2m^2+m$. For characteristic different from two, S is simple as it stands. If the characteristic is two and m is odd, $m \geqq 3$, the cube of S is a simple algebra of order $2m^2-m-1$. If m is even, $m \geqq 4$, the cube of S still has to be reduced modulo a one-dimensional center, and we get a simple algebra of order $2m^2-m-2$.

Exceptional algebras. Let C be the Cayley matrix algebra (i.e., the unique Cayley-Dickson algebra with no divisors of zero) over a field F. Let A be the algebra of derivations of C. Then A is always 14-dimensional, and for characteristic different from 3 it is simple. When the characteristic is 3, A acquires a 7-dimensional ideal I of inner derivations, and I is isomorphic to the algebra of 3 by 3 matrices of trace 0, reduced modulo the center. The algebra A/I of outer automorphisms of I is again isomorphic to I.

For characteristic 0 there are four more exceptional algebras, of orders 52, 78, 133, and 248. The analogues of these for characteristic p have yet to be investigated. Schafer and Tomber have proved that the obvious candidate for the 52-dimensional one (the derivations of the exceptional Jordan algebra) degenerates for characteristic two by acquiring a 26-dimensional ideal of inner derivations.

The Witt algebra and its generalizations. Let F be a field of characteristic p and B the (commutative associative) p-dimensional algebra over F generated by an element x satisfying $x^p = 0$. Let A be the algebra of derivations of B. If we write D_i for the derivation sending x into x^{i+1}, the D's form a basis of A with multiplication table $[D_iD_j] = (i-j)D_{i+j}$. B is a p-dimensional Lie algebra which is simple for $p \neq 2$; it was discovered by Witt.

Two generalizations have been given in the literature. Zassenhaus used the same multiplication table, with the subscripts allowed to range over any additive subgroup of F (instead of the integers mod p). Jacobson took the derivation algebra of the algebra generated by several indeterminates with pth power 0.

There is a family of algebras which includes both of these generalizations. Let V be a vector space over F, and G a total additive group of functionals on V. Let A be the vector space direct sum of copies of V, one for each member of G. The general element of A is expressible as a sum over G of elements (x, α), $x \in V$, $\alpha \in G$. We define

$$[(x, \alpha)(y, \beta)] = (\alpha(y)x - \beta(x)y, \alpha + \beta).$$

The resulting Lie algebra is simple except when V is one-dimensional and the characteristic is two; there is then a simple ideal with dimension smaller by one. Zassenhaus' algebra is the case where V is one-dimensional, while Jacobson's is the case where G consists of all integer-valued functionals.

I. KAPLANSKY

THE JUNE MEETING IN PORTLAND

The five hundred fourth meeting of the American Mathematical Society was held at Reed College, Portland, Oregon, on Saturday, June 19, 1954, following the meeting on Friday of the Pacific Northwest Section of the Mathematical Association of America.

Attendance at the meetings was approximately 85, including 63 members of the Society.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor V. L. Klee, Jr., of the University of Washington delivered an hour address entitled *Convex sets in linear spaces*. Professor Klee was introduced by Professor R. A. Beaumont.

There were two sessions for contributed papers, at which Professors A. T. Lonseth and Harold Chatland presided.

On Friday evening preceding the meeting there was a joint dinner of the Society and Association, at which the visitors were greeted by President Ballantine of Reed College.

Abstracts of papers presented at the meeting follow. Abstracts whose titles are followed by "*t*" were presented by title. Mrs. Lehmer was introduced by the Associate Secretary, Mr. Rall by Professor Lonseth, Professor Nash by Professor W. T. Martin, and Mr. Krabble by Professor C. B. Morrey.

ALGEBRA AND THEORY OF NUMBERS

588*t*. W. E. Barnes: *Primal ideals in noncommutative rings.*

In any associative ring R an element x is not right prime (nrp) to an ideal A if $yRx\subseteq A$ for some $y\notin A$. An ideal is primal if the elements nrp to it form an ideal. These definitions differ from those of Curtis (Amer. J. Math. vol. 76 (1952) pp. 687–700) but reduce to them for rings with unit and A.C.C. for ideals. They also reduce to Fuchs' (Proc. Amer. Math. Soc. vol. 1 (1950) pp. 1–8) for commutative rings. An ideal B is nrp to A if every element of B is nrp to A. Maximal nrp to A ideals always exist and their intersection is called the adjoint of A. In a class of rings, called uniform, the maximal nrp ideals of any ideal are prime. The A.C.C. implies uniformity, but not conversely. Results (similar to the classical Noether theory) on representations of an ideal as the intersection of primal ideals with prime adjoints are obtained which include those of Fuchs and Curtis. (Received May 3, 1954.)

589. W. E. Barnes: *Principal component ideals in noncommutative rings.*

If A and B are ideals in any associative ring such that $A\subseteq B$, the lower right isolated B-component of A, $L(A,B)$, is the ideal sum of all ideals Am^{-1}, where m is right prime to B (see previous abstract) and $Am^{-1}=\{x \mid xRm\subseteq A\}$. The upper right isolated B-component of A, $U(A,B)$ (which always contains $L(A,B)$) is the intersection of all ideals $C \supseteq A$ and such that every m right prime to B is right prime to C. If B is a maximal nrp to A ideal, then $L(A, B)$ and $U(A, B)$ are called lower and upper right principal components of A. For B a prime ideal in a commutative ring, these definitions reduce to those of W. Krull. It is shown that every ideal in an associative ring is the intersection of its lower right principal components, and under certain conditions is also the intersection of its upper right principal components. (Received May 3, 1954.)

590*t*. H. W. Becker: *Latin orthotopes*. Preliminary report.

These are rectangular figures in space of any number of dimensions, whose line elements are unrepetitive permutations in any direction parallel to the axes. $L^{m,n}_{i,f}$ is the number of LO with n finite axes of f letters, and m axes of i letters, $i \to \infty$. $W^{m,n}_{i,f} = i^{i^m f^n}$ is the corresponding number of word orthotopes, that is, $m+n$ dimensional crosswords = harmonies = permutations repetitive or not. $W \supset L$, and $L^{m,n}_{i,f} \sim (N/D)^{i^m f^n}$, where $N = N(\pi, i, m, f, n)$, $D = D(e, i, m, f, n)$, generalizing $L^{1,1}_{i,f} \sim (i!)^f/e^{f(f-1)/2}$, Amer. Math. Monthly vol. 50 (1943) p. 512; Erdős and Kaplansky, Amer. J. Math. vol. 68 (1946) p. 230. For $f = ig/h$, a correction gives $L^{1,1}_{i,ig/h} \sim N(i/e^{g/2})^{i^2 g/h}$; $L^2_i \sim (i!)^i/e^{i(i+1)/2}$. This approximation for L^2_7 is almost perfect, by Norton's enumeration as corrected by Sade (Marseilles), Ann. Math. Statist. vol. 22 (1951) p. 306, Mann and Sade, Math. Reviews vol. 10 (1949) p. 278, verified by Yamamoto (Fukuoka), Bull. of Math. Stat. vol. 5 (1952) p. 1; but for L^2_8 it exceeds the estimates of Norton and Sade. Since $L^m_i \sim N(i/e^{(m+1)/2})^{i^m}$, the chance of m-dimensional derangement is $L^{m,1}_{i,2}/(L^m_i)^2 \sim 1/e^{i^{m-1}}$. These results grew out of contacts with D. H. Lehmer and H. S. M. Coxeter (who coined the title, generalizing p. 192, Math. Recreations and Essays, 1940), Amer. Math. Monthly vol. 51 (1944) p. 307; see Riordan, ibid. vol. 49 (1952) p. 160. Like swinging 3 bats before stepping up to the plate, they make Latin $\boxplus$ seem easier by comparison; as a punching bag, they may serve to sharpen others' attack on the moot N/D. (Received April 22, 1954.)

591*t*. H. W. Becker: *New Diophantine double and triple systems*.

The system $x^2+y^2=u^2$, $u^2+y^2=\square$ is impossible (R. D. Carmichael, *Diophantine analysis*, p. 14). But $x^2+y^2+z^2=t^2$, $t^2+y^2=q^2$ has the solution $q, t=(6r^2+s^2)^2 \pm (2rs)^2$, $x=36r^4-s^4$, $z=(2rs)^2$, $y=4rs(6r^2+s^2)$. General solution depends on that of $R^4-6R^2S^2+S^4=2\boxplus=q^2-t^2$, an Eels quadrilateral (EQ) with long odd side $=\square$ and both Pythagorean $\blacktriangle$ primitive. A solution of $x^2+y^2+z^2=t^2$, $t^2+y^2=\square$, $x^2-z^2=\square$ is to choose $2rs=\square$ in t, $x=9r^2 \pm 4rs+s^2$; $y, z=2(3r \pm s)(2rs)^{1/2}$. Then $(t^4-y^4)(x^4-z^4)=\square$, discriminant of a Petrus vector, here the Desboves-Lenhart. The smallest example of the kind is $t, y, x, z=21, 20, 5, 4$; t, y and x, z are also the Pythagorean parameters of an EQ with both odd sides $\boxplus$ and the component $\blacktriangle$ both primitive, the smallest example being $9^2+40^2=41^2$, $41^2+840^2=841^2$. An explicit solution of such $X^2+Y^2=U^2$, $U^2+Z^2=T^2$ is $T, X=(36r^4 \pm 16r^2s^2+s^4)^2$; $Y, Z=8r^2s^2(6r^2 \pm s^2)(36r^4 \pm 8r^2s^2+s^4)$; $U=1296r^8-56r^4s^4+s^8$: whence cyclic quads. with 2 sides $\boxplus$, as 1681, $9 \cdot 841$, $41 \cdot 840$, $40 \cdot 841$, etc. A solution of Euler's problem $\iota^2(\zeta^2-\theta^2)$, $\iota^4-\zeta^2\theta^2=\boxplus$ (Bull. Amer. Math. Soc. vol. 60 (1954) p. 136) is $[\iota, \zeta, \theta]=[(TX)^{1/2}, U, T \text{ or } X]$ $=[r(9r^4 \pm 16r^2s^2+4s^4), 81r^8-56r^4s^4+16s^8, (9r^4 \pm 16r^2s^2+4s^4)^2]$, in terms of the other solution to the above EQ. Only tabular solutions of $t^2 \pm (y^2+z^2)=\square$ are known. Look up a congruent number $=2\boxplus$ (Dickson's *History* II, Chap. XVI), e.g., the first one, 5. Then $(5^2+4^2)^2 \pm 4 \cdot 5 \cdot 4(5^2-4^2)=41^2 \pm (12^2+24^2)=49^2, 31^2$. Can any pair of these t, y, z etc. be sides of a Pythagorean $\triangle$? (Received April 22, 1954.)

592*t*. H. W. Becker: *Parametric solutions of* (I) $(A^4-B^4)(C^4-D^4)=\square$.

None but numerical solutions are given in Dickson's *History* II explicitly: Euler, p. 448; Gerardin, p. 458. If $(a, b; c, d)=abcd(a^2-b^2)(c^2-d^2)=\square$, its 2nd Petrus transform is $((a^2+b^2)^2, 4ab(a^2-b^2); (c^2+d^2)^2, 4cd(c^2-d^2))$, from which a solution of (I) is $\{A, B; C, D\}=\{a^2\pm 2ab-b^2; c^2\pm 2cd-d^2\}$. Substituting Euler's forms of p. 668 ibid., one gets $\{r, s; r(r^8+6r^4s^4-3s^8), s(3r^8-6r^4s^4-s^8)\}$; Rolle's, $\{r^2\pm 2rs-s^2; r^8+s^8\pm 8rs(r^6-s^6)-12r^2s^2(r^4+s^4(\pm 8r^3s^3(r^2-s^2)+38r^4s^4\}$; Euler-Hillyer's, $A, D, B, C=9r^4+4r^2s^2\mp 4rs^3-s^4, 9r^4\mp 12r^3s-4r^2s^2-s^4$. Or, specialize Petrus vectors, to get: Euler-Fermat II, $\{27r^4+s^4, 4rs^3; 3r^4+s^4, 4r^3s\}$; Rolle, $\{r, s; r^4\pm 2r^2s^2-s^4\}$; Desboves-Lenhart, $A, C, B, D=9r^4\pm 8r^2s^2+4s^4, 4rs(3r^2\pm 2s^2)$. The two smallest numerical solutions, Euler's $\{3, 2; 9, 7\}$ and $\{3, 2; 11, 2\}$ are D-L and E-H P.V., but mavericks as to (I). Also mavericks as yet are their triad $\{9, 7; 11, 2\}$, and $\{7, 3; 17, 1\}$, $\{14, 5; 18, 1\}$, and $\{17, 9; 29, 11\}$ of Gerardin's table. From Gerardin's forms, ibid. p. 647, a Pythagorean tetrahedron is deduced, whose 1st Petrus transform gives the solution of (I): $\{r^6\pm 3r^5s-2r^4s^2+r^2s^4+s^6; r^6+r^4s^2-2r^2s^4\pm 3rs^5+s^6\}$. If $ijkl$, $(i^2+j^2)(k^2+l^2)=\square\square$, the 2nd Petrus transform of this Pyth. tet. is the solution of (I) $\{i\pm j; k\pm l\}$, e.g.: the Rignaux $\{r^8\pm 6r^6s^2-5r^4s^4\pm 12r^2s^6+4s^8; r^8-8r^4s^4-2s^8, r^8+4r^4s^4-14s^8\}$; D-L, $A, C, B, D=7r^4\pm 16r^3s+54r^2s^2\pm 32rs^3-s^4, r^4\mp 32r^3s-54r^2s^2\mp 16rs^3-7s^4$; D-L, $\{729r^4-54r^2s^2\pm 16rs^3+s^4; 9r^4\pm 16r^3s-6r^2s^2+s^4\}$; transformed D-L, $A, D, B, C=243r^5\pm 243r^4s-6r^3s^2\mp 14r^2s^3+7rs^4\mp s^5, 243r^5\pm 189r^4s+42r^3s^2\mp 2r^2s^3-9rs^4\pm s^5$. (Received May 3, 1954.)

593*t*. H. W. Becker: *The forms of E. T. Bell for modified Martin trefoils and Pythagorean tetrahedrons.*

A. Martin sought Pythagorean $\triangle$ whose 3 sides are hyps. of other Pyth. $\triangle$. In Scripta Mathematica, J. Ginsburg (June 1945) found an infinity of solutions, and E. T. Bell (March 1946) the general solution. Bell's forms easily modify, so 1, 2, or 3 of the sides are legs instead of hyps. of other Pyth. $\triangle$. If 1 or 3, and corresponding pairs of sides are equal, the trefoil folds into a Pyth. tet.: $U^2+Z^2=V^2+X^2=T^2$, $V^2-Z^2=U^2-X^2=Y^2$. If Y is more even than Z, the forms are those of Bell after the changes: $a_1^2+b_1^2\leftrightarrow a_1^2-b_1^2$; $p, q, r, s, t, u, v, w, x\rightarrow U, Y', X, Y, Z, V, U', Z', T$; $Y=Y'$, etc. If Y is less even than Z, additionally $2a_2b_2\leftrightarrow a_2^2-b_2^2$. If Y is odd, instead $2a_1b_1\leftrightarrow a_1^2+b_1^2$, $2a_4b_4\leftrightarrow a_4^2-b_4^2$, $U\leftrightarrow Y'$. The a_n, b_n are the sq. roots of the sq. factors of $(T\pm U)/2$, etc. There are 4 transforms E; F; G; H under which $U, V, X, Y, Z, T\rightarrow TY, VX, VY, XY, XZ, UV; UZ, TY, XZ, YZ, UY, UV; UV, TZ, TY, XZ, VZ, TV; TX, UV, UX, XZ, TY, TU$. Then there are 5 pairs of a_n, b_n, a different 4 of which are used in each of the Pyth. tets. I, EI, FI, GI, HI. The Euler-Hillyer $a_n, b_n=3r^2+s^2, 2rs; (3r\mp s)(r\pm s); 9r^4-s^4, 8r^2s^2; 9r^4\mp 2r^2s^2+s^4, 2rs(3r^2\pm s^2)$. The Desboves-Lenhart a_n, $b_n=(3r^2+6rs+2s^2)(9r^2-6rs-2s^2), 4r^2(3r+2s)^2; (3r^2+2rs-2s^2)\cdot(21r^2+14rs+2s^2), 2r(3r+2s)(3r^2+2rs+2s^2); (15r^2+10rs+2s^2)(3r^2+2rs+2s^2), 8rs(3r+s)(3r+2s); 9r^2+6rs+2s^2, 4(3r+2s); (21r^2+14rs+2s^2)(3r^2+2rs-2s^2), 9r^4+12r^3s+40r^2s^2+24rs^3+4s^4$. The Rolle a_n, $b_n=4rs(r^2+s^2), r^4-6r^2s^2+s^4; 2(r^4-s^4), r^4-6r^2s^2+s^4; r(r^4+6r^2s^2-3s^4), s(3r^4-6r^2s^2-s^4); r(r^4-2r^2s^2+5s^4), s(5r^4-2r^2s^2+s^4); r^8+20r^6s^2-26r^4s^4+20r^2s^6+s^8, 2(r^2+s^2)^2(r^4-6r^2s^2+s^4)$. (Received May 3, 1954.)

594*t*. R. L. Blair: *A note on f-regularity in rings.*

An element a of a ring A is *f-regular* (cf. Trans. Amer. Math. Soc. vol. 75 (1953) pp. 136–153) in case $a\in(a)^2$, where (a) denotes the ideal of A generated by a. An ideal of A is f-regular in case each of its elements is f-regular. The existence of a largest f-regular ideal $F(A)$ of A and the fundamental relation $F(A/F(A))=0$ follow from general considerations of Brown and McCoy (Trans. Amer. Math. Soc. vol. 69 (1950) pp. 302–311). The list of "radical-like" properties of $F(A)$ is rounded out by showing that if B is an ideal of A then $F(B)=B\cap F(A)$ and that, if A_n denotes the complete matrix ring of order n over A, then $F(A_n)=(F(A))_n$. The location of $F(A)$ with respect to the Jacobson radical $J(A)$ and the prime radical $N(A)$ of McCoy is considered. It is shown that $N(A)\cap F(A)=0$ but that the statements (i) "In any ring A, $J(A)\cap F(A)=0$" and (ii) "Any radical ring (in the sense of Jacobson) which contains no proper nonzero ideal is a zero ring" are equivalent. Whether or not (ii) is true is an open question. Finally, if A satisfies the descending chain condition for right ideals, $F(A)$ coincides with the largest regular ideal of A. (Received May 6, 1954.)

595*t*. T. A. Botts: *On lattice embeddings for partially ordered sets.*

A particularly simple suprema-preserving embedding of a partially ordered set in a complete distributive lattice is noted. Applied to the lattice of all additive topologies on a set S [J. Wada, *Lattices of spaces*, Osaka Math. J. vol. 5 (1953) pp. 1–12], this yields a lattice of rather interesting elements, termed *channel structures* on S [E. J. McShane, E. E. Floyd, and T. A. Botts, *Channel spaces*, to appear]. (Received April 29, 1954.)

596. J. L. Brenner: *An extension of Hadamard's condition that a matrix have nonzero determinant.*

If a square matrix $A=(a_{ij})$ of complex numbers is such that the relations $|a_{kk}| > \sum_{j\neq k} |a_{kj}|$ hold $(k=1, 2, \cdots)$, its determinant is not 0. This condition is generalized as follows. Let the rows of A be partitioned into two or more sets, and suppose for each set of rows it is true that the absolute value of the principal minor exceeds the sum of the absolute values of the $-1+C_r^n$ nonprincipal minors on those rows. Then det A is not 0. It is possible for A to satisfy the general condition for one partitioning but not for a finer or coarser partitioning. Also, bounds for det A are found. The theorem gives, as usual, location theorems for the characteristic roots of an arbitrary square matrix. As Hadamard's original theorem has been generalized in the literature in various directions, so can this theorem be generalized in the parallel directions. (Received January 18, 1954.)

597. J. R. Byrne (*p*) and R. A. Beaumont: *Distributive polynomial functions and the construction of rings.*

If $S_1, S_2, \cdots, S_k$ are subgroups of the additive group of a commutative ring R, then a distributive polynomial function f from $S_1\times S_2\times \cdots \times S_k$ into R is a function defined by a polynomial $f(x_1, x_2, \cdots, x_k)$ in $R[x_1, x_2, \cdots, x_k]$ which has the property that there exists an integer m $(1\leq m\leq k)$ such that (1) $f(s_1+t_1, \cdots, s_m+t_m; s_{m+1}, \cdots, s_k)=f(s_1, \cdots, s_m; s_{m+1}, \cdots, s_k)+f(t_1, \cdots, t_m; s_{m+1}, \cdots, s_k)$ and (2) an analogous condition in the last $k-m$ arguments is satisfied, for all s_i and t_i in S_i, $i=1, 2, \cdots, k$. The form of such a polynomial function is determined, and when R has no proper divisors of zero, necessary and sufficient conditions that f be distributive are found. These results are used in the construction of rings with additive group $S_1\oplus S_2\oplus \cdots \oplus S_k$, where $S_1, S_2, \cdots, S_k$ are ideals in R. (Received May 5, 1954.)

598*t*. G. E. Forsythe: *SWAC computes all 126 distinct semigroups of order 4.*

By a "semigroup" is here meant a set of elements closed under an associative multiplication, with nothing else assumed. Two semigroups are "distinct" if they are neither isomorphic nor anti-isomorphic. Let $f(n)$ be the number of distinct semigroups of order n. It has been known that $f(1)=1$, $f(2)=4$, $f(3)=18$. The National Bureau of Standards Western Automatic Computer (SWAC) has now determined that $f(4)=126$. The author's code caused an exhaustive search through the 4^{16} possible multiplication tables. In about 100 minutes (reduced to about 80 minutes by J. Selfridge) all 3492 semigroups of order 4 were punched out. SWAC later reduced these to an "alphabetized" list of normalized multiplication tables of the 126 distinct semigroups. Of these, 58 are commutative. A hectographed (unpublished?) list by K. S. Carman, J. C. Harden, and E. E. Posey of 121 distinct semigroups of order 4 was examined by SWAC and shown to be without error (though incomplete). (Received June 10, 1954.)

599. Burrowes Hunt: *A Farey-dissection of the plane and the corresponding simultaneous-approximation algorithm.*

The rational points in the plane are given homogeneous coordinates (x, y, z), relatively prime integers with z the denominator in nonhomogeneous coordinates. The Farey-dissection of order N of the unit square in the first quadrant, extended over the plane by translation, is a dissection into triangles with vertices at all rational points with $z \leqq N$. To the triangle T with vertices $V_i=(x_i, y_i, z_i)$, $z_1 \geqq z_2 \geqq z_3$, is assigned the matrix T whose rows are V_1, V_2, V_3, in this order. In the dissection of order z_2+z_3, T is split into two triangles, of matrices AT and BT, by drawing the line from V_1 to the mediant V_2+V_3 of the opposite side. A and B are the matrices which take the rows of T into V_2+V_3, V_1, V_3 or V_2+V_3, V_1, V_2, respectively. The entire dissection is determined by assigning matrices to the four triangles with bases the sides of the unit square and common "highest" vertex $(1, 1, 2)$, and using only the operations A and B. V_3 is called an approximation to every point in or on T. The approximation to $P=(\xi, \eta, 1)$ is periodic if for some T, and W some word in A and B, P is in W^nT for $n=0, 1, 2, \cdots$. If periodic, then ξ and η are numbers in the field of a root of a cubic equation over R. The converse proposition is as yet neither proved nor disproved. (Received May 3, 1954.)

600. D. H. Lehmer: *Remarks on powers of a rational number.*

The distribution of the fractional parts of the successive integral powers of a rational number $r>1$ is not completely known. When $r=3/2$ this problem prevents the complete solution of Waring's Theorem. By using the SWAC it is shown that Waring's Theorem for nth powers is true for $n \leqq 5000$ because the fractional part of $(3/2)^k$ is sufficiently large for no value of $k \leqq 5000$. It is proved on the other hand that the fractional parts of $(1+1/q)^k$, where q is a fixed integer, are not randomly distributed. It is conjectured that this is the case for $(p/q)^k$. The conjecture has been verified for $q<p<17$. (Received March 15, 1954.)

601. Emma Lehmer: *Period equations applied to residue difference sets.*

It has been shown by Hall and Ryser that in a difference set of k elements, modulo

p, each difference occurring λ times, all the prime divisors q of $k-\lambda$ are multipliers provided $q>\lambda$. The last proviso, although essential to the proof, does not appear to be necessary in all cases. For residue difference sets the multipliers coincide with the residues. In this paper we show that *all* divisors of $k-\lambda$ are multipliers for all known residue difference sets. The theorem is obvious for quadratic residue sets. For quartic residue sets, discussed by Chowla, it is a special case of a theorem stated without proof by Sylvester to the effect that all divisors of $(3p+x^2)/16$ are quartic residues of $p=x^2+4y^2$, y odd. For octic difference sets, discussed by the author, this follows from a theorem, proved in the present paper, that all the divisors of $(7p+a^2)/64$ are octic residues of $p=a^2+2b^2=9a^2+4y^2=16n+9$. The proofs are based on a theorem of Kummer about the divisors of numbers represented by the cyclotomic period equations. The cubic period equation yields a similar result, namely that the factors of $(p+B^2)/4$ are cubic residues of $p=4A^2+27B^2$. This can be applied with $B=1$ to mixed sextic residue difference sets of Marshall Hall, who gave a proof, in this special case, using the law of cubic reciprocity. (Received March 15, 1954.)

602. E. A. Maier and Ivan Niven (p): *On the Löwenheim-Müller verification theorem.*

The following generalization of the Löwenheim-Müller verification theorem (cf. L. Löwenheim, *Über die Auflösung von Gleichungen im logischen Gebietekalkul*, Math. Ann. vol. 68 (1910) p. 179) is proved. Write ab, $a+b$, a' and b' for the intersection, union, and complements of elements a and b of any Boolean algebra, and let 0 and 1 denote the null and universal elements. Let $f(x_1, \cdots, x_n)$ denote any polynomial in n indeterminates $x_1, \cdots, x_n$ and their complements. Then any inclusion relation $f(x_1, \cdots, x_n) \subset g(x_1, \cdots, x_n)$ holds for all elements of the algebra if it holds for the 2^n cases where $x_1=0$ or 1, $x_2=0$ or 1, $\cdots$, $x_n=0$ or 1. (Received May 3, 1954.)

603*t*. D. H. Wagner: *On free products of groups.*

The reduction procedure mentioned in the preliminary report (cf. Bull. Amer. Math. Soc. Abstract 58-1-10) is applied to prescribe a recursive procedure for deciding whether or not a given element of a free product belongs to the subgroup generated by a given finite subset. It is assumed that a procedure is at hand for deciding the corresponding question within each of the given free factors. (Received April 8, 1954.)

604*t*. N. A. Wiegmann: *Some theorems on matrices with real quaternion elements.*

Certain basic properties of real quaternion matrices have been developed in earlier papers (L. A. Wolf, Bull. Amer. Math. Soc. vol. 42 (1936) pp. 737–743; H. C. Lee, Proc. Royal Irish Academy vol. 52 (1949) pp. 253–260). Further properties of such matrices are developed here: After observing that an analogue of the Jordan normal form is implicit in the first reference, some properties of commutative quaternion matrices are obtained; it is shown that a polar representation is possible for quaternion matrices; certain properties of normal quaternion matrices are verified; if A is a quaternion matrix, it is shown that there exist unitary quaternion matrices U and V such that UAV is real and diagonal; and some results related to this latter transformation are obtained. (Received April 9, 1954.)

ANALYSIS

605t. Richard Arens and A. P. Calderón: *Analytic functions of several Banach algebra elements.*

Let A be a commutative Banach algebra with unit, M its space of maximal ideals, $a_1, \cdots, a_n$ a fixed set of elements of A. The set of all $(a_1(m), \cdots, a_n(m))$ for m in M is the *joint spectrum* $S(A)$ (of $a_1, \cdots, a_n$) relative to A. Theorem 1: If F is holomorphic in a neighborhood of $S(A)$, then there is an a in A such that $a(m) = F(a_1(m), \cdots, a_n(m))$ for all m. Theorem 2 extends this to (many-valued) analytic functions in a manner which for square roots comes down to this: if a continuous complex function f can be found so that $0 \neq f(m)^2 = a_1(m)$ for all m, then $f(m) = a(m)$ for some a in A, i.e., a_1 has a square root mod the radical. Use is made of the following lemma. Let $\mathcal{A}$ be the system of finitely generated subalgebras of A which contain $a_1, \cdots, a_n$, directed by anti-inclusion. Then $\lim_{B \in \mathcal{A}} S(B) = S(A)$. Thereby Theorem 1 is reduced to G. E. Silov's original theorem (Mat. Sbornik vol. 32 (1953) pp. 353–364), which is like Theorem 1 except that $a_1, \cdots, a_n$ (and 1) are supposed to generate A. (Received May 6, 1954.)

606. M. G. Arsove: *Some conditions ensuring absolute continuity of the mass distribution for a potential.*

The conditions stated herein apply the upper and lower Blaschke operators $\overline{\Delta}^B u$ and $\underline{\Delta}^B u$, defined respectively as the lim sup and lim inf of $(4/r^2)\left[(1/2\pi)\int_0^{2\pi} u(z+re^{i\theta})d\theta - u(z)\right]$ as $r \rightarrow 0$. Theorem: Let Ω be an open set and u a subharmonic function on Ω having the property that the set $\{z: u(z) = -\infty\}$ supports none of the mass of u; if $\overline{\Delta}^B u < +\infty$, except perhaps on a Borel F_σ set of capacity zero, then the mass distribution for u is absolutely continuous. Corollary: let w be continuous on Ω; if $\overline{\Delta}^B w$ and $\underline{\Delta}^B w$ are finite, except perhaps on an F_σ of capacity zero, and if $\underline{\Delta}^B w$ is locally essentially bounded below, then w can be represented as the difference of two continuous subharmonic functions having absolutely continuous mass distributions. These results, closely related to theorems of Rudin (Trans. Amer. Math. Soc. vol. 68 (1950) pp. 278–286), are proved by means of the Looman-Menchoff techniques (see the author's abstract in the Proceedings of the International Congress of Mathematicians, Amsterdam, 1954). (Received May 5, 1954.)

607t. M. M. Day: *Strict convexity and smoothness.* Preliminary report.

If B is a normed linear space, say that B is (a) sc, (b) sm, or (c) scm if B is isomorphic to a normed space B' in which the unit sphere is (a) strictly convex, (b) smooth, or (c) both. Clarkson (Trans. Amer. Math. Soc. (1936)) showed that every separable space is sc, and Klee (Trans. Amer. Math. Soc. (1953)) showed that if S is an arbitrary index set, then $l_1(S)$ is sc. It is proved here that: (1) Every separable B is scm. (2) $c_0(S)$ is sc and sm, all S. (3) $l_1(S)$ is not sm if S is uncountable. (4) $m_0(S)$, $=$ set of bounded real functions on S which vanish outside some countable set, is not sc if S is uncountable and is not sm if S is infinite. (Received May 7, 1954.)

608t. J. B. Diaz and G. S. S. Ludford: *On two methods of generating solutions of linear partial differential equations by means of definite integrals.*

(i) J. Le Roux (Ann. École Norm. (1895)) proved, among other things, that if $U(x, y; \alpha)$ is a solution, containing a parameter α, of the linear equation (*) $L(u) \equiv u_{xy}+a(x, y)u_x+b(x, y)u_y+c(x, y)u=0$, then $\int_{\alpha_0}^{x} U(x, y; \alpha)f(\alpha)d\alpha$ (with fixed limit α_0) generates solutions of (*) from arbitrary functions $f(\alpha)$ provided $U(x, y; \alpha)$ satisfies the characteristic condition $U_y+a(\alpha, y)U=0$ for $x=\alpha$. (ii) The kernel of S. Bergman's "integral operator" method is the theorem: If $E(x, y, t)$ is a solution of $(1-t^2)(E_{yt}+aE_t)-(E_y+aE)/t+2xtL(E)=0$ such that $(1-t^2)^{1/2}(E_y+aE)/xt$ is continuous for $t=0$ and tends to zero for each (x, y) as $t\rightarrow\pm 1$, then $\int_{-1}^{+1}E(x, y, t) \cdot f[x(1-t^2)/2]dt/(1-t^2)^{1/2}$ generates solutions of (*) from arbitrary functions f of a single variable. (iii) In this paper it is shown that the theorem in (ii) is equivalent to that in (i), and that in applications to the equations of plane compressible flow, the method of Le Roux yields simpler results (in particular, since in this case a, b, c are functions of $x-y$ alone, one need only look for a *single* solution $\tilde{U}(x, y)$ of (*) and then write $U(x, y; \alpha)=\tilde{U}(x-\alpha, y-\alpha)$). In all these considerations the independent variables x, y may be either real or complex. (Received April 12, 1954.)

609t. Jacob Feldman and R. V. Kadison: *The uniform closure of the set of regular operators in a ring of operators.*

A precise description is given for those operators uniformly approximable by regular operators in a ring R. It is proved that A is such an operator if and only if for each $\epsilon>0$ there exists a subspace M belonging to R, containing the null space N of A, such that $\|AM\|<\epsilon$ and M and $H\ominus(A(H\ominus M))$ are equivalent (in the sense of Murray-von Neumann, *On rings of operators*, Ann. of Math. vol. 37 (1936) pp. 116–229). In the special case where R is the ring of all bounded operators on a separable Hilbert space, this statement reduces to the following simple form: An operator A is not a uniform limit of regular operators if and only if it is the product of a regular operator and a partial isometry between subspaces with unequal codimensions. (Received March 17, 1954.)

610. R. D. James: *Summable trigonometric series.*

The author has previously defined a P^n-integral of Perron type which goes directly to a generalized nth primitive (Trans. Amer. Math. Soc. vol. 76 (1954) pp. 149–176). It is shown in the present paper that if a trigonometric series is summable (C, k), $k=0, 1, 2, \cdots$, to a function $f(x)$ for $0\leqq x\leqq 2\pi$, and satisfies certain other conditions, then $f(x)$ is necessarily P^{k+2}-integrable. Moreover, the coefficients of the trigonometric series are given by obvious modifications of the usual formulas for Fourier coefficients. (Received April 12, 1954.)

611t. G. L. Krabbe: *The Titchmarsh semi-group in Banach-space.*

Suppose $p>1$ and let T_α be the transformation defined as follows: if $\|a\|_p =(\sum_{\nu=-\infty}^{\infty} |a_\nu|^p)^{1/p}<\infty$, then $T_\alpha a$ is the sequence c such that $c_n=\sum_{\nu=-\infty}^{\infty} a_\nu(-1)^{n+\nu} \cdot \sin \alpha\pi/(n+\alpha-\nu)\pi$ $(n=0, \pm 1, \pm 2, \cdots)$. Theorem I: There exists a nondecreasing function f such that $\|T_\alpha a\|_p\leqq\|a\|_p f(|\operatorname{Im} \alpha|)$ when $\|a\|_p<\infty$. From this follows directly that T_α is an entire function of α, and $T_\alpha T_\lambda a=T_{\alpha+\lambda}\, a$, $T_0 a=a$ when $\|a\|_p<\infty$. E. Hille pointed out [Mathematical Reviews vol. 8 (1947) p. 659] that these results hold when $p=2$. A transformation G (related to Hilbert's form) is defined as follows: if $\|a\|_p<\infty$, then Ga is the sequence x such that $x_n=\sum_{\nu=-\infty}^{\infty} a_\nu(-1)^{n+\nu}(n-\nu)^{-1}$, $\nu\neq n$ $(n=0, \pm 1, \pm 2, \cdots)$. Among further immediate consequences of Theorem I is derived the fact that G is a bounded operator; G is the infinitesimal generator of the semi-group $\{T_\alpha;\alpha\}$, and $T_\alpha=\exp \alpha G$ for all complex α. The proof of Theorem I generalizes the method initiated by E. C. Titchmarsh in a special case [Math. Zeit. vol. 25 (1926) pp. 321–333]. (Received April 16, 1954.)

612*t*. A. J. Lohwater: *The boundary values of pseudo-analytic functions.*

Let P denote the class of pseudo-analytic functions having bounded dilatation coefficient in $|z|<1$ for which $\lim_{r\to 1}|f(re^{i\theta})|=1$ for almost all $e^{i\theta}$ on $|z|=1$. It is shown that if $|f(z)|<1$, then every omitted value α with $|\alpha|<1$ is an asymptotic value. Extensions are made to nonbounded functions of P, and applications are made to the theory of the distribution of values of functions of P. (Received May 6, 1954.)

613*t*. J. S. MacNerney: *Stieltjes integrals in linear spaces.*

Let S be a linear normed complete space (norm $x=\|x\|$), and B the LNC space of continuous linear transformations from S into S (norm $T=|T|=\text{LUB }\|Tx\|$ for $\|x\|\leq 1$). Stieltjes-type integrals among functions from the real numbers into S and into B are considered; boundedness and bounded variation for functions from the real numbers to S or to B are defined in terms of the respective norms. H. S. Wall's theory of harmonic matrices [Bull. Amer. Math. Soc. Abstract 60-1-132] is extended to a theory of harmonic operators and associated integral equations. Application is also made to R. Nevanlinna's derivative concept [Math. Scand. vol. 1 (1953) pp. 104–112]: if D is a locally convex, connected set in S and G is a function from D to B which is continuous in the norm topology of B then, in order that there exist a function f from D to S such that $f'=G$ on D, it is necessary and sufficient that $\int_0^1 G(s)\cdot ds=0$ for each continuous function s from the number interval $[0,1]$ to D, of bounded variation on $[0,1]$, such that $s(0)=s(1)$. (Received April 29, 1954).

614*t*. John Nash: *C^3 isometric imbeddings.*

Given a closed n-manifold with C^3 metric one obtains a C^3 isometric imbedding in $E^{2n^2/2+3n/2}$ by a process of successive approximations. First one imbeds it in $n(n+1)/2+2n$ dimensions in such a way as to be susceptible to the perturbation process. Then one employs $n(2n+1)$ new dimensions to absorb almost all of the metric correction needed. The perturbation involves only the first $n(n+1)/2+2n$ imbedding functions. Each point of the manifold is moved in a direction orthogonal to the imbedding. This makes the metric effect depend on the second derivative matrix of the $n(n+1)/2+2n$ functions. This gives rise to a system of linear equations in the perturbation vector components. A systematic global solution of these gives the perturbation as a function of the metric change desired. In connection with use of successive perturbations smoothing techniques must be applied to the functions involved in order to achieve convergence. The low frequency part of the correction must be made first. If the metric is C^k, $k\geq 3$, one obtains a C^k imbedding. Undoubtedly the dimensionality $2n^2/2+3n/2$ can be reduced and probably the process will extend to the C^2 and analytic cases. (Received June 8, 1954.)

615*t*. Vikramaditya Singh and W. J. Thron: *On the number of singular points, located on the unit circle, of certain functions represented by C-fractions.*

The following theorem has been proved: Let the continued fraction $1+K_{n=1}^{\infty}(d_n z^{\alpha_n}/1)$, where $d_n\neq 0$, α_n a positive integer for all $n\geq 1$, satisfy conditions

(i) $\lim_{n\to\infty} (4|d_n|)^{1/\alpha_n}=1$ and (ii) $\lim_{k\to\infty} \alpha_{n_k}=\infty$ where $\lim_{k\to\infty} n_k/k<1/2$. Introduce $\liminf_{n\to\infty} \rho_n/(h_n-\rho_n)=k$ where $h_n=\sum_{\nu=1}^{n+1}\alpha_\nu$ and ρ_n is the maximum of the degrees of $A_n^*(z)$ and $B_n^*(z)$ where $A_n^*(z)/B_n^*(z)$ denotes the nth approximant of the continued fraction if the d_n are replaced by their moduli. Then the function represented by the continued fraction has at least two singularities on the unit circle if $1/2\leq k<1$, and for $0<k<1/2$ the number of singularities on the unit circle is $[(\pi/2)\cos^{-1}(1/(1+2k))]+1$. Thus in particular, if $k=0$, the function has the unit circle as a natural boundary. This is an improvement of an earlier result by one of the authors [W. J. Thron, Duke Math. J. vol. 20 (1953) pp. 195–198]. (Received May 3, 1954.)

616*t*. D. A. Storvick: *On meromorphic functions of bounded characteristic.*

Let $f(z)$ be meromorphic with bounded characteristic in $|z|<1$ and let the radial limits of $f(z)$ have modulus 1 for almost all z on $|z|=1$. Assume that $f(z)$ takes on values of modulus both greater than and less than 1. It is proved that, unless $f(z)$ is a schlicht function which maps $|z|<1$ onto the w-plane slit along an arc of $|w|=1$, every value of modulus 1 which is assumed only finitely often (or omitted) by $f(z)$ in $|z|<1$ is an asymptotic value of $f(z)$. This extends results of Hössjer [Acta. Univ. Szeged. vol. 5 (1930)] and Seidel [Trans. Amer. Math. Soc. vol. 36 (1934)], and is related to a result of Lehto [Ann. Acad. Sci. Fennicae (A) vol. 160 (1953)]. (Received April 22, 1954.)

617. Bertram Yood: *Periodic mappings on Banach algebras.*

Let B be a semi-simple real Banach algebra. An automorphism or anti-automorphism T of B is called periodic with period n if n is the smallest positive integer such that $T^n=I$, the identity. It is shown that if $n=3$ or 2^m and if the set $\{x\in B \mid (I+T+\cdots+T^{n-1})(x)=0\}$ is closed, then T is continuous. Also if every T of period two on every closed semi-simple sub-algebra of B is continuous, then every T of period 2^m on B is continuous. (Received May 6, 1954.)

APPLIED MATHEMATICS

618*t*. Elvy L. Fredrickson: *Applications of the Schmidt theory to nonlinear integral equations.*

The method of Erhard Schmidt [Math. Ann. vol. 65 (1908) pp. 370–430] for solving nonlinear integral equations is here applied to specific problems. Approximate solutions are obtained for: (1) Hamel's problem of the vibrating pendulum, (2) Duffing's equation, (3) an equation of Chandrasekhar which governs the angular distribution of radiation at the surface of a plane parallel stellar atmosphere. Work sponsored by U. S. Army Office of Ordnance Research under Contract DA-04-200-ORD-177. (Received May 10, 1954.)

619*t*. F. J. Murray: *Mechanisms and robots.*

A mechanism is a device with N states $p_1, \cdots, p_N$ which change automatically and this change $p_i\to p_j$ is given by a function $Ri=j$. The behavior of a mechanism is specified by an iterative structure which is independent of the enumeration of the states. The number of such structures can be obtained using methods of G. Pólya (Acta Math. vol. 68 (1938) pp. 145–254). The relation with Boolean algebra is readily obtained. If the change of state is dependent on the environment, the mechanism is termed a robot. Robots which have specified probability responses to changes in the environment are described. A general method for constructing mechanisms and robots is indicated. Mathematical notions are used rather than the methods of formal logic and consequently much irrelevant discussion is avoided. (Received April 12, 1954.)

620t. L. B. Rall: *Error bounds for iterative solutions of Fredholm integral equations.*

The integral equation of Fredholm type and second kind is regarded as an operational equation $y = (I - \lambda K)x$ in the space (C) of functions which are real and continuous on a closed interval $[a, b]$. A general iterative method $x_n = x_{n-1} - P(I - \lambda K)x_{n-1} + Py$, P an operator in (C), converges for arbitrary $x_0 \in (C)$ provided that $M(I-Q) < 1$ or Q is positive definite and $M(Q) < 1$, where $Q = P(I - \lambda K)$, and $M(R)$ denotes the upper bound of the operator R in (C). Error estimates for the norm $\|x - x_n\|$ are obtained for the general process; this process is specialized to obtain the iterative methods due to C. Neumann, G. Wiarda, H. Buckner, P. Samuelson, and C. Wagner, and explicit error estimates for these methods. The general error bounds are extended to include the error introduced by the use of methods of approximate integration, and are illustrated by their application to a numerical example. [Work sponsored by U. S. Army, Office of Ordnance Research, Contract DA-04-200-ORD-177.] (Received May 10, 1954.)

621. W. M. Stone (p) and Ray Lee: *On the mean value Doppler frequency spectrum of the return from a plane earth.*

Consider a radar moving above a plane earth along a line in a plane vertical to the earth. The cone of radio energy, directed along a second arbitrarily oriented line, illuminates a conic section on the earth's surface. Under the assumption that each element of the illuminated area radiates a mean return signal in all directions with an appropriate Doppler shift in frequency, it is possible to integrate the resulting inverse fifth power function over the area of the conic section. The spectrum function is somewhat lengthy but is surprisingly elementary in form. Four independent parameters are involved, the three direction cosines associated with the line of motion and the central line of the cone of energy, plus the generating angle of the cone. Various properties involving the slope of the spectrum function, total power, etc., are obtained by elementary means. (Received May 4, 1954.)

622t. Alexander Weinstein: *A unified theory of the radiation problem and of the Tricomi-Germain problem.*

Let (1) $L_k u = u_{xx} - u_{yy} - (k/y)u_y$. The solution $u = u^k(x, y)$ of $L_k u = 0$, $-\infty < k < 1$, which satisfies the conditions (2) $u^k(x, 0) = f(x)$ and (3) $u^k(x, y) = 0$ for $x = y$ is given by the formula (4) $u^k(x, y) = \Gamma^{-1}(1-k)(\partial/\partial x)^{n+1}\int_0^{x-y} g(\xi)[(x-\xi)^2 - y^2]^{-k/2} d\xi$ where $g(\xi)$ denotes the Riemann-Liouville integral $I^{n+k}f(\xi)$. Here $-n$ denotes the greatest integer not exceeding k; $n = 0, 1, 2, \cdots$. In order to fulfill all conditions of the problem, $f(x)$ has to satisfy certain differentiability conditions and has to vanish together with a certain number of derivatives at $x = 0$. This problem is for $k = -n$ a new formulation of the radiation problem for the wave equation in a space of $3+n$ dimensions. The cylindrical waves are obtained in the limiting case $k = 1$. The new treatment furnishes a complete solution of a problem discussed in Courant-Hilbert, vol. 2, pp. 403–408, 415–416. For $k = 1/3$ the above unified formula gives a new solution of the Tricomi problem in which the unknown function is prescribed on the parabolic line and vanishes on one characteristic. An explicit solution of this last problem was recently given by P. Germain and R. Bader (Rend. Circ. Mat. Palermo vol. 2 (1953) pp. 53–69, especially p. 63). Their solution, however, can be extended only to the range $0<k<1$. In this range both solutions coincide in view of a uniqueness theorem of C. Morawetz (Bull. Amer. Math. Soc. vol. 59 (1953) p. 352). This research was supported by the Office of Scientific Research, U.S.A.F. (Received May 6, 1954.)

GEOMETRY

623. D. B. Dekker: *Twisted curves and the mean-value proposition.*

The mean-value proposition in the small states that between any two sufficiently close points of a curve there exists a third point of the curve at which the tangent is parallel to the chord joining the two points. The mean-value proposition in the large states that for each chord of a curve there exists a parallel tangent. Both of these propositions are obviously true for any plane differentiable curve. By use of Taylor formulas with remainders it is shown that any curve in E_3 belonging to class C^3 with nonvanishing curvature at each point is a plane curve if and only if the mean-value proposition in the small is true. Also it is shown that any curve in E^3 belonging to class C^1 with a nonvanishing tangent vector at each point and with a spherical indicatrix of tangents of dimension less than or equal to one is a plane curve if and only if the mean-value proposition in the large is true. The proof here involves a consideration of the dimension of the spherical indicatrix of chords. (Received May 3, 1954.)

624*t*. V. L. Klee: *Common secants for plane convex sets.*

A *secant* of a plane set A is a line which intersects A. Theorem: Suppose F is a family of connected plane sets, either F is finite or each member of F is compact, and there is a line in the plane none of whose parallels intersects more than one member of F. Then if each three members of F have a common secant, there is a secant common to all members of F. (With "three" replaced by "four" this has been proved by Vincensini. With "connected plane sets" replaced by "mutually parallel line segments," it has been proved by Santalo and by Rademacher and Schoenberg.) The paper will appear in Proc. Amer. Math. Soc. (Received April 26, 1954.)

LOGIC AND FOUNDATIONS

625*t*. A. R. Schweitzer: *Connecting links between Grassmann's space analysis and integral equation theory.*

Grassmann's space analysis is concerned with the concepts determinant (outer product), quadratic form (inner product), function space (Grassmann's concept of units). That a type of function space was known to Grassmann is clearly indicated by his phrase, "wenn Funktionen als Einheiten gesetzt werden" (A_2, Abschnitt II, Kapitel 1, §5, Nr. 392). The preceding concepts have counterparts in integral equation theory. Transition from the former class of concepts to the latter involves passage from the finite to the infinite. Another link between Grassmann's theory and integral equation theory is the concept functional equation (Fréchet, Nouv. Ann. de Math. (4) vol. 9, p. 150). Reference is made to G. Kowalewski, *Einfuhrung in die Determinantentheorie einschliesslich der Fredholmschen Determinanten* (Berlin, 1942); G. Kowalewski, *Ueber Funktionenräume* (Sitzungsber. der kaiserl. Akad. der Wiss. in Wien. Mathem-naturw. Klasse; Bd. CXX Abt. IIa. Jänner 1911); D. Hilbert, *Quadratische Formen: Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*. Reference is also made to Encyk. der Mathem. Wiss., Bd. II, Teil 3, Heft 9 (p. 1434, note); Heft 7 (pp. 1025–1028). (Received April 23, 1954.)

TOPOLOGY

626*t*. Professor S. T. Hu: *The canonical spaces of associative algebras.*

For any given associative algebra A over a commutative field F with a unity element and for each topology T in F, a topological space X is constructed in this paper and is called the canonical space of the algebra A over the commutative field F with topology T. X turns out to be a Hausdorff space provided F is such under the topology T. If A is the residue algebra of the polynomial ring $K[x_1, \cdots, x_n]$ modulo an ideal I, and if T is the usual topology of the topological field $F=K$ of complex numbers, then it is proved that the canonical space X is homeomorphic with the algebraic variety V in the n-dimensional complex affine space K^n which consists of the common zeros of all polynomials in the ideal I. This implies that the algebraic variety V is determined up to a homeomorphism by the purely algebraic structure of the residue algebra $K[x_1, \cdots, x_n]/I$ and the topology of the field K. It also gives the algebraic mechanism of this determination. If F is the field R of all real numbers with the usual topology and if A is the algebra $C(S, R)$ of all continuous real-valued functions defined on a Hewitt space S, then it turns out that the canonical space X is homeomorphic with S. (Received May 3, 1954.)

627*t*. V. L. Klee: *Some topological properties of convex sets.*

This paper continues the study of topological properties of infinite-dimensional normed linear spaces and their convex subsets. In addition to results stated in earlier abstracts (Bull. Amer. Math. Soc. Abstracts 59-3-321, 59-4-435), the following is proved: (1) Suppose E is an infinite-dimensional Banach space, X is a compact subset of E, and f is a homeomorphism of X into E. Suppose that either E is Hilbert space or X is finite-dimensional. Then there is an isotopy η of E onto E such that η_0 is the identity map on E and $\eta_1 | X = f$. By elaboration of a mapping-technique introduced by Keller, the following theorems are proved: (2) For each positive integer n, the Hilbert parallelotope is n-point homogeneous. (3) If C is a locally compact closed convex subset of a normed linear space, then there are cardinal numbers m and n with $0 \leq m \leq \aleph_0$ and $0 \leq n < \aleph_0$ such that C is homeomorphic with either $[0, 1]^m \times]0, 1[^n$ or $[0, 1]^m \times [0, 1[$. (Received April 26, 1954.)

628*t*. Deane Montgomery and Hans Samelson: *A theorem on fixed points of involutions in S^3.*

Let T be an orientation preserving semi-linear homeomorphism of period two of S^3 onto itself; further let T be different from the identity and have at least one fixed point. It follows from work of P. A. Smith that F, the set of all fixed points, is a simple closed curve but it is not known whether F is necessarily unknotted. This paper shows that, at any rate, F cannot belong to a certain class of knots including the ordinary cloverleaf. Let C_1 and C_2 be two disjoint simple closed polygons in S^3; C_1 is called a parallel knot of order 2 if there exists a polyhedral Möbius band with C_1 as edge and C_2 as a middle curve, that is, C_2 is a generator of the fundamental group of the band. The theorem proved is as follows: Suppose the fixed point curve F of the semi-linear involution T of S^3 is a parallel knot of order two of C. Then $\pi_1(S^3 - C)$ is infinite cyclic and the linking number of F and C is ± 1. (The work of the second author was supported by the National Science Foundation.) (Received May 4, 1954.)

629t. F. P. Peterson: *Hopf Theorem* Mod *C*.

Let C be a "class" of abelian groups in the sense of Serre (Ann. of Math. vol. 58, pp. 258–294) satisfying also his axiom (II$_B$), namely, $A \in C$ implies $A \otimes B \in C$ and Tor $(A, B) \in C$. The following dual (concerning cohomotopy and cohomology) to the Hurewicz theorem is proved. Let (X, A) be a compact pair with dimension $X = N$. If $H^i(X, A) \in C$ for $i > n$ $(n > (N+1)/2)$, then $\pi^i(X, A) \in C$ for $i > n$ and $\pi^n(X, A) \rightarrow H^n(X, A)$ is a C-isomorphism. Using facts about $\pi_i(S^n)$, the conclusion of the above can be strengthened. Let α_C be the largest number such that $\pi_{i+j}(S^i) \in C$ for $0 < j < \alpha_C$ (within the stable range of $\pi_{i+j}(S^i)$). The conclusion then reads that $\pi^i(X, A) \rightarrow H^i(X, A)$ is a C-isomorphism for $i > \text{Max}\ ((N+1)/2, n - \alpha_C)$. With this relative theorem, it is easy to prove the following dual to the Whitehead isomorphism theorem. Let X, Y be compact spaces of dimensions M, N respectively. Let $f: X \rightarrow Y$ be a map, and let $n > \text{Max}\ ((M+2)/2, (N+1)/2)$. Then $f^*: H^i(Y) \rightarrow H^i(X)$ is a C-isomorphism for $i > n$ and C-onto for $i \geqq n$ if and only if $f^\#: \pi^i(Y) \rightarrow \pi^i(X)$ is a C-isomorphism for $i > n$ and C-onto for $i \geqq n$. (Received May 3, 1954.)

630t. A. D. Wallace: *Partial order and indecomposability.*

Let X be a space and let R be a closed subset of $X \times X$. For our present purpose it is convenient to write $x \leqq y$ if $(x, y) \in R$ and to assume that $\leqq$ is reflexive and transitive. It is natural to ask how the structure of R is affected by the structure of X. In a note to appear in the Proceedings of this Society we prove the following: (i) For each $a \in X$ let $\{x \mid x \leqq a\}$ be connected; (ii) For each $x, y \in X$ let there exist $z \in X$ such that $z \leqq x$ and $z \leqq y$; (iii) Let X be an indecomposable continuum—then $R = X \times X$. (Received April 5, 1954.)

631t. A. D. Wallace: *Two separation theorems.*

Theorem 1. Let $\mathcal{T}$ be a family of open sets in the space X which satisfies: (i) If $T_1, T_2 \in \mathcal{T}$, then $T_1 \cup T_2 \in \mathcal{T}$; (ii) If U is open and if $F(U) \subset F(T)$ and if $T \in \mathcal{T}$, then $U \in \mathcal{T}$. Let A and B be subsets of X such that $F(A)$ and $F(B)$ are compact and such that the interior of A does not meet B and vice versa. Suppose that, if $a \in F(A)$ and if $b \in F(B)$, $a \in T$ and $b \in X \backslash T^*$ for some $T \in \mathcal{T}$. Then $A^* \subset T$ and $B^* \subset X \backslash T^*$ for some $T \in \mathcal{T}$. Theorem 2. Let $\mathcal{A}$ be a family of subsets of the space X which satisfies: (i) If $A_1, A_2 \in \mathcal{A}$, then $A_1 \cup A_2 \in \mathcal{A}$; (ii) if U is open and if $F(U) \subset A \in \mathcal{A}$, then $U \cup A \in \mathcal{A}$. If $p \in X$ let $x \in C(p)$ if for no $A \in \mathcal{A}$, does one have $x \in A^*$ and $p \in X \backslash A^*$. Let X be a rim compact Hausdorff. Then $C(p)$ is closed and if $F(C(p))$ is compact it follows that $C(p)$ is connected. To prove Theorem 2, a corollary of Theorem 1 is used. Let X be rim compact Hausdorff, let A be closed with $F(A)$ compact, and let U be an open set about A. Then there is an open set V about A with $F(V)$ compact and with $V^* \subset U$. The above theorems are novel only in their generality and are useful only as a unifying device. (Received April 5, 1954.)

632t. L. E. Ward, Jr.: *A note on dendrites and trees.*

A partial order, $\leqq$, on a space X is said to be semicontinuous if $L(x) = \{a : a \leqq x\}$ and $M(x) = \{a : x \leqq a\}$ are closed sets, for each $x \in X$. We shall say that X is a *tree* if X is a compact connected Hausdorff space in which every pair of distinct points is separated by a third point. Thus a dendrite is simply a metric tree. Theorem: Let X be a compactum; then a necessary and sufficient condition that X be a dendrite is that X admit a partial order, $\leqq$, satisfying (i) $\leqq$ is semicontinuous, (ii) $\leqq$ is order dense, (iii) $L(x)\cap L(y)$ is a non-null chain for each $x, y\in X$, (iv) $M(x)-x$ is an open set for each $x\in X$. If the words "compact Hausdorff space" and "tree" are substituted for "compactum" and "dendrite," the theorem remains valid. (Received April 20, 1954.)

633t. G. W. Whitehead: *Homotopy groups of joins and the groups $A_q(\pi; G)$.*

Let X, Y be spaces, $X * Y$ their join. Suppose that X is $(m-1)$-connected, Y $(n-1)$-connected. Then there is a spectral sequence $\{E_r\}$ such that $E_2^{p,q} = H_{n+p}(Y; \pi_{m+q}(X))$ for $q \leqq m-2$ and E_∞ is the graded group of $\sum_n \pi_n(X * Y)$ with respect to a suitable filtration. In particular, if $X = K(\pi, n)$, $Y = K(G, n)$, then $\pi_{2n+s+1}(X * Y) \approx H_{n+s}(\pi, n; G)$ for $s \leqq m-3$. This gives a simple proof of the unpublished theorem of H. Cartan that the stable Eilenberg-MacLane groups satisfy the condition $A_q(\pi; G) \approx A_q(G; \pi)$. (Received April 23, 1954.)

634t. G. W. Whitehead: *On the homology suspension.*

Let B be an n-connected space, F its space of loops, A an abelian group. Then for $q \leqq 3n-1$ there is an exact sequence $H_{q-1}(F; A) \xrightarrow{\sigma} H_q(B; A) \xrightarrow{\tau} G_q \xrightarrow{\rho} H_{q-2}(F; A) \xrightarrow{\sigma} H_{q-1}(B; A)$, where $G_q \approx H_q(B\times B, B \vee B; A) \approx H_{q-2}(F\times F, F \vee F; A)$, σ is the suspension, $\tau: H_q(B; A) \rightarrow H_q(B\times B, B\vee B; A)$ is induced by the diagonal map, and $\rho: H_{q-2}(F\times F, F\vee F; A) \rightarrow H_{q-2}(F; A)$ is naturally related to the operation of path multiplication in F. A dual result holds for cohomology. If $\theta: H^n(K; \pi) \rightarrow H^q(K; A)$ is a universally defined cohomology operation, u the element of $H^q(\pi, n; A)$ which corresponds to θ, and if $q \leqq 3n-1$, then θ is additive if and only if u is the suspension of an element of $H^{q+1}(\pi, n+1; A)$. The latter result has been conjectured by Eilenberg and MacLane to hold for all q. (Received April 23, 1954.)

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BOOK REVIEWS

Linear analysis. By A. C. Zaanen. New York, Interscience; Amsterdam, North-Holland; Groningen, Noordhoff, 1953. 7+600 pp. $11.00.

This book goes a long way (but not all the way) toward filling the gap in the mathematical literature caused by the fact that Banach's book has been out of date for several years. The book is divided into three parts: measure theory, operator theory, and integral equations. The overlap with the material in Banach's book is, of course, in the second part, which is the longest and most important one.

The writing is clear and well organized; the author is an excellent expositor. A conspicuous and pleasing feature is the quality and quantity of examples. Not only is there an adequate supply of exercises at the end of each chapter, but throughout the body of the book there are many detailed discussions of standard and non-standard examples: sequence spaces, the Orlicz generalization of L_p spaces, sequential transformations, integral kernels, etc. There is some emphasis, but not a disproportionate amount, on the author's own work on symmetrisable transformations and kernels.

Hilbert spaces receive much more attention here than in Banach, but the treatment is more from the point of view of Banach spaces than would be the case in a book devoted to Hilbert space. The spectral theorem is not proved.

An unusual aspect of the book is the author's explicitly stated desire to avoid use of the well-ordering theorem, because of its "controversial" nature. This, apparently, is not done in the intuitionistic spirit; the author's proofs are like all ordinary mathematical proofs, and, in particular, they make free use of the principle of excluded middle. The axiom of choice, at least for countably many choices from arbitrarily large sets, is used, more or less explicitly, several times.

As a consequence of the avoidance of transfinite methods, many standard theorems appear in the book either for separable spaces only, or else accompanied by the *assumption* of the validity of the Hahn-Banach extension theorem. Presumably as a further consequence of the same philosophy, weak *convergence* is systematically preferred to weak *topology*, and, for instance, the Tychonoff-Alaoglu theorem on the weak compactness of the unit sphere in a conjugate space appears in its sequential form only.

A fair idea of the material covered can be obtained from the titles of the seventeen chapters; they are as follows. (1) Point sets. Euclidean space. (2) Measure. Measurable functions. (3) Integration. (4) Additive set functions. (5) The Lebesgue spaces L_p and the Orlicz spaces L_ϕ. (6) Banach space and Hilbert space. (7) Bounded linear transformations. (8) Banach spaces of finite dimension. (9) Bounded linear transformations in Hilbert space. (10) Range, null space and spectral properties of bounded linear transformations. (11) Compact linear transformations. (12) Compact symmetrisable, self-adjoint and normal transformations in Hilbert space. (13) General theory of non-singular linear integral equations. (14) Integral equation with normal kernel. (15) Integral equation with a symmetrisable kernel, expressible as the product of a kernel of finite double-norm and a bounded non-negative function. (16) Integral equation with Marty kernel. (17) Integral equation with Garbe kernel or Pell kernel.

Despite its self-imposed limitations, the book contains so much material, and treats its topics so thoroughly, that it is a welcome addition to the literature of functional analysis; it is recommended as both a reference for the expert and a text for the student.

PAUL R. HALMOS

Principles of numerical analysis. By A. S. Householder. New York, McGraw-Hill, 1953. 10+274 pp. $6.00.

With the age of supersonic aircraft, hydrogen bombs, large automatic control systems, and so on, has come a large increase in the volume and importance of scientific computation. The procession of automatically sequenced digital computers following J. W. Mauchly's construction of the ENIAC during the war is providing an enormous capacity to solve these computing problems and others. But many users and programmers of these machines know relatively little mathematics, while mathematicians are often quite unaware of the mathematical literature on computing methods. Much of the newer literature is found only in journals of diverse fields, or in reports of various research projects.

To cope with the situation, here and there serious mathematicians have been formed into groups with numerically inclined physicists and others, in order to study computing methods and devise new ones. The mathematicians on such a team are likely to call themselves *numerical analysts*. But there has been no agreed definition of numerical analysis as these people use the words, and no standard reference work. Books by Milne, Scarborough, Hartree, and others are primarily oriented toward desk calculating machinery, and most are written for mathematical amateurs or undergraduate students.

Dr. Householder is Chief of the Mathematics Panel at Oak Ridge National Laboratory. Presumably to help fill the gaps mentioned above, he gave a course in Oak Ridge for the University of Tennessee in 1950, and distributed some lecture notes which have been much appreciated. The book now reviewed grew out of these notes, with many additions.

Above all, the contribution of this book is to exhibit a substantial number of the tools of the new numerical analysis, and give them a scholarly mathematician's treatment. This is a large and difficult undertaking, and we congratulate Dr. Householder on its success. In preparing the book, the author has digested a surprising amount of the world literature on numerical methods. Numerical analysts in machine laboratories have now received their first standard reference.

The book is in eight chapters. Each is concluded with a section, Bibliographic Notes, relating the different sections of the chapter to the titles in the book's central bibliography. In the first chapter, The Art of Computation, the author attempts the classification of errors and their estimation. The chapter mainly develops the details of the "rounded-off" arithmetical operations with "digital" (i.e., rounded-off) numbers characteristic of digital computations. While the slavery of these estimates is certainly necessary for precise error bounds, one is thankful that most programming proceeds well enough without it.

Chapter 2 on matrices and linear equations, together with Chapter 4 on matrix eigenvalues, comprise almost half the text. For the most part the discussion of vectors, matrices, linear transformations, metrics, projection operators, norms, and so on, is descriptive and not abstract. Proofs are generally included. The author discusses several iterative methods for solving systems of linear equations and inverting matrices, and unifies them in terms of either projection matrices or a general linear iteration formula. Several methods are presented for the first time in a book—for example, the orthogonalizations of residuals devised by Lanczos, Hestenes, and Stiefel. A brief discussion of round-off errors is followed by a presentation of elimination procedures. The author summarizes the numbers of operations required for different methods.

Next to linear algebra, the most attention is given to the solution of nonlinear equations and systems, treated in Chapter 3. The chapter begins with standard material on polynomials in one variable, symmetric functions, and separation of roots, and carries on to König's theorem on zeros of analytic functions. With these tools the Graeffe and Bernoulli methods for getting polynomial zeros are described. Then follows a long section on functional iteration, from which Newton's method and higher order iterations are developed, both for equations and systems of equations. Aitken's δ^2 process is described here.

The chapter on matrix eigenvalues first sketches the theory of minimal polynomials for general matrices over the real field, and relates eigenvalues to the field of values of a matrix. The rest of the chapter is a gold mine of hard to find information on iterative methods of finding eigenvalues: pure iterations, iterations with Chebyshev polynomials in a matrix, deflating a matrix to make intermediate eigenvalues dominate, separating nearly equal eigenvalues, and the Jacobi rotation method of diagonalizing a matrix. The material on direct methods is just as valuable: solving Newton's identities, escalator methods, minimized iterations, and reduction to triple-diagonal form.

Interpolation, the subject of Chapter 5, is an older and more available field; the author claims to confine himself to a few general principles and some remainder formulas old and new. We also find optimal-interval interpolation. In Chapter 6 least squares approximation, trigonometric interpolation, and other problems are unified from a rather abstract point of view. Chapter 7 treats numerical integration and differentiation with a standard approach. Chapter 8 gives the Monte Carlo method of numerical integration a once over lightly, concluding with the "author's opinion that the method has proved and will prove most useful for the intrinsically stochastic physical problems."

Now follows a consolidated bibliography of over 400 titles, surely a great help to numerical analysts. It is surprising how many of the authors cited are first class mathematicians, considering how unfashionable numerical analysis has been in the memory of most of us. Several pages of exercises and a good index conclude the book.

As indicated, the book abounds with material densely packed into 246 pages of text. A few topics of numerical analysis have been omitted—for example, differential equations, game theory, and inequalities. These are large topics, probably best left to specialized books.

In the preface the author announces that this is a textbook, and that he tried to make the discussion accessible to one who has had a course in calculus and a little probability theory. The reviewer doubts that the author has succeeded in that object—the book is no place for the uninitiated to learn basic mathematical concepts. Many of the descriptions of methods will be perfectly clear to the student. But to be able to comprehend the theory, in the reviewer's opinion, the typical reader will need the equivalent of a master's degree in mathematics, and preferably more. This is partly because the material is fairly difficult in itself, but partly because the theoretical exposition is not lucid. The author goes from topic to topic with few signposts to help the reader, and none for the scanner. Although there are many derivations and proofs, results are nowhere summarized as theorems. The reviewer found much of the theory hard going, and occasionally suspected the author of formal manipulation without any justification within the reader's presumed knowledge (for example, in replacing a polynomial in a scalar by a polynomial in a matrix in equation (2.06.5) and after (4.0.7)).

The publishers have done a fine job in preparing the book. The few misprints noted were minor. The equation numbers might well have been simplified; why have (2.1322.5) or (2.201.10)?

In summary, the reviewer considers the book as a first definitive treatise on the subject matter and spirit of modern numerical analysis. As such, it is unique in a new area, and gives background, essential ideas, and references for a great number of methods never before brought together in a book. It is oriented toward the exploitation of automatic digital computers. While hard to read in spots, it will be a welcome addition to the library of every one interested in digital computation.

George E. Forsythe

Tables of integral transforms. Prepared under the direction of A. Erdélyi. New York, McGraw-Hill, 1954. 20+391 pp. $7.50.

This is the fourth of a series of five volumes prepared in part from notes left by the late Harry Bateman. The first three are entitled *Higher transcendental functions* and are devoted to a description of the properties of such functions; the volume under review together with a fifth to follow form a table of integrals involving such functions and intended as "companions and sequel" to the first three. The whole work is dedicated to the memory of Harry Bateman, and was prepared under the direction of A. Erdélyi with the collaboration of Research Associates: W. Magnus, F. Oberhettinger, F. G. Tricomi; Research Assistants: D. Bertin, W. B. Fulks, A. R. Harvey, D. L. Thomsen, M. A. Weber, E. L. Whitney; and Vari-typist, R. Stampfel. The project was performed at the California Institute of Technology, supported by a grant from the Office of Naval Research.

The integrals of the present volume are classified in seven chapters under the following types of transforms: Fourier cosine, Fourier sine, exponential Fourier, Laplace, inverse Laplace, Mellin, inverse Mellin.

Preceding Chapter I, there is a very brief introduction to the first three chapters, calling attention to the relations existing between the various types of transforms of the book and giving references to the more important treatises on Fourier transforms. Similar introductions to the Laplace tables and to the Mellin tables appear before Chapters IV and VI, respectively. Each chapter except the last begins with a short section of general transforms involving arbitrary functions, the material sometimes called the "grammar" of the transform. Then begins the "dictionary," or list of special functions and their images. This list is subdivided in a fixed order: algebraic, exponential, logarithmic, etc. This order refers to the function to be transformed appearing in the left-hand column of each page (the other column carries the image). Each entry appears in a ruled box which includes the range of validity and any special feature such as the need for Cauchy principal value. The general transform under discussion appears at the top of each page. In a few cases where the image seems too complicated to print in the appropriate box a reference to the literature is given instead. Infrequently a statement like the following appears in the body of the tables: "For similar integrals see Titchmarsh, E. C. . . ." *Ad hoc* notations are explained where they occur.

As an appendix there is a 22-page listing of definitions of special functions. Although notations for many of these functions have become standard, those for others remain confused in the literature, so that this is an indispensable part of the book. The user will also be grateful for an index to the appendix.

In the reviewer's opinion, this table represents an important addition to the existing tables of transforms. It may become one of the invaluable tools for the engineer or indeed for any worker in the field of integral transforms. Many analysts who have chafed under the notation of Campbell and Foster, an otherwise excellent table, will welcome the more familiar notation of this new table of Fourier transforms. Another welcome feature of the present volume is the inclusion of separate chapters for inverse transforms. This greatly improves the usefulness of the book, for without these chapters a given function might present itself in either of two columns, only the first of which is arranged "alphabetically." [Have you ever tried to discover in the Sunday *New York Times* whether your team won at football when you didn't know the name of the opposing college? Easy if your team won; hard otherwise!]

In several weeks of use, the reviewer has found only one misprint (definition of principal value on p. 367), but of course this is small guarantee against the existence of others. One mildly adverse criticism could be made on account of possible ambiguity in the use of the solidus. For example, in formula (50), p. 93, does $(1/2\pi)$ mean $1/(2\pi)$ or $(1/2)\pi$? Probably the latter, though the fact that "1" is printed at a higher level than "2π" might suggest the former. In (17), p. 31 the use of $\pi\beta^{-1}$ and $(\pi/\beta - ac/\beta)$ in the same formula makes one wonder what the guiding principle in the employment of the solidus should be.

D. V. WIDDER

Vorticity and the thermodynamic state in a gas flow. By C. Truesdell. (Memorial des Sciences Mathématiques, no. CXIX.) 4+53 pp. Paris, Gauthier-Villars, 1952.

In present-day researches on the flow of a gas, the vorticity has come to play an increasingly important role. It is only necessary to cite, for instance, the investigations on the motion of the earth's atmosphere which have stressed more and more the importance of the vertical component of the atmospheric vorticity. In his monograph, Professor Truesdell shows how closely connected is the vorticity with the thermodynamic variables such as the pressure, temperature, entropy, enthalpy, etc., of the gas. Gas flows are divided into two mutually exclusive classes: complex-laminar (Kelvin's complex-lamellar) in which the vorticity and velocity vectors are perpendicular; and Beltrami motions where they are parallel. Irrotational motion is a special case of complex-laminar flow. The properties of these flows are expressed in twenty-six theorems of a general type; for example, it is shown why and under what conditions a gas-flow may be instantaneously barotropic, or isentropic, or such that all the variables of state and the speed are constant upon each stream-line. As an illustration, Theorem 10 may be quoted, viz: "In an inviscid flow in continuous motion as an inert mixture devoid of heat flux, if the extraneous force be zero or normal to the velocity and if the pressure field be steady, then both the entropy and total enthalpy of each particle remain constant." The last ten pages of the text are devoted to "Prim gases" in which the density is a product of a function of the pressure and a function of the specific entropy. The reading of the monograph is much facilitated by the provision of an index of definitions and by an index of symbols.

This work will be an invaluable reference book to those already acquainted with hydrodynamics and gas dynamics. The beginner may however well ask: where in nature or in the laboratory do piezotropic fluids, Beltrami and Hamel flows, Prim gases, etc., occur? Am

I reading a work on pure mathematics or are there physical situations to which these theorems apply, and, if so, what are they? Brief answers to these questions could have been given without unduly lengthening the text and it may be hoped that, in some later edition, this minor defect will be remedied. In the meantime, the expert will be grateful for so clear and condensed a compendium of the thermodynamic properties of a gas in motion.

G. C. McVittie

Höhere Mathematik für Mathematiker, Physiker, Ingenieure. Part VI. *Integration und Reihenentwicklung im Komplexen. Gewöhnliche und partielle Differentialgleichungen.* By R. Rothe and I. Szabó. Stuttgart, Teubner, 1953. 251 pp. 17.60 DM.

This book is a continuation of Rothe's *Höhere Mathematik für Mathematiker, Physiker und Ingenieure.* A further volume is planned dealing with eigenvalue problems and with the calculus of variations.

Contents: I. Functions of a complex variable. Regularity, Cauchy's Theorem, residues, power series, isolated singularities, the Γ-function, asymptotic expansions. II. Linear differential equations. Elementary general theory, solution in power series, classification of singularities, behavior near a singular point, solution by Laplace transforms. III. Special differential equations. Hypergeometric equation, Legendre's equation, Bessel functions, Mathieu functions. IV. Partial differential equations, Cauchy's problem; elliptic, hyperbolic, parabolic. Riemann's method. Separation of variables.

This book is primarily intended for engineers, but its standards of rigor leave nothing to be desired. The presentation is very clear and by careful organization of the material a large amount of information has been condensed into small space without any bad effect on the readability. General theorems are followed by interesting illustrations and well selected (solved) exercises are provided; many of them develop the theory given in the text a little further and give at the same time an idea of the various possible applications, from engineering problems to quantum mechanics. The confluent hypergeometric series and Mathieu functions receive a comparatively detailed treatment, in line with their increasing use in Applied Mathematics. The few pages on partial differential equations give a surprisingly large amount of information, not just the usual collection of particular solutions.

To avoid a review without any adverse comment the reviewer should like to question the feasibility of computing $J_1(1000)$ from Hansen's integral by means of Simpson's formula.

W. H. J. Fuchs

Analytic functions. By S. Saks and A. Zygmund. Trans. by E. J. Scott. (Monografie Matematyczne, vol. 28.) Warsaw, Nakladem Polskiego Towarzystwa Matematycznego, 1952. 8+452 pp.

The theory of analytic functions has figured as a standard topic in the curriculum of a mathematics student for many years and there is a fairly clearcut agreement on what makes up the minimal contents of a first course on the theory of functions of a complex variable. However, when one examines the large number of texts on the subject, it is evident that standards of rigor and generality vary considerably. It is also clear that one can discern two quite distinct threads running through the fabric: first, the presence of arguments and methods which are very general—such as the use of topological notions (connectedness, compactness, interiority, etc.)—and are not peculiar to the theory of analytic functions; second, the presence of results and methods which are due to the particular features of the theory of analytic functions and give the theory its special color. In exposing a classical discipline, it is highly desirable that one should seek to isolate on the one hand the general tools and methods which are of constant use but are not peculiar to the discipline and on the other hand those features of the discipline which are special to it.

Such a program is envisaged by the book under review, the *Analytic functions* of Saks and Zygmund translated into English by E. J. Scott. Ever since its appearance in 1938, the original *Funkcje Analityczne* has evoked considerable interest far beyond the Polish mathematical public; tantalizing reference was frequently made to the manner in which certain topics were treated, a notable example being the elementary treatment of the Runge theorem concerning the approximation of analytic functions by polynomials and the exploitation of this result to prove the Cauchy integral theorem for simply-connected regions of the finite plane (i.e. regions having connected complement with respect to the extended plane) and to pave the way for the treatment of the Riemann mapping theorem.[1]

We learn from the authors' preface that it was their goal to take the middle road between a strictly "arithmetic" account of the theory along Weierstrassian lines and a "geometric" treatment which introduced certain intuitive geometric concepts without rendering them precise. "By no means renouncing the application of the auxiliary apparatus of Geometry and the Theory of Sets, they tried to confine it to a domain in which it could be justified and made precise without undue difficulty for the beginner." This program has been brilliantly carried out and achieved without the sacrifice of any of the standard results that properly belong to a first course in the theory of analytic functions. Their approach was on the one hand to marshal up at the outset the general methods which are of constant application and to expound them in a general setting, and on the other hand to avoid appeal to the more delicate arguments of plane topology by employing adequate (for applications) elementary configurations. An example of the emphasis on elementary considerations is the proof of the Runge theorem cited above which is based on the use of square nets and the Cauchy integral formula for rectangles.

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[1] We remark that the Runge approach was employed by Walsh in 1933 to prove with considerably more sophisticated apparatus (in fact, conformal mapping methods) the Cauchy-Goursat theorem for functions which are continuous in a closed Jordan region with rectifiable boundary and are analytic in the interior.

The Introduction is concerned with set theory, topological and metric spaces, elementary aspects of the topology of the plane. The first chapter deals with general questions of continuity, convergence, normal families, Ascoli's theorem, the elementary transcendental functions, and linear fractional transformations. From the outset exercises are introduced in abundance. Some are merely illustrative, others develop the theory further (e.g. Gauss' theorem on the zeros of the derivative of a polynomial, a very extensive and instructive set of problems on linear fractional transformations, Osgood's theorem on the pointwise limit of a sequence of analytic functions (in Chap. 2), etc.). The second chapter is concerned with the Cauchy theory treated in the elementary setting of systems of rectangles and the consequences of this restricted theory. The third chapter treats power series, meromorphic functions, isolated singularities, the Rouché and Hurwitz theorems, the interiority of meromorphic maps, and culminates in an introductory account of functions of two complex variables including the Weierstrass "Vorbereitungssatz." A dominant idea of the fourth chapter, *Elementary geometric methods*, is the possibility of approximating an analytic function by a rational function (which leads by the standard translation of pole technique to the Runge theorem in the simply-connected case) and the exploitation of this fact. The Cauchy theorem is now treated under more general hypotheses and residue theory is formulated with the aid of the notion of the order of a point with respect to a closed curve. The fifth chapter treats the Riemann mapping theorem and the Schwarz-Christoffel formulae. Chapter Six gives an excellent account of analytic continuation and puts to good use the abstract preliminaries of the introductory section. The use of meromorphic rather than analytic elements merits comment as does the proof of the monodromy theorem which is based on the Riemann mapping theorem. The selection of material from a rich field always involves a question of choice. An account of Riemann's classic work on the hypergeometric functions would have found a fitting niche in this chapter (cf. Ahlfors, *Complex analysis*) but its omission is certainly understandable.

The seventh chapter treats entire functions and meromorphic functions in the finite plane. The material treated includes the well-known expansion theorems of Weierstrass and Mittag-Leffler, growth questions, and the Picard theorems treated via the Bloch theorem. The remaining two chapters treat elliptic functions, the gamma and zeta functions, and Dirichlet series.

This brief account of the book indicates its scope and point of view. As we have remarked there is an abundance of exercises on which the good student may sharpen his mathematical teeth. He will have more than one occasion to test his skill with category arguments. On the other hand, the reader will note an absence of the treatment of the more delicate boundary problems which appeal either to a refined use of the topology of the plane or to methods involving Lebesgue integration. This is of course in accord with the stated program and intent of the book.

This book is a very welcome addition to the collection of texts on the theory of analytic functions which are now available in English. It will be a rewarding experience to the earnest student.

MAURICE HEINS

The stability of rotating liquid masses. By R. A. Lyttleton. Cambridge University Press, 1953. 10+150 pp. $6.50.

Ever since Newton deduced from his theory of gravitation that the shape of the earth must be an oblate spheroid, there has been intensive research into the question of the possible equilibrium shapes of rotating liquids. Maclaurin and Clairaut showed that for any value of angular momentum a spheroid is a possible equilibrium form. In 1834 Jacobi showed that if the angular momentum is greater than a certain amount an ellipsoid with three unequal axes is also a possible form of relative equilibrium.

The question of the stability of these equilibrium forms was first investigated by Poincaré in 1885. There are two different kinds of stability possible for rotating systems, known as "secular" and "ordinary" stability. To explain the distinction, consider a system rotating with constant angular velocity ω and assume it has n degrees of freedom $q=(q_1, q_2, \cdots, q_n)$ relative to a set of axes rotating with angular velocity ω.

In rotating systems the usual potential energy V must be replaced by the quantity $V-\omega^2 I/2$ where I is the moment of inertia of the system. The equilibrium conditions are

$$\frac{\partial}{\partial q_n}\left(V - \frac{1}{2}\omega^2 I\right) = 0.$$

For small displacements from equilibrium we may assume that

$$V - \frac{1}{2}\omega^2 I = (q, Bq)$$

where B is a real symmetric matrix and the parenthesis denotes the usual real scalar product. We shall assume also that the kinetic energy relative to the rotating axis is

$$T = \frac{1}{2}(\dot{q}, A\dot{q})$$

where the dots denote differentiation with respect to the time t and A is a real symmetric, positive definite matrix. If there are no external forces and if friction is neglected, then the equations of motion will be

$$A\ddot{q} + \omega G\dot{q} + Bq = 0 \tag{1}$$

where G is a real skew-symmetric matrix. The term $\omega G\dot{q}$ is the so-called gyroscopic term. Its presence is due to the fact that the equations of motion are written in a rotating, and not a static, set of coordinate axes.

The free oscillations of the system can be found by substituting $q=e^{\lambda t}q_0$ in equation (1). We get

$$(\lambda^2 A + \omega\lambda G + B)q_0 = 0. \tag{2}$$

This will have a nontrivial solution if

$$\det(\lambda^2 A + \omega\lambda G + B) = 0. \tag{3}$$

Since the transpose of $\lambda^2 A+\omega\lambda G+B$ is $\lambda^2 A-\omega\lambda G+B$, and since the determinant of a matrix and its transpose are equal, it follows that $-\lambda$ is a root of (3) if λ is; consequently (3) is an polynomial equation in λ^2.

The rotating system is said to be ordinarily stable if all the values of λ^2 which satisfy (3) are real and negative. In that case, all free oscillations are bounded and the system oscillates in the neighborhood of equilibrium in response to any disturbance.

Write (2) as follows:

$$(4)\qquad (\lambda A + \omega G + B\lambda^{-1})q_0 = 0.$$

Since A, B, and G are matrices with real elements, the complex conjugate of this set of equations is

$$(5)\qquad (\bar{\lambda} A + \omega G + B\bar{\lambda}^{-1})\bar{q}_0 = 0.$$

Add the scalar product of (4) by $\bar{q}_0$ to the scalar product of (5) by q_0. We get

$$(6)\qquad (\lambda + \bar{\lambda})(\bar{q}_0, A\bar{q}_0) + (\lambda + \bar{\lambda})\,|\lambda|^{-2}(\bar{q}_0, Bq_0) = 0$$

because $(\bar{q}_0, Gq_0) = -(q_0, G\bar{q}_0)$

A rotating system for which B is positive-definite is said to be secularly stable. In this case (6) shows that $\lambda+\bar{\lambda}$ must be zero, that is, λ is purely imaginary; consequently, a system which is secularly stable is also ordinarily stable. On the other hand, if the system is secularly unstable, that is, if B is not positive-definite, then equation (6) may be satisfied by putting

$$|\lambda|^2 = -(\bar{q}_0, Bq_0)/(\bar{q}_0, Aq_0).$$

If these values of λ are also purely imaginary, the system will still be ordinarily stable; consequently, a system may be secularly unstable but ordinarily stable.

The difference between secular and ordinary stability becomes evident when friction is taken into account. In that case the equations of motion become

$$A\ddot{q} + (F + \omega G)\dot{q} + Bq = 0$$

where F is a real, positive-definite matrix. If we put $q = e^{\lambda t}p$, then just as before we can show that

$$(7)\qquad [\lambda + \bar{\lambda}][(\bar{p}, Ap) + |\lambda|^{-2}(\bar{p}, Bp)] + (\bar{p}, Fp) = 0.$$

If B is positive-definite, then this equation implies that $\lambda+\bar{\lambda}$ is negative, that is, the real part of λ is negative; consequently, if the system is secularly stable, friction will dampen all disturbances from equilibrium.

If B is not positive-definite, the bracket in (7) will become negative and then the real part of λ will be positive and its magnitude will depend on the strength of the frictional forces; consequently, when a rotating system is ordinarily stable, frictional forces may increase the disturbances from equilibrium and so the system will not stay near equilibrium. Since this displacement from equilibrium depends upon the strength of the frictional forces, the system may stay near equilibrium for an extremely long time. For example, the lunar orbit is ordinarily stable but secularly unstable and its change from equilibrium under the influence of frictional forces is extremely slow.

Poincaré showed that, as the angular momentum increased, the MacLaurin spheroids which were originally secularly stable become secularly and ordinarily unstable at that value of angular momentum for which Jacobi proved the existence of a rotating ellipsoid. Using ellipsoidal harmonics, Poincaré showed further that the Jacobi ellipsoids are secularly stable until a certain value of angular momentum at which point there exist certain pear-shaped figures of equilibrium.

Darwin suggested that as the angular momentum increased the furrow in these pear-shaped figures might deepen until the liquid split in two independent bodies. This fission process could be the origin of binary stars, planets, and satellites. It was proved by Jeans, Liapounoff, and E. Cartan that these pear-shaped figures were secularly unstable.

The crux of the problem, however, is whether these figures are ordinarily stable. Because of the small frictional forces involved in astronomical phenomena, ordinary stability would be sufficient to permit the formation of binary systems by the above method. E. Cartan eventually proved that these figures are ordinarily unstable.

The book under review presents the mathematical details of the theory outlined above and also discusses their cosmological implications. The account starts with a discussion of the stability of rotating system and follows with a discussion of the spheroidal and ellipsoidal figures of equilibrium. Next, the theory of ellipsoidal harmonics and Lamé functions is developed. This theory is used to present Poincaré's and Cartan's discussion of the stability of the equilibrium figures. Finally, the author shows that the theory of binary fission advocated by Darwin and Jeans is untenable.

The author is to be commended for preparing this clear, logical account of a subject which has excited the interest of the greatest mathematicians.

BERNARD FRIEDMAN

RESEARCH PROBLEMS

21. Richard Bellman: *Integral equations.*

Solve the integral equation $\lambda f(x) = \int_0^1 e^{-xy} f(y)dy$. (Received May 21, 1954.)

22. Richard Bellman: *Dynamic programming.*

Solve the functional equation $f(x) = \text{Max}\ (g(x)+f(ax),\ h(x)+f(bx))$, given that $0<a,\ b<1$; $h(x),\ g(x)>0$; $h(0)=g(0)=0$; $h'(x),\ g'(x)>0$; $h''(x),\ g''(x)>0$. (Received May 21, 1954.)

23. Richard Bellman: *Calculus of variations.*

Given the problem of determining y so as to maximize $J(y) = \int_0^T F(x, y)dt$, subject to the relation $dx/dt = G(x, y)$, $x(0) = c$, and the constraint $0 \leq y \leq x$; determine the conditions upon F and G which will ensure that the solution has the form $y = x$ in $[0, t_1)$, $0<y<x$ in (t_1, t_2), $y=0$ in $[t_2, T]$, where t_1 may be zero, t_2 may be t_1, and t_2 may be T. (Received May 21, 1954.)

24. Richard Bellman: *Stability theory.*

Necessary and sufficient conditions that a polynomial possess only roots with negative real parts were given by Hurwitz in the form of determinantal inequalities where the elements of the determinants are the coefficients in the polynomial. The problem of determining necessary and sufficient conditions that a matrix A possess only characteristic roots with negative real parts may be reduced to the above by calculating the characteristic polynomial of A. Can one avoid this complicated computation and obtain determinantal criteria directly in terms of the elements of A? (Received May 21, 1954.)

W I L E Y

THE FOUNDATIONS OF STATISTICS*

By Leonard J. Savage, *University of Chicago.* The author concentrates on the personal aspect of statistical probability. He offers a theory of decision-making under uncertainty in which both statistics and probability are imbedded in a general system of rational behavior. Emphasizes the role of economics in statistics. *1954. 294 pages. $6.00.*

THEORY OF EQUATIONS

By C. C. MacDuffee, *University of Wisconsin.* This work includes an unusually complete treatment of systems of equations of higher degree, together with an introduction to the theory of numbers and a discussion of the theory of polynomials. *1954. 120 pages. $3.75.*

THEORY OF GAMES AND STATISTICAL DECISIONS*

By David Blackwell, *Howard University, and* M. A. Girshick, *Stanford University.* Opens with a rigorous treatment of the relevant parts of game theory—normal form, minimax theory, utility theory, etc.—and then presents the applications of this theory to problems of statistical decision. *1954. 355 pages. $7.50.*

ECONOMIC ACTIVITY ANALYSIS

Edited by Oskar Morgenstern, *Princeton University.* Contains thirteen studies by twelve economic researchers working in the field of input-output analysis and linear programming. In this work these systems are analysed on a much broader basis than has so far been attempted. *1954. 554 pages. $6.75.*

DECISION PROCESSES

Edited by Robert M. Thrall, Clyde H. Coombs, and Robert L. Davis, *all of the University of Michigan. 1954. 332 pages. $5.00.*

INTRODUCTION TO ELLIPTIC FUNCTIONS WITH APPLICATIONS

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THE THEORY OF DYNAMIC PROGRAMMING

RICHARD BELLMAN

1. Introduction. Before turning to a discussion of some representative problems which will permit us to exhibit various mathematical features of the theory, let us present a brief survey of the fundamental concepts, hopes, and aspirations of dynamic programming.

To begin with, the theory was created to treat the mathematical problems arising from the study of various multi-stage decision processes, which may roughly be described in the following way: We have a physical system whose state at any time t is determined by a set of quantities which we call state parameters, or state variables. At certain times, which may be prescribed in advance, or which may be determined by the process itself, we are called upon to make decisions which will affect the state of the system. These decisions are equivalent to transformations of the state variables, the choice of a decision being identical with the choice of a transformation. The outcome of the preceding decisions is to be used to guide the choice of future ones, with the purpose of the whole process that of maximizing some function of the parameters describing the final state.

Examples of processes fitting this loose description are furnished by virtually every phase of modern life, from the planning of industrial production lines to the scheduling of patients at a medical clinic; from the determination of long-term investment programs for universities to the determination of a replacement policy for machinery in factories; from the programming of training policies for skilled and unskilled labor to the choice of optimal purchasing and inventory policies for department stores and military establishments.

It is abundantly clear from the very brief description of possible applications that the problems arising from the study of these processes are problems of the future as well as of the immediate present.

Turning to a more precise discussion, let us introduce a small amount of terminology. A sequence of decisions will be called a *policy*, and a policy which is most advantageous according to some preassigned criterion will be called an *optimal policy*.

The classical approach to the mathematical problems arising from the processes described above is to consider the set of all possible

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sequences of decisions, which is to say, the set of all feasible policies, compute the return from each such feasible policy, and then maximize the return over the set of all feasible policies.

It is evident that, straightforward and reasonable as such a procedure is, it is often not practical. For processes involving even a moderate number of stages and a moderate range of choices at each stage, the dimension of the resultant maximization problem will be uncomfortably high, with continuous processes requiring maximization over function space.

If we momentarily re-examine the situation, not as a mathematician, but as a "practical man," we see that this price of excessive dimensionality—a price that occasionally makes even a modern computing machine cringe—arises from a demand for too much information. How much information is actually required to carry out a multi-stage decision process?

Do we require a knowledge of the complete sequence of decisions, those to be performed at the present stage, those at the next stage, and so on? Not at all! It is sufficient to furnish a general prescription which determines at any stage the decision to be made in terms of the current state of the system. In other words, if at any *particular* time we know what to do, it is never necessary to know the decisions required at subsequent times.

Donning our mathematical cap again, we see that this commonsense attitude reduces the dimension of the problem to its proper level, namely the dimension of the decision problem that confronts one at any particular time.

For the case of deterministic processes, which is to say, those where the initial state and the decision uniquely determine the outcome, both viewpoints are possible. For the case of stochastic processes, where a decision determines only a distribution of outcome states, the classical enumerative approach is virtually impossible.

2. **The fundamental approach.** As stated above, the basic idea of the theory of dynamic programming is that of viewing an optimal policy as one determining the decision required at each time in terms of the current state of the system. Following this line of thought, the basic functional equations given below describing the quantitative aspects of the theory are uniformly obtained from the following intuitive

PRINCIPLE OF OPTIMALITY. *An optimal policy has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decisions.*

The functional equations we shall derive are of a difficult and fascinating type, wholly different from any encountered previously in analysis. Nonetheless, as we shall see below, they may be utilized to provide an entirely new approach to some classical problems.

3. **Mathematical formulation—I. A discrete deterministic process.** To illustrate the type of functional equation that arises from an application of the principle of optimality, let us begin with the simplest case of a deterministic process where the system is described at any time by an M-dimensional vector $p=(p_1, p_2, \cdots, p_M)$, constrained to lie within some region D. Let $T=\{T_k\}$, where k runs over a set which may be finite, enumerable, or continuous, be a set of transformations with the property that $p\in D$ implies that $T_k(p)\in D$ for all k.

Let us assume that we are considering an N-stage process to be carried out to maximize some scalar function, $R(p)$ of the final state. We shall call this function the N-stage return. A policy consists of a selection of N transformations, $P=(T_1, T_2, \cdots, T_N)$, yielding successively the states

$$\begin{aligned} p_1 &= T_1(p), \\ p_2 &= T_2(p_1), \\ &\cdots\cdots\cdots, \\ p_N &= T_N(p_{N-1}). \end{aligned} \tag{3.1}$$

If D is a finite region, if each $T_k(p)$ is continuous in p, and if $R(p)$ is a continuous function of p for $p\in D$, it is clear that an optimal policy exists. The maximum value of $R(p_N)$, determined by an optimal policy, will be a function only of the initial vector p and the number of stages N. Let us then define

$$\begin{aligned} f_N(p) &= \operatorname*{Max}_P R(p_N) \\ &= \text{the } N\text{-stage return obtained using an optimal policy starting from the initial state } p. \end{aligned} \tag{3.2}$$

To derive a functional equation for $f_N(p)$, we employ the principle cited above. Assume that we choose some transformation T_k as a result of our first decision, obtaining thereby a new state $T_k(p)$. The maximum return from the following $(N-1)$ stages is, by definition, $f_{N-1}(T_k(p))$. It follows that k must now be chosen so as to maximize this. The result is the basic functional equation

$$f_N(p) = \operatorname*{Max}_k f_{N-1}(T_k(p)), \qquad N = 2, 3, \cdots. \tag{3.3}$$

It is clear that a knowledge of any particular optimal policy, not necessarily unique, will yield $f_N(p)$, which is unique. Conversely, given the sequence $\{f_N(p)\}$, all optimal policies may be determined.

We thus have a duality between the space of functions and the space of policies which is of great theoretical and computational importance. This point will be discussed again below.

4. Mathematical formulation—II: Discrete stochastic case. Let us now consider the case where the transformations are stochastic rather than deterministic. A choice of a transformation T_k now yields a stochastic vector z as the new state vector with an associated vector distribution function $dG_k(p, z)$.

It is clear that it is now in general meaningless to speak of maximizing *the* return. We must agree to measure the value of a policy in terms of some average value of the function of the final state. Let us call this expected value the N-stage return.

We now define $f_N(p)$ as before in terms of the N-stage return. If z is the state resulting from any initial transformation T_k, the return from the last $(N-1)$ stages will be $f_{N-1}(z)$. The expected return as a result of the choice of T_k is

$$\int_{z\in D} f_{N-1}(z)dG_k(p, z). \tag{4.1}$$

Hence, the functional equation for $f_N(p)$ is

$$f_N(p) = \operatorname*{Max}_k \int_{z\in D} f_{N-1}(z)dG(p, z), \qquad N = 2, 3, \cdots. \tag{4.2}$$

Note that the deterministic process may be considered to be merely a particular case of a stochastic process.

5. Mathematical formulation—III: Infinite stochastic process. For mathematical purposes, it is frequently useful to consider the fictitious infinite process in which there are an unbounded number of stages. In that case, the sequence $f_N(p)$ is replaced by the single function

$$f(p) = f_\infty(p),$$

and the formal equivalent of (3.2) is

$$f(p) = \operatorname*{Max}_k \int_{z\in D} f(z)dG_k(p, z). \tag{5.1}$$

6. Mathematical formulation—IV: Continuous deterministic proc-

ess. If we consider a continuous process where a decision must be made at each point of a time interval, we are led to maximization problems over function spaces. The simplest examples of these problems are furnished by the calculus of variations. As we shall show below, our approach leads to a new view of this classical theory.

Defining

(6.1) $f(p; T)$ = the return obtained over a time interval $[0, T]$ using an optimal policy starting from an initial state p,

the analogue of the functional equation of (3.3) is

$$f(p; S + T) = \operatorname*{Max}_{D[0,S]} f(T_S(p); T), \tag{6.2}$$

where the maximum is taken over all allowable decisions made over the initial interval $[0, S]$.

As soon as we consider infinite processes, we are confronted by the difficulty of showing that the maximum is actually attained. Consequently, in general, we must initially replace (6.2) by the rigorous equation

$$f(p; S + T) = \operatorname*{Sup}_{D[0,S]} f(T_S(p); T) \tag{6.3}$$

and then show, under various assumptions, that the extremum is attained.

As will be shown below, the limiting form of (6.3) as $S \to 0$ yields a partial differential equation.

We shall not discuss here the corresponding problem for the case of stochastic processes since a number of interesting and difficult conceptual questions arise which have not as yet been fully resolved.

7. Some examples—I: An allocation problem. Before proceeding any further with our general discussion, let us illustrate these ideas by means of a number of examples, of both stochastic and deterministic type, which are representative of the types of problems which fall within the domain of the general theory.

PROBLEM 1. We are given a quantity $x > 0$ that may be divided into two non-negative parts, y and $x - y$. From y we obtain a return of $g(y)$, at the expense of reducing y to ay where $0 < a < 1$; from $x - y$ we obtain a return of $h(x - y)$ at the expense of reducing $x - y$ to $b(x - y)$ where $0 < b < 1$. The process is now repeated with the new initial quantity $ay + b(x - y)$, and so on indefinitely. How does one allocate at each stage so as to maximize the total return obtained over the entire process?

This is a very simple prototype of a large class of important allocation and investment problems which occur in a number of diverse activities.

Let

(7.1) $f(x)$ = the total return obtained employing an optimal policy.

Arguing as above, it is readily seen that $f(x)$ satisfies the functional equation

$$(7.2)\qquad \begin{aligned} f(x) &= \operatorname*{Sup}_{0 \leq y \leq x} \left[g(y) + h(x - y) + f(ay + b(x - y))\right], \qquad x > 0, \\ f(0) &= 0. \end{aligned}$$

For a discussion of the various ways in which this equation can arise, and some of the analytic results which can be obtained, we refer the reader to [**4**; **6**; **11**; **12**].

Treatment of the closely related optimal inventory problem may be found in [**2**; **29**; **15**].

8. **Some examples—II: Stochastic gold mining.** Let us now consider the following example:

PROBLEM 2. We are fortunate enough to possess two gold mines, Anaconda and Bonanza, and a sensitive gold-mining machine with the following characteristics: If the machine is used in Anaconda, it will mine, with probability p, a fixed fraction r of the gold there and be undamaged; with probability $(1-p)$ it will mine nothing and be damaged beyond repair. If the machine is used in Bonanza, it will mine, with probability q, a fixed fraction s of the gold there and be undamaged; with probability $(1-q)$ it will mine nothing and be damaged beyond repair.

At each stage, as long as the machine is undamaged, we have our choice of using the machine in Anaconda or Bonanza. Given the initial amounts, x and y respectively in each mine, what sequence of choices maximizes the expected amount mined before the machine is damaged?

Let

(8.1) $f(x, y)$ = the expected amount of gold mined before the machine is damaged using an optimal policy, starting with x in Anaconda and y in Bonanza.

It is easily seen that $f(x, y)$ satisfies the functional equation

$$(8.2)\qquad f(x, y) = \operatorname{Max} \begin{bmatrix} A: p\left[rx + f((1 - r)x, y)\right] \\ B: q\left[sy + f(x, (1 - s)y)\right] \end{bmatrix}.$$

The solution has the following simple structure:

$$\begin{aligned} &\text{a. For } prx/(1-r) > qsy/(1-s), \text{ choose } A,\\ &\text{b. For } prx/(1-r) < qsy/(1-s), \text{ choose } B,\\ &\text{c. For } prx/(1-r) = qsy/(1-s), \text{ choose either.}\end{aligned} \tag{8.3}$$

Using this prescription, $f(x, y)$ may be computed recurrently. The boundary curve between the two decisions regions is the locus of points where immediate expected gain over immediate expected loss is the same for both choices. Unfortunately, as a counterexample of Karlin and Shapiro [36] shows, this simple and intuitive rule is not valid generally in more complicated decision processes.

For a discussion of further results and extensions of both discrete and continuous types, see [3; 9; 11; 25; 26].

9. Some examples—III: A problem in the calculus of variations. A simple example of a continuous decision process is furnished by the following problem in the calculus of variations:

PROBLEM 3. Maximize $\int_0^T F(x, y)dt$ over all y where x and y are connected by the relation $dx/dt = G(x, y)$, $x(0) = c$.

The classical technique in the calculus of variations, patterned directly after the technique used in maximization problems in finite-dimensional spaces, consists of considering the function yielding an extremum as a point in function space. This point is now characterized by means of variational properties, of which the most important is the Euler equation.

This approach corresponds to finding y as a function of t. Instead, we shall view the problem as a continuous decision process and seek to determine y at any time as a function of the two state parameters, c and T. Let us then set

$$f(c, T) = \operatorname*{Max}_{y} \int_0^T F(x, y)dt. \tag{9.1}$$

We shall in what follows proceed completely formally, assuming the maximum is attained, that all functions have the requisite number of continuous derivatives, and so on. Using the principle of optimality, we see that $f(c, T)$ satisfies the equation

$$\begin{aligned} f(c, S + T) &= \operatorname*{Max}_{y[0,S]} \left[\int_0^S F(x, y)dt + \int_S^{S+T} F(x, y)dt \right] \\ &= \operatorname*{Max}_{y[0,S]} \left[\int_0^S F(x, y)dt + f(c(S), T) \right], \end{aligned} \tag{9.2}$$

where $c(S)$ is x at $t=S$. Assuming that y is continuous, we obtain after a simple computation the limiting form of (9.2) as $S \rightarrow 0$

$$f_T = \text{Max}\, [F(c, v) + G(c, v) f_c], \tag{9.3}$$

where $v = y(0)$. Proceeding formally, we have for the determination of the maximum

$$F_v + G_v f_c = 0. \tag{9.4}$$

Eliminating f between (9.3) and (9.4), we obtain the first-order partial differential equation

$$\left(-\frac{F_v}{G_v}\right)_v v_T = \left(\frac{FG_v - GF_v}{G_v}\right)_v v_c + \left(\frac{FG_v - GF_v}{G_v}\right)_c. \tag{9.5}$$

The characteristics of this equation lead directly to the Euler equation obtained by the usual variational approach:

$$G_y \frac{d}{dt}\left(\frac{F_y}{G_y}\right) = \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}. \tag{9.6}$$

The same is true in the multi-dimensional problem where x, y and $G(x, y)$ are vectors and $F(x, y)$ is a scalar function. The case where the integrand contains t explicitly can always be reduced to the above by the introduction of a new dependent variable.

If we add to our original problem a constraint such as $0 \leq y \leq x$, one which occurs frequently in connection with allocation and investment problems, the functional equation is replaced by

$$f_T = \underset{0 \leq v \leq c}{\text{Max}}\, [F(c, v) + G(c, v) f_c]. \tag{9.7}$$

Various conditions under which this problem has a solution of particularly simple structure are given in [17]. We might note in passing that the difficulty induced by a constraint of the type above is due to the fact that free variation is not permitted whenever y has an extreme value of 0 or x, and consequently inequalities replace equalities.

Further discussion of these techniques will be found in [10; 16; 17; 18].

10. **Some examples—IV: An eigenvalue problem.** This functional-equation approach is also applicable to eigenvalue problems associated with differential equations of the form

$$\frac{du^2}{dt^2} + \lambda^2 \phi(t) u = 0, \qquad u(0) = u(1) = 0, \tag{10.1}$$

where we are interested in the values of λ^2 which yield nontrivial solutions u.

Under suitable conditions upon $\phi(t)$, this problem is equivalent to that of determining the successive minima of $\int_0^1 u'^2 dt$ subject to the constraints $\int_0^1 \phi(t) u^2 dt = 1$, $u(0) = u(1) = 0$. In order to employ the functional equation, we imbed the problem within the more general problem of determining the successive minima of

$$J(u) = \int_a^{a+t} u'^2 dS \tag{10.2}$$

subject to the constraints

$$\begin{aligned} &\text{(a)}\quad u(a) = u(a+t) = 0, \\ &\text{(b)}\quad \int_a^{a+t} \phi(S) u^2 dS + k \int_a^{a+t} \phi(S)(a+t-S) u(S) dS = 1. \end{aligned} \tag{10.3}$$

Writing $\text{Min}_u\, J(u) = f(a, k, t)$, we can derive a partial differential equation for f, which is nonlinear. Using the fact that ϕ may be considered constant, and equal to $\phi(a)$, for small t, this equation may be used to determine the eigenvalues computationally (see [10; 16; 18]).

11. **Some examples—V: Games of survival.** As our last example, let us consider a particularly interesting example of a multi-stage game, the so-called "game of survival."

Let us assume that two players, A and B, are playing a zero-sum game determined by the matrix

$$A = (a_{ij}), \qquad i, j = 1, \cdots, N,$$

and that A starts initially with an amount of money x, and B starts initially with y. Both are playing the game with the purpose of ruining the other. How should both play?

Let us define, for x and y positive,

(11.1) $f(x\ y) =$ the probability that A ruins B, given that A starts with x, and B with y, and both play optimally.

It is clear that A wishes to maximize this probability and B wishes to minimize it.

For other values of x and y, $f(x, y)$ is defined as follows:

$$f(x, y) = \begin{cases} 0, & x \leqq 0,\ y > 0, \\ 1, & y \leqq 0,\ x > 0. \end{cases} \tag{11.2}$$

It is now clear that $f(x, y)$ satisfies the functional equation

$$
(11.3)\qquad
\begin{aligned}
f(x, y) &= \operatorname*{Max}_{p} \operatorname*{Min}_{q} \left[\sum_{i,j=1}^{N} p_i q_j f(x + a_{ij}, y - a_{ij}) \right] \\
&= \operatorname*{Min}_{q} \operatorname*{Max}_{p} [\cdots].^{1}
\end{aligned}
$$

Since the total sum of money in the game remains constant, it is clear that we can replace $f(x, y)$ by a function of one variable, x.

For further developments, we refer the reader to [3; 37], and to a recent paper by Shapley [39].

12. **Approximation in policy space and monotone convergence.** The functional equations we have derived above are, in the main, analytically intransigent. The theoretical and numerical properties of the solutions must then be derived by use of that general factotum of analysis, the method of successive approximations. If our functional equation has the form

$$(12.1)\qquad f(p) = T(f(p)),$$

as do those above, we choose an initial function $f_0(p)$, and obtain a sequence of functions by means of the algorithm

$$(12.2)\qquad f_{n+1}(p) = T(f_n(p)), \qquad n = 0, 1, \cdots .$$

The physical background will usually provide precisely the conditions required for geometric convergence of this sequence to the solution of (12.1), where the uniqueness will be equally guaranteed by the same conditions. This technique we call approximation in function space.

Let us recall, however, that in a sense the function $f(p)$ is not of paramount importance. Rather, it is the optimal policies which yield $f(p)$ that are the most important. It follows that it may be wiser to approximate to optimal policies rather than to approximate directly to maximum returns.

In many ways this is a simpler and more natural technique, as well as more practical in applications. The principal theoretical advantage lies in the fact that we now obtain monotone convergence.

To illustrate this in its simplest form, let us consider the functional equation discussed in

$$(12.3)\qquad f(x) = \operatorname*{Max}_{0 \le y \le x} [g(y) + h(x - y) + f(ay + b(x - y))].$$

Perhaps the simplest initial guess is to assume that $y = 0$ continually.

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[1] This approach is also applicable to nonzero sum games and yields a new rationale for optimal play, see [47].

This yields as our initial approximation to the maximum return the function $f_0(x)$ satisfying the functional equation

$$f_0(x) = h(x) + f_0(bx). \tag{12.4}$$

It is now clear that the function $f_1(x)$ determined by

$$f_1(x) = \operatorname*{Max}_{0 \leq y \leq x} \left[g(y) + h(x - y) + f_0(ay + b(x - y))\right] \tag{12.5}$$

is always greater than or equal to $f_0(x)$. Hence, inductively, if

$$f_{n+1}(x) = \operatorname*{Max}_{0 \leq y \leq x} \left[g(y) + h(x - y) + f_n(ay + b(x - y))\right], \quad n = 0, 1, \cdots, \tag{12.6}$$

we have

$$f_{n+1}(x) \geqq f_n(x) \tag{12.7}$$

and thus monotone convergence; see [**3**; **8**].

A completely analogous technique is applicable to continuous processes, and in particular the calculus of variations. The results are particularly interesting in connection with eigenvalue problems where we obtain monotone convergence (see [**16**; **18**]).

13. **Further results.** We have not the space here to discuss any of a number of other interesting and important problems in dynamic programming. For those interested in bottleneck problems occurring in multi-stage production processes, we refer to [**7**; **14**; **27**].

Those interested in scheduling problems may consult [**22**; **23**; **33**].

A number of mathematical problems occurring in connection with the control of engineering and economic systems are discussed in [**20**; **21**].

Finally, we should like to mention a number of papers concerned with the very difficult mathematical problems occurring in the general theory of learning processes [**32**; **34**; **35**; **24**].

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44. ———, *Dynamic programming and a new formalism in the theory of integral equations*, Proc. Nat. Acad. Sci. U.S.A., to appear.

45. ———, *A problem in the sequential design of experiments*, The RAND Corporation, Paper P-586.

46. ———, *Decision-making in the face of uncertainty*—I, Navy Quarterly of Logistics, September 1954.

47. ———, *Decision-making in the face of uncertainty*—II, Navy Quarterly of Logistics, December 1954.

48. ———, *Dynamic programming and multi-stage decision processes of stochastic type*, Symposium on Linear Programming, Washington, D. C., December, 1954.

49. R. Bellman, I. Glicksberg, and O. Gross, *On the optimal inventory equation*, The RAND Corporation, Paper P-572.

50. ———, *On some mathematical problems arising in the theory of optimal inventory and stock control*, The RAND Corporation, Paper P-580.

THE RAND CORPORATION

THE SUMMER MEETING IN LARAMIE

The fifty-ninth Summer Meeting of the American Mathematical Society was held at the University of Wyoming, Laramie, Wyoming, from August 31 to September 3, 1954, in conjunction with meetings of the Mathematical Association of America and the Canadian Mathematical Congress.

Over 500 people registered for the meeting including 234 members of the Society.

The Committee to Select Hour Speakers for Annual and Summer Meetings invited five speakers. Tuesday afternoon Professor R. D. James of the University of British Columbia addressed the Society on *Integrals and summable trigonometric series*. Professor L. V. Ahlfors presided. Professor Charles Loewner of Stanford University addressed the Society Wednesday morning on *Some transformations of systems of partial differential equations* at a session presided over by Professor D. H. Lehmer. Thursday morning Professor R. S. Phillips of the University of Southern California addressed the Society on *Semi-groups of operators* and Thursday afternoon Professor J. G. Wendel addressed the Society on *Semi-groups in algebra* at sessions presided over by Professor B. J. Pettis and Professor W. L. Duren respectively. Dr. R. E. Bellman of The Rand Corporation addressed the Society Friday morning on *Theory of dynamic programming*. Professor C. B. Tompkins presided.

There were fourteen sessions for contributed papers presided over by Professors M. A. Basoco, R. W. Brink, J. W. Hurst, V. L. Klee, C. G. Phipps, G. B. Price, O. F. G. Schilling, Nathan Schwid, M. F. Smiley, Leonard Tornheim, A. W. Tucker, V. J. Varineau, R. M. Winger, and C. R. Wylie.

The Committee on Arrangements provided an enjoyable program for those attending the meeting. Included in these events were: a tea for the ladies on Monday afternoon, movies Monday evening, a trip to Vedauwoo on Tuesday afternoon, a trip to Snowy Range and a wild game steak fry on Wednesday, and a square dance Thursday evening.

On Wednesday evening, at the wild game steak fry, President George Duke Humphrey of the University of Wyoming welcomed the mathematical organizations to Wyoming. Professor W. L. Duren responded for the organizations by presenting resolutions thanking the University of Wyoming and all responsible individuals for their generous and effective contributions to the success of the meeting.

The Council met on Tuesday evening, August 31, 1954.

The Secretary announced the election of the following ninety-two persons to ordinary membership in the Society:

Mr. David Samuel Adorno, Haller, Raymond and Brown, Inc., State College, Pennsylvania;
Mr. Robert Joseph Arms, New York University;
Mr. Donald Gary Aronson, Massachusetts Institute of Technology;
Mr. Pierre Belanger, 83 Casot Avenue, Quebec, Quebec, Canada;
Mr. Stoughton Bell, University of California, Berkeley;
Mr. John William Blattner, University of California, Los Angeles;
Mr. Mark Clinton Breiter, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland;
Mr. Delmar Lee Boyer, University of Kansas;
Mr. Robert W. Butcher, University of Michigan;
Mr. Roderick Peter Campbell Caldwell, University of Illinois;
Mr. Benjamin Ralph Cato, Jr., University of Maryland;
Dr. Moody Lee Coffman, Convair Division, General Dynamics Corp., Fort Worth, Texas;
Dr. Louis J. Cote, Columbia University;
Mr. Glen Jacob Culler, University of California, Berkeley;
Mr. Hubert M. Custer, Elizabethtown College;
Mr. Don Allen Davis, University of Washington;
Mr. Joseph DeMilia, U. S. Naval Gun Factory, Washington, D. C.;
Mr. Gus DiAntonio, University of Pittsburgh;
Mr. Russell D. F. Dineen, Board of Public Education, Wilmington, Delaware;
Miss Mary Louise Edwards, Lincoln University;
Professor John Arthur Englund, Creighton University;
Mr. William Paul Evans, Caterpillar Tractor Company, Peoria, Illinois;
Miss Louisa Sonia Grinstein, University of Michigan;
Mr. John Wesley Hamblen, Purdue University;
Miss Florence Ruth Harwich, Columbia University;
Mr. Simon Hellerstein, Syracuse University;
Mr. Robert Bennett Herrera, Los Angeles City College;
Mr. Frederick William Hoagland, 1157 East 57th Street, Chicago, Illinois;
Mr. Harry Hochstadt, W. L. Maxson Corp., New York, New York;
Mr. Henry F. Hunter, Knolls Atomic Power Laboratories, Schenectady, New York;
Mr. Dale Harper Husemoller, Harvard University;
Mr. Amnon Jakimovski, Tel-Aviv, Israel;
Mr. Guy Johnson, Jr., Rice Institute;
Mr. David Hunter Jones, University of Washington;
Mr. John Jones, Jr., Mississippi Southern College;
Mr. Edgar David Kann, New York University;
Lieutenant Jack Herman Kaplan, United States Army, Fort Eustis, Virginia;
Mr. Arthur Alonzo Karwath, Saint Ambrose College, Davenport, Iowa;
Miss Catherine Shannon Kay, North Central College;
Mr. Donald Allen Kearns, University of Maine;
Mr. Maurice Kennedy, California Institute of Technology;
Mr. Richard Frederick King, University of Illinois;
Mr. Walter Charles Koroluk, Olin Industries, New Haven, Connecticut;
Dr. Gregers Louis Krabbe, University of California, Berkeley;

Mr. Gene Levy, University of Oklahoma;
Professor Chio-Shih Lin, University of the Philippines;
Mr. Richard Gene Long, University of Washington;
Mr. William Caroll Long, Consolidated Vultee Aircraft Corp., Fort Worth, Texas;
Mr. Gerald E. Mahoney, Boston University;
Mr. Mark Edward Mahowald, University of Minnesota;
Dr. Allan Dean Martin, Oberlin College;
Mr. Samuel Melamed, McGill University;
Mr. Roger Edwin Messick, Carnegie Institute of Technology;
Mr. Richard John Mihalek, Illinois Institute of Technology;
Dr. Cordell Bridges Moore, Consolidated Vultee Aircraft Corp., Fort Worth, Texas;
Mr. Hal G. Moore, Carbon Senior High School and Carbon College, Price, Utah;
Professor Mikao Moriya, Okayama University, Okayama, Japan;
Professor David Eugene Muller, University of Illinois;
Mr. Lewis Amedeus Ondis, II, Westinghouse Electric Corp., Pittsburgh, Pennsylvania;
Mother Harriet Ann Padberg, College of the Sacred Heart, Grand Coteau, Louisiana;
Mr. Frank Joseph Palas, Oklahoma University;
Professor Henry Parkus, Michigan State College;
Miss Constance Miller Parson, Fort Valley State College;
Mr. Edward William Payne, William J. Bryan University;
Mr. Arthur Ferris Payton, National Union Radio Corp., Orange, New Jersey;
Professor George W. Peglar, Iowa State College;
Mr. Paul John Persiani, Saint Louis University;
Dr. Lloyd Lewis Philipson, Rand Corporation, Santa Monica, California;
Mr. Bennie H. Pierce, Lees-McRae College;
Mr. José Fortunato Pita de Macedo, Lisbon High School, Lisbon, Portugal;
Dr. Gerald Cowles Preston, Purdue University;
Dr. Royal William Randall, Jr., Purdue University;
Mr. Thomas David Riney, Purdue University;
Dr. Leon C. Robbins, Jr., Burroughs Corp., Philadelphia, Pennsylvania;
Mr. Willy Werner van Roosbroeck, Bell Telephone Laboratories, Murray Hill, New Jersey;
Mr. Paul John Schillo, University of Buffalo;
Mr. Ralph Edgar Schwartz, University of Massachusetts;
Dr. Binyamin Schwarz, Ministry of Defense, Tel-Aviv, Israel;
Mr. Cecil Fay Shelton, Atlantic Refining Company, Caracas, Venezuela;
Dr. Bella Manel Shiffman, Ramo-Wooldridge Corporation, Los Angeles, California;
Mr. Lewis Joseph Simonoff, Syracuse University;
Dr. Frank William Sinden, Aurora, Illinois;
Mrs. Doris Skillman Stockton, University of Massachusetts;
Mr. Malcolm Smith, Cook Research Laboratories, Skokie, Illinois;
Mr. Newton Birrell Smith, 228 Welch, Ames, Iowa;
Mr. Charles Edwart Steward, Boeing Airplane Company, Seattle, Washington;
Mr. Charles Russell Strain, Engineering Research Associates, Inc., Arlington, Virginia;
Professor Lee C. Teng, University of Wichita;
Mr. Louis George Vargo, Convair Division, General Dynamics Corp., Fort Worth, Texas;
Mr. Andries Cometis de Wilde, Detroit Institute of Technology;

Professor Samuel Watkins Williams, Arkansas College;
Mr. Bevin K. Youse, University of Georgia.

It was reported that the following five persons had been elected to membership on nomination of institutional members as indicated:

University of Florida: Mr. George Van Zwalenberg.
University of Illinois: Mr. James Harman Abbott
Metropolitan Life Insurance Company: Mr. Joseph Thomas Gannon
University of Missouri: Mr. William Robert Abel and Mr. James Fredrik Jakobsen

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: German Mathematical Society: Professor Guido Hoheisel, University of Köln, Köln, Germany; Indian Mathematical Society: Dr. V. K. Balachandran, Mr. E. de St. Q. Isaacson, and Dr. M. Venkataraman; Polish Mathematical Society: Professor Henry Hiz, Pennsylvania State University; The Swedish Mathematical Society: Professor Harold Bror Evert Bergström, Chalmers University of Technology, Goteborg, Sweden; Wiskundig Genootschap: Dr. Gerard Willem Decnop, University Library, Amsterdam, Holland, and Dr. Herman van Rossum, The Amsterdam Lyceum.

The following appointments by the President were reported: R. V. Churchill appointed Chairman of the Editorial Committee for Applied Mathematics Symposium Proceedings for the period July 1, 1954–June 30, 1955; F. J. Murray appointed a member of the Editorial Committee for Applied Mathematics Symposium Proceedings for a period of three years beginning July 1, 1954 (committee now consists of R. V. Churchill (Chairman), A. E. Heins, and F. J. Murray); J. L. Doob appointed Chairman of the Committee on Translations from Russian and Other Foreign Languages for a period of one year beginning July 1, 1954; R. E. Bellman and Hans Samelson appointed as members of the Committee on Russian and Other Foreign Languages for a period of three years beginning July 1, 1954 (committee now consists of J. L. Doob (Chairman), R. E. Bellman, R. P. Boas, Irving Kaplansky, and Hans Samelson); F. B. Jones appointed Chairman of the Organizing Committee for Summer Institutes; Edwin Hewitt and J. J. Stoker appointed as members of the Organizing Committee for Summer Institutes for a period of three years beginning July 1, 1954 (committee now consists of: F. B. Jones (Chairman), A. A. Albert, Salomon Bochner, S. S. Chern, Edwin Hewitt, and J. J. Stoker); John von Neumann (Chairman), H. W. Bode, Richard Courant, D. H. Lehmer, and T. Y. Thomas appointed as a Program Committee for a Symposium in Applied Mathematics to be held on April 14–15, 1955, at Polytechnic Institute of Brooklyn; R. M. Foster (Chairman), L. W. Cohen, S. P. Hoffman, Jr., J. P. Russell and Seymour Schuster appointed as a Committee on Local Arrangements for a meeting to be held at the Polytechnic Institute of Brooklyn, April 14–16, 1955; Professors B. P. Gill and C. H. W. Sedgewick as auditors for 1954.

The following appointments of representatives of the Society were reported: at the inauguration of Francis H. Horn as President of Pratt Institute on May 15, 1954: Professor Samuel Borofsky; at the inauguration of David Hitchens Morgan as President of the Texas Agricultural and Mechanical College on May 20, 1954: Professor E. C. Klipple; at the inauguration of Benjamin Franklin Hilbun as President of Mississippi State College on July 14, 1954: Professor Arthur Ollivier.

The Secretary reported that the following persons had accepted invitations to deliver addresses during 1954: Charles Loewner, Laramie, Wyoming, Summer Meeting 1954; J. G. Wendel, Laramie, Wyoming, Summer Meeting 1954; Arne Beurling, Cambridge, Massachusetts, October 30, 1954; Nathan Jacobson, University of Alabama, November 26–27, 1954; Lipman Bers, Pittsburgh, Pennsylvania, Annual Meeting 1954; and Samuel Eilenberg, Pittsburgh, Pennsylvania, Annual Meeting 1954.

The Secretary reported that Professor K. O. Friedrichs had accepted an invitation to deliver the Josiah Willard Gibbs Lecture in December, 1955 and that Professor Salomon Bochner had accepted an invitation to deliver the Colloquium Lectures in 1956.

It was reported to the Council that the Executive Committee and the Trustees had approved a recommendation that Russian Translations be issued in bound volumes of about two hundred to three hundred pages, each containing (whenever feasible) papers on related subjects. The volumes will be issued at irregular intervals, sold separately, and kept in stock, like any other series of books published by the Society. About four volumes a year are contemplated.

The following actions taken by mail vote of the Council were reported: The Riverside Campus of the University of California was elected to institutional membership in the Society. Dr. John H. Curtiss was elected Executive Director of the Society for a term of two years beginning September 1, 1954.

It was reported that the Executive Committee had voted to schedule a joint session of the Society and Section A of the AAAS at the meeting of the AAAS in Berkeley, California, December, 1954.

The final report of the Executive Director Dr. H. M. MacNeille was presented. In this report the activities at the Headquarters of the Society during the past five years were reviewed.

The Council voted to set meetings on February 26, 1955 at Columbia University and June 18, 1955 at the University of British Columbia. The Council elected Professors A. M. Gleason and J. L. Kelley as representatives of the Society on the Board of Editors of the Annals of Mathematics for a term of three years beginning January 1, 1955. The Council elected Professors D. H. Lehmer and N. E. Steenrod as representatives of the Society in the Division of Mathematics of the National Research Council for a term of three years beginning July 1, 1955.

The Council voted to extend Professor Saunders MacLane's leave of absence as a member of the Editorial Committee for the Transactions and Memoirs to November 10, 1954.

A Business Meeting of the Society was held in Education Auditorium at 10:15 a.m. September 1, 1954, with Vice President D. H. Lehmer presiding. The Secretary reported on the affairs of the Society.

The members present voted to amend the by-laws as follows:

To amend Article VIII, Section 2 to read: There shall be four classes of members, namely ordinary, contributing, corporate and institutional.

To amend Article VIII, Section 5 to read: A university or college, or a firm, corporation, or association interested in the support of mathematics may be elected a corporate or an institutional member.

To amend Article IX, Section 5 to read: The minimum dues of a corporate member shall be one hundred dollars.

To amend Article IX, Section 6 to read: The privileges of a corporate or an institutional member shall depend on its dues in a manner to be determined by the Council, subject to approval by the Board of Trustees. These privileges shall be in terms of Society publications to be received by the institution and of the number of persons it may nominate for ordinary membership in the Society.

To amend Article IX, Section 11 to read: The annual dues of ordinary, contributing, and corporate members shall be payable after October first of the preceding year and due by January first of the year to which they apply. The Society shall submit bills for dues. If the annual dues of any member remain undischarged beyond what the Board of Trustees deem to be a reasonable time, his name shall, after due notice, be removed from the list of members. If a member wishes to discontinue his membership at any time, he shall submit his resignation in writing to the Society.

The abstracts of the papers follow. Those having the letter "*t*" following the abstract number were read by title. Where a paper, presented in person, has more than one author, the symbol (p) follows the name of the author who presented it. Paper number 774 was read by Dr. T. C. Doyle. Professor Jehle and Dr. Bade were introduced by Professor H. B. Ribeiro, Dr. Peaceman and Dr. Rachford by Dr. Jim Douglas, Jr., Mr. Hanf by Professor Alfred Tarski and Dr. Knothe by Professor J. A. Ward.

ALGEBRA AND THEORY OF NUMBERS

635. T. M. Apostol (p) and Abe Sklar: *Identities related to approximate functional equations of certain Dirichlet series.*

The authors derive identities expressing any Hecke series $\phi(s)=\sum a(n)n^{-s}$ of signature $(\lambda, \kappa, \gamma)$ in terms of its partial sums and a series involving integrals of Bessel functions. A similar identity is obtained for the product of two Riemann zeta-functions $\zeta(s)\zeta(s+p)$. These identities can be used to obtain approximate functional equations for the Dirichlet series in question. The method employs inverse Mellin transforms and is similar to the technique used by one of the authors [Duke Math. J. vol. 18 (1951) pp. 517–525] to obtain identities of a different type for the same functions. (Received July 19, 1954.)

636*t*. Rafael Artzy: *On loops with a special property.*

Loops G with the property $xy\cdot x'=y$ for all $x, y\in G$ are dealt with, x' being defined by $xx'=1$. As a special device the author uses "cycles," i.e. sequences $x_1, \cdots, x_n$ such that $x_i'=x_{i+1}$ mod n. The main results are the following: (1) G cannot consist only of 1 and of m cycles of equal length n, unless n divides $2m$. Corollary: If $m=1$, right and left inverses coincide. (2) There exist infinitely many nonisomorphic loops G of infinite order whose elements (besides 1) form an infinite cycle. (3) The unit element and the elements of a cycle of length n, together with the elements of all the cycles whose lengths are factors of n, form a subloop of G. (4) If G has a cycle of length $n>2$, G has another cycle whose length divides n. (5) If an infinite G contains finite cycles, then 1 and all elements of the cycles form a subloop of G. (6) Finite loops G of identical structure, as regards the number of cycles and their lengths, need not be isomorphic. (Received July 12, 1954.)

637*t*. H. W. Becker: *Duals of the classic rational cuboids, and Hero and Pythagorean triangles with two sides in common.*

R. D. Carmichael, *Diophantine analysis*, p. 103, prob. 9, made the even power case of Fermat's Last Theorem depend on (1) $x^2+y^2=z_1^2$, $x_1^2+y^2=z^2$, $z^2+y_1^2=z^2$, a tetrahedron with 3 rt. $\measuredangle$ each at a different vertex. Dualize the Saunderson cuboid, M. Kraitchik, Scripta Math. vol. 11 (1945) p. 322, by reversing alternate signs, then apply $r, s\rightarrow r\pm s$. (1) is thus satisfied by (2): $x, z=(r^2\mp s^2)(r^4\pm r^2s^2+s^4)$; $y=2rs(r^4-s^4)$; $x_1, z_1=(r^2\pm s^2)(r^4\mp 3r^2s^2+s^4)$; $y_1=2r^3s^3$; $=63, 65, 60, 25, 87, 16$ in the smallest example. The tet. is physically unrealizable unless $x_1+y_1>z_1$, met by $r, s=4, 3$, etc.; each side of the inequality is divisible by $r-s$. FLT further requires $x_1, y_1, z_1=p$th powers, $p>1$: it is conjectured that at most 2 of the 6 values can be such. The Euler transforms of (1), (2) are the tets. with 2 rt. $\measuredangle$ at one vertex symmetrical to the 3rd rt. ∠ (3) $X^2+Y^2=Z_1^2$, $Z^2=Y^2-X_1^2=X^2-Y_1^2$, and (4) $X, Z=2rs(r^2\pm s^2)(r^4\mp r^2s^2+s^4)$; $Y=\prod(r^4\pm r^2s^2+s^4)$; $X_1, Z_1=(r^4\pm r^2s^2+s^4)(r^4\mp 3r^2s^2+s^4)$; $Y_1=4r^4s^4$; $X_1+Y_1=(r^4-r^2s^2+s^4)^2$; $=260, 252, 273, 105, 377, 64, 169$ in the least example. In the least real tet. $r, s=5, 4$. Removing Z_1 and dividing out $r^4-r^2s^2+s^4$, the tet. flattens down to a Hero $\triangle\ k_1, b, k_2$ of alt. $=a$ and base-segments $=b_1, b_2$, having 2 sides in common with a Pyth. $\triangle\ k_1, b, k$. With $b=b_1+b_2$, $k=(k_1^2+b^2)^{1/2}$, and H, P the respective areas, one gets (5), $a, k_1=2rs(r^2\mp s^2)$; $b, k_2=r^4\pm r^2s^2+s^4$; $b_2, k=r^4\mp 3r^2s^2+s^4$; $b_1=4r^2s^2$; $H, P=rs(r^2\mp s^2)(r^4+r^2s^2+s^4)$, generalizing the Hero and Pythagorean triangle pair 20, 21, 13 and 20, 21, 29. (Received May 20, 1954.)

638t. H. W. Becker: *Hero and Diophantus quadrilaterals with edges squares.*

Their diagonals are respectively $2(ijkl)^{1/2}=$ int., and $\{(i^2+j^2)(k^2+l^2)\}^{1/2}$; they have the same 4 integer edges $ik\pm jl$, $il\mp jk$. All 4 are $\boxplus$ in the Euler (5) and Gerardin solutions of $A^4+B^4=C^4+D^4$, Dickson's *History* II, pp. 646–647; i, j, k, l are resp. $f^8+20f^6g^2+86f^4g^4+20f^2g^6+g^8$, $4fg(f^2+g^2)(f^4-18f^2g^2+g^4)$, $4f^6+f^4g^2+10f^2g^4+g^6$, $f^6+10f^4g^2+f^2g^4+4f^6$, and $(f^2+g^2)(f^4+3f^2g^2+g^4)$, $6f^3g^3$, $f^2(f^6-2f^4g^2+f^2g^4+g^6)$, $g^2(f^6+f^4g^2-2f^2g^4+g^6)$. These are the same quads. under $f, g\to f\pm g$; but are based on distinct solutions of $pq(p^2+q^2)=rs(r^2+s^2)$. "$2379^4+27^4=729^4+577^4$" is an erratum; the intended meaning was the quad. of Desboves[1[illegible]] and its self-transform $A\pm D$, $C\pm B=2379, 27, 729, 577=p', q', r', s'$. In Fauquembergue's[178] solution of $A^2+B^4=C^4+D^4$, ibid., the last 3 of $ik\mp jl$, $il\pm jk$ are $\boxplus$, with $i, j, k, l=f^2+16g^2$, $f^2+12fg+4g^2$, $6f(f-g)$, $10f^2+6fg+g^2$. To make $ik-jl$, $il-jk$ $\boxplus$, put $i, k, j, l=f(f\pm g)$, $g(f\mp g)$. Or, let $i, j, k, l=I^2, J^2, K^2, L^2$; I, J, K, L parameters of a quad. with 4 edges $\boxplus$. Or, satisfy $ab(c^2-d^2)$, $cd(a^2-b^2)=\boxplus$, met by parameters of any Petrus vector or its double; perhaps the simplest solution of this type is the Euler-Hillyer $i, j, k, l=(3r^2+s^2)(3r^2-s^2)^2(3r\pm s)(r\pm s)$, $16r^2s^2(3r\mp s)(r\mp s)$. From one numerical solution of $A^2+B^2=C^4+D^4$, hence $ik+jl$, $il-jk=\boxplus$, another is derived, etc. ad inf., by Carmichael's transform, *Diophantine analysis*, p. 47. Thus $39^2+20^2=6^4+5^4\to 42961^2+54080^2=119^4+260^4$. Parametric solution is a desideratum. So are DQ with 2 or more edges nth powers. (Received May 20, 1954.)

639t. H. W. Becker: *Transforms from automedian triangles with two sides squares to Pythagorean tetrahedrons.*

(1) $2T^4-S^4=R^2$, an $A\triangle$ with 2 sides $\boxplus$; recurrence solutions, Dickson's *History* II, pp. 620–627 (errata: p. 622 last line, ξ, η should be y, x; p. 625 last line, σ^2x^2 should be σ^2x; p. 626 line 34, $8\lambda hk$ should be $8\lambda\mu hk$). (2) $v^2+x^2=u^2+z^2=t^2$, $v^2-z^2=u^2-x^2=y^2$, a PT. Landen, ibid. p. 447 (erratum: "The case $f=2$ gives Bumpkin's answer" isn't true till 2 sentences later) last method, is generalized to a transform (1)→(2) $x=TRS^2$, $z=(R^2-S^4)T^2/2$; $y, u=(R^2\mp S^2T^2)S^2/2$; $v, t=(T^2R^2\mp S^6)/2$. Thus $T, S, R=13, 1, 239\to x, z, y, u, v, t=3107, 4826640, 28476, 28645, 4826724, 4826725$; this specimen is unique, in that t and v (and most of the important parameter pairs) differ by unity. Frenicle's problem ibid. pp. 184–186: (3) $X^2+Y^2=T^2-Z^2=U^2$, $W=X\pm Y=Z\pm U$ (in some combo), $S=X\mp Y$, $V=Z\mp U$, depends on (1). A solution of (4) $abcd(a^2-b^2)(c^2-d^2)=\square=y^2/16$ is $a, b, c, d, a\pm b, c\pm d=Z, U, UV, ZW, W, V, T^2, S^2$. E.g. if $T=S=R=1$, then $U, X, Y, Z, T=5, 3, 4, 12, 13$ and $a, b, c, d=12, 5, 85, 84$, parameters of a PT. This triadizes with the Petrus transform of the Landen PT, whose $a, b, c, d, a\pm b, c\pm d=UV, ZW, T^4, 4ZUVW, T^2, S^2, R^2, S^4$. (3) is a Petrus transform (5) $T, U=t^2\pm y^2$; $U, X=x^2\pm z^2$; $Y=2xz$, $Z=2ty$ of a $3D$ Diophantine vec-

tor (6) $x^2+y^2+z^2=t^2$, such that $z(x\pm z)=ty$, $x^2\pm 4ty=w^2$. With (1′) $S'^4-2T'^4=R'^2$, and subscript 0 denoting the next smaller solution of (1), (3), then (7) $T_0'/S_0'=(T_0S_0\pm R_0)/(2T_0^2+S_0^2)$; $w, x, y, z, t=T_0T_0'\pm S_0S_0', S_0T_0', T_0T_0', T_0S_0'$. Fermat's $P\triangle$ is (8) $P^2+Q^2=T^4$, $P\pm Q=S^2$; $P, Q=(S^2\pm R)/2=2ZU$, VW in some order. The $A\triangle$ needn't be realizable. (Received July 1, 1954.)

640*t*. Garrett Birkhoff and R. S. Pierce: *Lattice ordered rings.*

A po-ring is an (associative) ring which is partly ordered such that $x\geqq y$ implies $a+x\geqq a+y$ for all a, and $a\geqq 0$, $b\geqq 0$ implies $ab\geqq 0$. An l-ring is a po-ring which is a lattice under its partial ordering. An f-ring is an l-ring which satisfies: $b\wedge c=0$ implies $ab\wedge c=ba\wedge c=0$ for all $a\geqq 0$. *Lemma*: An l-ring is an f-ring iff $(a\vee 0)\wedge(-a(b\vee 0)\vee 0)=0$ and symmetrically for all a and b. *Corollary*: The class of f-rings is closed under the formation of subalgebra, direct union and homomorphic image. *Theorem*: Any l-ring [σ-complete l-ring] with a unity which is a strong [weak] order unit is an f-ring. *Theorem*: An f-ring is subdirectly indecomposable iff it is a (simply) ordered ring. *Corollary*: Any f-ring is isomorphic to a subdirect union of ordered rings and, conversely, any subdirect union of ordered rings is an f-ring. (Received April 5, 1954.)

641. R. H. Bruck and L. J. Paige (p): *Loops whose inner mappings are automorphisms.*

Let G be a loop and let I^* be the inner mapping group of G. If every element of I^* is an automorphism of G, G is said to be an A-loop. A-loops have the following properties: (i) Subloops and quotient loops are A-loops; (ii) Characteristic subloops are normal; (iii) Every maximal associative (commutative) subset is a subloop; (iv) A-loops are power associative. A necessary and sufficient condition that every isotope of an A-loop is again an A-loop is that $N\supseteq G'$, where N is the nucleus and G' the derived loop of G. The similarity between diassociative A-loops and Moufang loops is shown. (Received July 14, 1954.)

642*t*. Leonard Carlitz: *Weighted quadratic partitions over* $GF[q, x]$.

For $\alpha\in GF(q)$, $q=p^n$, $p>2$, define $e(\alpha)=e^{2\pi i t/p}$, where $t=\alpha+\alpha^p+\cdots+\alpha^{p^{n-1}}$. For $\xi=\sum_{-\infty}^{m}\alpha_i x^i$, $\alpha_i\in GF(q)$, define $e(\xi)=e(\alpha_{-1})$. Now construct the sum $S=\sum e(\xi_1U_1+\cdots+\xi_rU_r)$, where the summation is over all $U_i\in GF[q, x]$ of degree $<k$ such that $\alpha_1U_1^2+\cdots+\alpha_rU_r^2=M$. In the present paper it is shown that S can be evaluated in terms of Gauss and Kloosterman sums. For the special case of weighted quadratic partitions over a finite field see Canadian Journal of Mathematics vol. 5 (1953) pp. 317–323. (Received July 12, 1954.)

643*t*. Leonard Carlitz: *Some applications of weighted quadratic partitions.*

Making use of the explicit results for weighted quadratic partitions in a finite field (Canadian Journal of Mathematics vol. 5 (1953) pp. 317–323) it is shown for example how to determine the number of solutions $\xi_1, \cdots, \xi_r$ of the equation $\alpha=\alpha_1(\xi_1^p-\xi_1)^2+\cdots+\alpha_r(\xi_r^p-\xi_r)^2$, where $\alpha, \alpha_i, \xi_i\in GF(q)$. (Received July 12, 1954.)

644*t*. Chen-Chung Chang: *Two theorems on direct decompositions of relations*. Preliminary report.

A relation R is said to be *connected* if $R\neq 0$ and whenever $R=S\cup T$ where the fields of S and T are disjoint, then either $S=0$ or $T=0$. The following theorems are proved: (I) If R is a connected reflexive relation which is anti-symmetric, then the set of factor congruence relations over R forms a Boolean algebra under relative multiplication and set intersection. (See B. Jónsson and A. Tarski, Bull. Amer. Math. Soc. Abstract 59-1-114.) (II) If R is a connected reflexive relation which has at least one anti-symmetric element x, i.e., such that the formulas xRy and yRx imply $x=y$, then R has the refinement property, that is to say, if $R \cong \mathfrak{P}_{i\in I} S_i \cong \mathfrak{P}_{j\in J} T_j$, then there are relations $U_{\langle i,j\rangle}$, $\langle i, j\rangle \in I \times J$, such that $S_i \cong \mathfrak{P}_{j\in J} U_{\langle i,j\rangle}$ for every $i \in I$ and $T_j \cong \mathfrak{P}_{i\in I} U_{\langle i,j\rangle}$ for every $j \in J$. It is known that for every relation R the conclusion of (I) implies the conclusion of (II); also the conclusion of (I) implies R has at most one decomposition into indecomposable factors while the conclusion of (II) implies R has up to isomorphism at most one such decomposition. Theorems (I) and (II) improve various known factorization theorems for relations (J. Hashimoto, Ann. of Math. vol. 54 (1951) pp. 315–318; Jónsson and Tarski, op. cit.; Tarski, *Cardinal algebras*, p. 309). Similar results for algebras $\mathfrak{A} = \langle A, +\rangle$ have also been obtained. (Received July 19, 1954.)

645t. Harvey Cohn: *Finiteness of representations by binary abelian forms*. Preliminary report.

As an illustration let the form in (x, y) be $N(\omega)$ where $\omega = x - \theta y$ for θ an integer in an abelian field contained in $K_f = R(\exp 2\pi i/f)$ for f a prime. Then ω lies in K_f and is in constant ratio to another integer Ω of K_f with zero trace. Hence any bound on $N(\omega)$ would yield from Ω normal bases of submodules of integers with bounded index. Such Ω are then bounded in size and number, by normal bases. (Received July 19, 1954.)

646. W. R. Cowell: *Quasinormal kernels of loops*.

Let G be a loop and K be the kernel of a homomorphism of G upon a groupoid. Then K is said to be quasinormal in G if the cosets of the homomorphism have the form $x\Delta(K)$, where $\Delta(K)$ is the semigroup of mappings of G generated by $x \to kx$, $x \to xk$, where $x \in G$, $k \in K$. If $C(x)$ is the coset containing x in any homomorphism of G and K is quasinormal, then $C(x)\Delta(K)$ is the coset of a homomorphism of G and $C(x)\Delta(K) = C(x\Delta(K))$. The product of two quasinormal kernels H and K is defined to be $P(H, K) = H\Delta(K) = K\Delta(H)$ and it is shown that $P(H, K)$ is quasinormal in G. If $H \cap K = 1$, then $P(H, K) = H \times K$. For any two series of quasinormal kernels of G, there exist kernels (not necessarily quasinormal) of the subloops in the series which may be intercalated to produce isomorphic refinements. If M is a groupoid possessing an identity and having the property that if $a, b \in M$ the equations $ax = b$, $ya = b$ have solutions (not necessarily unique) $x, y \in M$, then M is the homomorphic image of at least one loop such that the kernel of the homomorphism is quasinormal. (Received July 19, 1954.)

647. R. B. Crouch: *Monomial groups*.

Let U be a set. Let H be a group. The set of monomial substitutions on the elements of U over H form a group Σ. The set V of pure multiplications forms a normal subgroup, the basis group. The set S of pure permutations forms a subgroup, the permutation group. Σ splits over V; $\Sigma = V \cup S$, $V \cap S = E$, where E is the identity of Σ. Ore [Trans. Amer. Math. Soc. vol. 5, pp. 15–64] has found all representative groups of the splitting over V, all normal subgroups of Σ, and all automorphisms of Σ when U is finite. This paper extends his investigation by allowing U to be of arbitrary order. All representative groups of the splitting over V are found for Σ and some of its subgroups, including the case when S is replaced by A, the infinite alternating group. With a rather strong restriction on elements of V and S or A all normal subgroups are determined. All representative groups of the splitting over V_n for $\Sigma_{n,A} = V_n \cup A_n$, where A_n is the finite alternating group, are found. The normal subgroups of $\Sigma_{n,A}$, $n \geqq 5$, are determined. That the basis group is characteristic is proved for the cases where the normal subgroups are known. (Received June 17, 1954.)

648*t*. D. O. Ellis: *IPIC representation of lattice automorphisms.*

It may be concluded from previous studies (Bull. Amer. Math. Soc. Abstracts 60-1-89, 60-1-88) that the IPIC mappings of a topolattice form a group under "dual convolution" and that this group is isomorphic to the automorphism group of the topolattice. In the present paper this result is extended to arbitrary bounded lattices (the boundedness being no restriction from the automorphism viewpoint). One may remark that the Boolean algebra of subsets of an infinite set has the same automorphism group as a certain nonmodular lattice and, hence, that automorphisms determine the algebraic character of a lattice very little. (Received June 16, 1954.)

649. Paul Erdős, Leonard Gillman and Melvin Henriksen (p): *An isomorphism theorem for real-closed fields.*

It is well known that every real-closed field has a unique ordering (which is preserved under isomorphism). An ordered *set* L is called an *η_α-set* if (i) no $A \subset L$ of power $< \aleph_\alpha$ is cofinal or coinitial with L, and (ii) if $A, B \subset L$, with $|A|, |B| < \aleph_\alpha$ and $A < B$, then there is an $x \in L$ such that $A < x < B$. All η_α-sets of power $\aleph_\alpha$ are similar (Hausdorff, *Grundzüge der Mengenlehre*, pp. 180–182). *Theorem* 1: All real-closed fields that are η_α-sets of power $\aleph_\alpha$, where $\alpha > 0$, are isomorphic. Next, let M be any maximal ideal of the ring $C(X)$ of all continuous real-valued functions on a completely regular space X. Then the residue class field $C(X)/M$ is a real-closed field containing the real field R; if $C(X)/M$ contains R properly, then M and $C(X)/M$ are called *hyper-real* (Hewitt, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 45–99; Isbell, Proc. Amer. Math. Soc. vol. 5 (1954) (to appear)). Theorem 1 is used in obtaining *Theorem* 2: If $\mathfrak{c} = \aleph_1$, then all hyper-real fields of power $\mathfrak{c}$ are isomorphic. *Corollary*: If $\mathfrak{c} = \aleph_1$, then all the nonreal residue class fields of maximal ideals of a countable complete direct sum of real fields are isomorphic. (Received July 19, 1954.)

650*t*. Paul Erdős, Leonard Gillman and Melvin Henriksen: *Hyper-real fields of various cardinals.*

Terminology is as in preceding abstract. Several results in general set theory, concerning decompositions of sets, are derived, and are applied to obtain the following theorems. *Theorem* 1: Let X be any infinite discrete space; then there exist hyper-real ideals M of $C(X)$ for which $|C(X)/M| = \mathfrak{c}$. (If $\mathfrak{c} = \aleph_1$, then by Theorem 2 of the preceding abstract, all such fields $C(X)/M$ are isomorphic.) *Theorem* 2: Let X denote the discrete space of power $\mathfrak{c}$, and let M be any maximal ideal of $C(X)$ such that for every $f \in M$, the subset of X on which f vanishes is of power $\mathfrak{c}$; then $\mathfrak{c} < |C(X)/M| \leqq 2^{\mathfrak{c}}$. *Theorem* 3: Let $\mathfrak{m}$, $\mathfrak{n}$ satisfy $\mathfrak{c} \leqq \mathfrak{n} \leqq \mathfrak{m}$, and let X denote the discrete space of power $\mathfrak{m}$; then there is a maximal ideal M of $C(X)$ for which $\mathfrak{n} < |C(X)/M| \leqq 2^{\mathfrak{n}}$. *Corollary:* If X is the discrete space of power $\mathfrak{m} \geqq \mathfrak{c}$, then not all hyper-real residue class fields $C(X)/M$ are isomorphic. (Received July 19, 1954.)

651*t*. William Hanf: *A result concerning isomorphism of Boolean algebras.* Preliminary report.

The following result (solving a problem of A. Tarski) has been established: (I) $\mathfrak{B}_1$,

$\mathfrak{B}_2$ *being two-element Boolean algebras, there exists a Boolean algebra* $\mathfrak{A}$ *such that* (i) $\mathfrak{A}$ *is isomorphic to the direct product* $\mathfrak{A}\times\mathfrak{B}_1\times\mathfrak{B}_2$ *and* (ii) $\mathfrak{A}$ *is not isomorphic to* $\mathfrak{A}\times\mathfrak{B}_1$. In fact, $\mathfrak{A}$ is the set-algebra consisting of all those sets of integers which are unions of an arbitrary finite set and of arbitrarily many pairs $\{2n, 2n+1\}$. In topological translation (I) yields: (I′) *There exists a 0-dimensional compact space S, with infinitely many isolated points, such that* (i′) *S is homeomorphic to each of its subspaces obtained by removing two isolated points, and* (ii′) *S is not homeomorphic to any subspace obtained by removing one isolated point.* (I) is related to an earlier result of Shin-ichi Kinoshita who constructed three infinite, denumerable Boolean algebras $\mathfrak{A}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$ satisfying (i), (ii). R. L. Vaught observed that no compact separable space, with infinitely many isolated points, satisfies (ii′); hence $\mathfrak{B}_1$, $\mathfrak{B}_2$ being two-element Boolean algebras, no denumerable Boolean algebra $\mathfrak{A}$ satisfies both (i) and (ii). Using a different method, B. Jónsson recently constructed a compact nonseparable space, with infinitely many isolated points, satisfying (ii′); his space however is not 0-dimensional. (Received July 23, 1954.)

652*t*. W. E. Jenner: *On integral subrings of Lie algebras.*

It is shown that a maximal integral subring of a semisimple Lie algebra over the quotient field of a Dedekind ring of characteristic zero is not necessarily a multiplicative unit operator on its ideals. For terminology see Bull. Amer. Math. Soc. Abstract 60-2-194. Let k be the rational numbers, $\mathfrak{o}$ the rational integers, and $\mathfrak{L}$ the Lie algebra of $n\times n$ matrices of trace zero with $n\geqq 4$. Let $\mathfrak{O}$ be the integral subring of $\mathfrak{L}$ spanned over $\mathfrak{o}$ by the elements e_{ij} $(i\neq j)$ and $e_{ii}-e_{nn}$ $(i\neq n)$ where the e_{rs} are the canonical basis elements for $[k]_n$. Let $\alpha=\sum_{i=1}^{n-1}(e_{ii}-e_{nn})$ and set $\mathfrak{A}=n\cdot\mathfrak{O}+\mathfrak{o}\{\alpha\}$. Then $[\mathfrak{O}, \mathfrak{A}]\subset\mathfrak{A}$. If $\beta=n^{-1}\alpha$ then $\mathfrak{O}_m=\mathfrak{O}+\mathfrak{o}\{\beta\}$ is the unique maximal integral subring containing $\mathfrak{O}$ and $[\mathfrak{O}_m, \mathfrak{O}_m]\subset\mathfrak{O}_m$. Thus neither $\mathfrak{O}$ nor $\mathfrak{O}_m$ is a multiplicative unit operator on its ideals. (Received July 14, 1954.)

653*t*. Bjarni Jónsson: *There exist nonrepresentable integral relation algebras.*

(Received July 15, 1954.)

654*t*. Irving Kaplansky: *The Zariski topology and algebraic matrix groups.*

Assume: (1) a T_1-space with descending chain condition on closed sets, (2) a group with continuous inverse and multiplication *separately* continuous, (3) for fixed a, the mapping $x\rightarrow x^{-1}ax$ continuous. Matrix groups under the Zariski topology satisfy these axioms, and a portion of their theory is obtainable this way. Among the byproducts there are several apparently new results such as: in any matrix group there is an upper bound to the size of the finite conjugate classes. (Received July 19, 1954.)

655*t*. Joachim Lambek and Leo Moser: *Some associative operations on integers.*

Let $f(0)=0, f(1), f(2), \cdots$ be an increasing sequence of non-negative integers, such that $\Delta f(a)+\Delta f(b)\leqq\Delta f(a+b)+1$, where $\Delta f(m)=f(m+1)-f(m)$. Let $g(n)$ be the largest m for which $f(m)\leqq n$. Then the operation $m\circ n=m+n+f(g(m)+g(n))-f(g(m))-f(g(n))$ is associative. This is shown by considering the (1-1) mapping $n\rightarrow(g(n), n-f(g(n)))$ of the non-negative integers into the first quadrant of the Cartesian plane, under which $m\circ n$ corresponds to the vector sum of the images of

m and n. An example is $m\circ n=m+n+2\{m^{1/2}\}\{n^{1/2}\}$, where $\{x\}$ is the closest integer to x. (Received July 6, 1954.)

656. W. J. LeVeque: *The error term in the distribution* (mod 1) *of* $\{2^k t\}$.

For $0<x<1$ and $0<t<1$, let $N_n(x,t)$ be the number of positive integers $k\leqq n$ such that $(2^k t)\leqq x$, where $(u)=u-[u]$. It is well known that the sequence $\{(2^k t)\}$ is uniformly distributed (mod 1), so that $\lim_{n\to\infty}N_n(x,t)/n=x$, for almost all t. Put $y_n(x,t)=n^{1/2}(N_n(x,t)/n-x)$. It is shown that if t is regarded as a random variable with uniform distribution in (0, 1), then $\lim_{n\to\infty}\Pr\{\int_0^1 y_n^2(x,t)dx<\alpha\}$ exists and equals $\Pr\{\int_0^1 y^2(x,t)dx<\alpha\}$, $y(x,t)$ a Gaussian stochastic process with a rather complicated covariance function. A similar result is obtained for $\lim_{n\to\infty}\Pr\{\int_0^1\phi_\beta(y_n(x,t))dx<\alpha\}$, where $\phi_\beta(u)$ is 1 or 0 according as $u\geqq\beta$ or $u<\beta$. Some support is given for the conjecture that the classical discrepancy $\sup_{0<x<1}|y_n(x,t)|$ also has a nondegenerate limiting distribution. (Received July 19, 1954.)

657. Gene Levy: *Representation theory for generalized p-rings.*

A ring R is called a q-ring if (1) $x\in R$ implies $x^q=x$, and (2) $t<q$ implies there is a $y\in R$ such that $y^t\neq y$. The characteristic is finite. Using a theorem of Birkhoff's, it is shown that every q-ring is isomorphic to a subdirect sum of Galois fields; the converse is true if the set of distinct Galois fields is finite. For a given q there may be no q-ring; the necessary and sufficient condition for the existence of a q-ring is that there be primes p_i and positive integers α_{ij} such that $q=1+\text{l.c.m.}_{i=1,2,\cdots,n;\,j=1,2,\cdots,m}(p_i^{\alpha_{ij}}-1)$. A q-ring of characteristic c is called a (q,c)-ring. For these rings the p_i are the prime factors of c. Consider the set S of all divisors of $(q-1)$ of the form $(p_i^{\alpha_i}-1)$, $p_i|c$. If $(p_i^{\alpha_i}-1)\in S$, there is a (q,c)-ring whose representation includes $GF_{p_i^{\alpha_i}}$. Every representation of a (q,c)-ring will include GF_{p_i} if and only if $(p_i-1)\in S$, $(p_i^{\alpha_i}-1)\notin S$, $\alpha_i>1$; it will include $GF_{p_i^{\alpha_i}}$ if and only if l.c.m. $\{S-(p_i^{\alpha_i}-1)\}<(q-1)$. (Received May 18, 1954.)

658. R. J. Mihalek (p) and L. R. Wilcox: *The Jordan chain theorem for certain nonmodular lattices.*

It is known that the Jordan theorem, to the effect that principal chains with the same end points have equal lengths, holds in any modular lattice, also in any lattice called semi-modular by Birkhoff (*Lattice theory*, rev. ed., pp. 100, 66–67). This same theorem is now proved for any lattice in which modularity is symmetric, whether of finite length or not. Various consequences, already proved for such lattices of finite length, notably covering relations involving special elements such as points (elements covering 0), lines (elements covering points), and hyperplanes (elements covered by 1) are shown to hold in this somewhat more general environment. (Received July 9, 1954.)

659t. Leo Moser and Max Wyman: *On an array of Aitkin.*

Let $A_{m,n}$ be defined by (i) $A_{m,n}=A_{m-1,n+1}+A_{m-1,n}$ $(m\geqq 1)$, (ii) $A_{0,n+1}=A_{n,0}$, (iii) $A_{0,0}=a$, $A_{0,1}=b$. The case $a=b=1$ has been considered by A. C. Aitkin, *A problem in combinations*, Math. Notes Edin. vol. 21 (1933) pp. 18–23, and others. Generally, if $A_{0,n}=B_n$ then $A_{m,n}=(B+1)^m B^n$, symbolically, and we need only study the B's. The general case is reduced to the case $a=0$, $b=1$. In this case the following results are obtained: (1) $B_{n+1}=(B+1)^n-1$. (2) $\sum_{n=0}^{\infty}B_n x^n/n!=e^{e^x}\int_0^x e^{-e^t}dt$. (3) B_n

$=\sum_{r=1}^{n}\Delta^r 0^n(0!-1!+2!-\cdots+(-1)^{r-1}(r-1)!)$. (4) For p a prime $B_p\equiv -(1/1!+1/2!+\cdots+1/(p-1)!)\pmod p$. (Received July 19, 1954.)

660t. Leo Moser and Max Wyman: *On solutions of $x^d=1$ in symmetric groups.*

Let $A_{n,d}$ denote the number of solutions of $x^d=1$ in the symmetric group of degree n. A relation is established between $A_{n,2}$ and the Hermite polynomials $H_n(x)$. This leads to a refinenent of some results of Chowla, Herstein, and Moore (Canadian Journal of Mathematics vol. 3 (1951) pp. 328–334). By another method, asymptotic formulae for $A_{n,p}$ are obtained. Some arithmetic properties of $A_{n,d}$ are proved, extending results of the paper mentioned above and of results proved by E. Jacobsthal (Norske Vid. Selsk. vol. 21 (1949) pp. 49–51) and Chowla, Herstein, and Scott (Norske Vid. Selsk. vol. 21 (1952) pp. 49–51). It is shown how some of these results can be extended to alternating groups. (Received July 19, 1954.)

661. H. B. Ribeiro: *Boolean algebras with operators and topological groups.* Preliminary report.

Complex algebras of groups, as T_1 spaces, have been discussed as, respectively, special relation algebras and special closure algebras. (For this and terminology see Chin-Tarski, *Distributive and modular laws in the arithmetic of relation algebras*, University of California Publications in Mathematics, 1951, and Jónsson-Tarski, *Boolean algebras with operators*, I and II, Amer. J. Math. vols. 73 and 74.) Now, the problem of placing the concept of topological group within the framework of the arithmetic of Boolean algebras with operators is that of expressing the continuity conditions. If $^{\sim}$ designates the closure operator, then the continuity of $^{\smile}$ is obviously given by $\breve{x}^{\sim}=\tilde{x}^{\smile}$ for any x, and that of ; is expressible as follows: for any x, y, z if $(x; y)\ z\neq 0$ then there are u and v such that $x\cdot u^{-}\neq 0$, $y\ v^{-}\neq 0$, and $(u^{-}; v^{-})+z^{-}=z^{-}$. Using this it follows $\tilde{x}; \tilde{y}\leq(x; y)^{\sim}$ for any x, y, which easily gives that the closure of any subgroup is a subgroup as well as many other known properties. (Received July 19, 1954.)

662. Alex Rosenberg and Daniel Zelinsky (p): *Galois theory of continuous transformation rings.*

A continuous transformation ring $L=L(M, N)$ is the ring of all continuous linear transformations on a pair of dual spaces (M, N) over a division ring. A subring A of L is (strongly) Galois in case A is the ring of invariants under a regular group G of automorphisms of L ("regular" defined as in Nakayama [Trans. Amer. Math. Soc. vol. 73 (1952) pp. 276–292]). Such an A automatically satisfies (i) A is a continuous transformation ring, hence has a socle, S; (ii) both M and N are completely reducible A-modules (i.e., $MS=M$ and $NS^*=N$) and the dimensions of M and N over A are equal and finite; (iii) the centralizer of A in L is simple. Then the fundamental theorem of Galois theory holds for L in the form: there is the usual 1-1 correspondence between the regular subgroups of G and the subrings of L that contain A and satisfy (i), (ii), and (iii). Likewise, every isomorphism between two subrings of L which is the identity on A can be extended to an automorphism of L. These theorems are direct generalizations of the corresponding theorems of Nakayama [op. cit.] for the ring of all linear transformations on a finite-dimensional space. The proofs lean heavily on the methods and results of a paper by Rosenberg [Bull. Amer. Math. Soc. Abstract 60-4-471] and an unpublished book by N. Jacobson. (Received July 19, 1954.)

663*t*. Alex Rosenberg and Daniel Zelinsky: *Extension of derivations in continuous transformation rings.*

We first give a simpler proof of the following theorem of Jacobson: Let E be the ring of all endomorphisms of an abelian group M, and let C be a primitive subring of E with nonzero socle S such that $MS=M$. Then any derivation of C into E can be extended to an inner derivation of E. Now let $L=L(M, N)$ be a continuous transformation ring, and let C be a primitive subring of L with nonzero socle S. Suppose further that $MS=M$, $NS^*=N$, and S is contained in the socle of L. Then any derivation δ of C into L can be extended to an inner one of L if either (i) C is an algebra over the center Φ of L, the division ring of C has finite dimension over Φ, and δ is Φ-linear; or (ii) there is a subring A of C annihilated by δ, and whose centralizer in E has a basis over the division ring of L consisting of (not necessarily continuous) semilinear transformations. The latter is a generalization of a theorem of Nakayama [Duke Math. J. vol. 19 (1952) pp. 51–63]. (Received July 19, 1954.)

664*t*. E. V. Schenkman: *The splitting of certain solvable groups.*

Let G be a finite group and let H be the least member of the descending central series, that is, G/H is nilpotent and $[G, H]=H$. Then if H is Abelian: (1) there is a complement X of H in G, that is, $G=HX$, and $x\cap H=E$; (2) if X and Y are two complements of H in G, then $X=hYh^{-1}$ for some h in H; (3) if G has no center then G and A the group of automorphisms of G are contained in the holomorph of H. (Received July 12, 1954.)

665. Abe Sklar: *A generalization of a summation formula of Ferrar.*

Ferrar [Compositio Math. vol. 4 (1937) pp. 394–405] has derived a class of summation formulae related to certain Dirichlet series, in particular the Riemann zeta-function. The author generalizes these and shows that the generalized formulae can be related to a wider class of Dirichlet series, including the Hecke-functions of signature $(\lambda, \kappa, \gamma)$. Some of the results of Apostol and Sklar [see Abstract 635] can be gotten as special cases of the generalized formulae. (Received July 19, 1954.)

666*t*. Andrew Sobczyk: *Canonical form for a real matrix.*

Any real nonsingular matrix T may be reduced, by unimodular change of basis, to a diagonal block form $D_1, A_1, \cdots, D_v, A_v, H_1, B_1, \cdots, H_w, B_w$, where there is one pair of blocks D_k, A_k for each real eigenvalue t_k, of multiplicity p_k, and one pair of blocks H_k, B_k for each pair of conjugate complex eigenvalues $(x_k \pm iy_k)$, of multiplicity q_k. Let r_k (at least one) be the number of independent eigenvectors which correspond to eigenvalue t_k; then block D_k is of the form t_kI, where I is the identity matrix of dimension (r_k-1), and A_k is a superdiagonal block of dimension (p_k-r_k+1), having only one eigenvector τ_k. Let s_k be the number of pairs of conjugate eigenvectors $\xi_1 \pm i\eta_1$, $\xi_2 \pm i\eta_2, \cdots$, corresponding to eigenvalue $x_k \pm iy_k$, which are such that $\xi_1, \eta_1, \xi_2, \eta_2, \cdots$, are linearly independent in the real vector space E_n; then block H_k has (s_k-1) blocks of the form $\begin{pmatrix} x_k & y_k \\ -y_k & x_k \end{pmatrix}$ on the diagonal, all other elements of H_k being zeros; block B_k has similar blocks on the diagonal, with all elements below the diagonal blocks being zeros. Any two nonconjugate eigenvectors of B_k are complex linearly dependent. If T is singular, of rank m, the canonical form is similar, with an additional diagonal block Z which is superdiagonal with zeros on the diagonal, and of which the elements of the last $(n-m)$ rows are all zeros. (Received July 15, 1954.)

667*t*. Irving Sussman: *On rings in which* $a^{n(a)}=a$. Preliminary report.

Rings characterized by the property that for each nonzero element not the identity, $a^{n(a)}=a$, $n(a)$ an integer exceeding 1, and not necessarily fixed or bounded, form a subclass of the encompassing class of associate rings hitherto treated (Bull. Amer. Math. Soc. Abstracts 59-6-589 and 60-1-151). Rings belonging to this subclass are named simply periodic. Algebras of this type where $n(a)$ is bounded, or perhaps a fixed integer, have been treated under other names by several authors (see, for example, N. Jacobson, *Algebraic algebras of bounded degree*, Ann. of Math. vol. 46 (1945); also, where $n(a)$ is a fixed prime, see work by Foster, McCoy, Tarski, et al.). Every finite ring without proper nilpotents is simply periodic. Among other interesting properties developed for simply periodic rings: they necessarily have identities if there are non-zero-divisors, and each non-zero-divisor is a unit; the ring decomposes uniquely into local multiplicative groups, and is a field if and only if it contains no proper idempotent; it is necessarily finite if there are a finite number of idempotents; the ring is a certain composition of isomorphic Boolean rings; it is a subdirect sum of finite fields, etc. Other characterizations of these rings are given. It is shown by example that the theory is not pertinent to semigroups. (Received July 14, 1954.)

668*t*. Alfred Tarski: *Some remarks and problems concerning isomorphism of algebras*. Preliminary report.

The result (I) of the abstract of W. Hanf (Abstract 651) implies the following: (II) *There are Boolean algebras* $\mathfrak{A}$ *and* $\mathfrak{C}$ *such that* $\mathfrak{A}\times\mathfrak{A}$ *and* $\mathfrak{C}\times\mathfrak{C}$ *are isomorphic while* $\mathfrak{A}$ *and* $\mathfrak{C}$ *are not*. In fact, $\mathfrak{A}$ is the same as in (I), and we put $\mathfrak{C}=\mathfrak{A}\times\mathfrak{B}_1$, where $\mathfrak{B}_1$ is a two-element algebra. In topological translation (II) yields: (II′) *There is a compact 0-dimensional space* S *with four subspaces* S_1-S_4 *such that* (i) *each of the subspaces* S_1-S_4 *is closed in* S; (ii) S *is the union of* S_1 *and* S_2 *as well as of* S_3 *and* S_4; (iii) S_1, S_2 *are both disjoint and homeomorphic, and so are* S_3, S_4; (iv) S_1 and S_3 *are not homeomorphic*. Using an argument of the author in his book *Cardinal algebras*, New York, 1949, p. 209 ff., the results (I) and (II) are carried over to commutative semigroups; in (I) one takes for $\mathfrak{B}_1$ and $\mathfrak{B}_2$ the additive semigroup of natural numbers. Open problems: (1) Are there denumerable Boolean algebras $\mathfrak{A}$ and $\mathfrak{C}$ satisfying the conditions of (II)? (2) Is it possible to extend (I) and (II) to groups and in particular to Abelian groups? (Received July 23, 1954.)

669. Leonard Tornheim: *The inessential discriminant divisors of cubic and certain quartic fields*.

The primes occurring in the inessential discriminant divisor of an algebraic number field of degree n over the rational field are known to be less than n. When $n=3$, 2 occurs to at most the first power and a necessary and sufficient condition for this to happen expressed in terms of a defining equation is obtained by means of a minimal basis. For the quartic subfield of the field generated by a primitive pth root of unity with p a prime and $p\equiv 1 \pmod 4$, Carlitz [Proc. Amer. Math. Soc. vol. 3 (1952) pp. 688–692] determined when 2 and 3 appear. These also occur only to the first power. (Received July 19, 1954.)

670*t*. A. D. Wallace: *The topological invariance of ideals in mobs*.

Let S be a compact connected mob (=Hausdorff semigroup) and let G be an abelian group. Let the codimension of S (relative to G) be at most n and let N be a closed set in S with $N \subset t_1 N$ for some $t_1 \in S$. Then (i) if $H^n(N, G) \neq 0$ the minimal ideal K of S is also a minimal right ideal; (ii) if A is a floor for an element of $H^n(N, G)$, then A is a left ideal contained in K. It follows that if also $N \subset N t_2$ for some $t_2 \in S$, then K is a floor for each element $H^n(N, G)$ and thus K is a topological invariant of S. Moreover K is a (topological) subgroup of S. If N is a subgroup of S with $H^n(N, G) \neq 0$, then $K = N$. (Received April 12, 1954.)

671*t*. L. M. Weiner: *The algebra of semi-magic squares.*

A square matrix $A = (a_{ij})$ is called an S-matrix if $\sum_{i=1}^{n} a_{ij} = \sum_{i=1}^{n} a_{ji} = S(A)$ for each j. The set of all S-matrices of order n forms a subalgebra of the total matric algebra of degree n. This algebra is the direct sum of the one-dimensional ideal M consisting of all matrices A for which $a_{ij} = a$, a constant, and the $(n-1)^2$-dimensional ideal N consisting of all matrices A for which $S(A) = 0$. A basis for the ideal N is given by the matrices A_{ij} $(i, j = 2, 3, \cdots, n)$, where A_{ij} is the matrix which has 1 in the first row and first column, -1 in the first row and jth column, -1 in the ith row and first column, and 1 in the ith row and jth column. (Received July 8, 1954.)

672. A. L. Whiteman: *A sum connected with the series for the partition function.*

In the formula for the number $p(n)$ of unrestricted partitions of an integer n there appears the sum (1) $A_k(n) = \sum \omega_{h,k} \exp(-2\pi i h n/k)$, where h ranges over those numbers which are less than k and relatively prime to k. The numbers $\omega_{h,k}$ are certain $24k$th roots of unity which arise in the theory of modular functions. Some years ago Atle Selberg proved (but did not publish) the result that $A_k(n)$ may be expressed alternatively in the form (2) $A_k(n) = (k/3)^{1/2} \sum (-1)^l \cos [(6l+1)\pi/6k]$, where l runs over integers in the range $0 \leq l < 2k$ which satisfy the congruence $(3l^2+l)/2 \equiv -n \pmod{k}$. In this paper Selberg's formula is proved by showing that (1) is the finite Fourier series expansion of (2). Selberg's formula is also used to provide a new approach to the evaluation and factorization of the $A_k(n)$. (Received May 20, 1954.)

673. L. R. Wilcox: *An imbedding theorem for semi-modular lattices.* Preliminary report.

Let M be a complemented modular lattice of length $\geqq 3$ in which join and meet are $\cup, \cap$. Let $L \subset M$ satisfy (1) $0, 1 \in L$; (2) $a, b \in L$ implies $a \cup b \in L$; (3) every $a, b \in L$ possess a g.l.b. $ab \in L$ with respect to elements of L; (4) if $a \in M$, $b \in L$, $a \leqq b$, then there exists $x \in L$ with $a \cup x = b$, $a \cap x = 0$. L is then called a ***regular subsystem*** of M. It is shown to be a left complemented lattice, i.e., in it modularity is a symmetric relation, and it possesses a strong complementation property (L. R. Wilcox, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 453–457), though it need not be a sublattice of M or modular. It is shown that under certain additional assumptions, primarily concerning complementation, an abstract left complemented lattice L of length $\geqq 6$ may be imbedded into a complemented modular lattice M so that L is (isomorphically) a regular subsystem of M. The principal tools used in the imbedding are a general theory of independence and a theory of *quasi-dual-ideals*, i.e., subsets S of L satisfying (a) $x \in S$, $y \geqq x$ implies $y \in S$, and (b) if $z = x \cap y$ with $x, y \in S$ and (x, y) modular, then $z \in S$. (Received July 9, 1954.)

674. K. G. Wolfson: *Annihilator rings.*

The structure of annihilator algebras (which generalize dual algebras) [Bonsall and Goldie, Proc. London Math. Soc. vol. 4 (1954) pp. 154–167] is generalized to arbitrary topological rings. Let K be a topological ring in which the intersection of the closed regular maximal right ideals is zero, and which possesses the annihilator property (*) a left (right) ideal J has zero right (left) annihilator if, and only if $\bar{J}=K$. Then K is the closure of the direct sum of simple rings with the annihilator property. For each of the simple components K_γ there exists a pair of topological vector spaces (A_γ, B_γ) over a topological division ring D_γ, which are dual relative to a continuous "inner product" such that each space is the set of all continuous linear functionals on the other. Then K_γ is the closure of the ring of all finite-valued linear transformations on A_γ which have adjoints on B_γ. In particular, a semi-simple discrete annihilator ring is a direct sum of simple rings with minimum condition, and is therefore a dual ring. The converse question of when dual spaces give rise to simple annihilator rings is examined. (Received July 19, 1954.)

ANALYSIS

675. H. A. Antosiewicz: *Stable systems of differential equations with integrable forcing term.*

Vector equations $dx/dt=p(x, t)$ (*), defined for all x and $t\geqq 0$, are considered subject to perturbations (i) $q(x, t)$ where $\|q(x, t)\|\leqq\|x\|k(t)$ for $\|x\|\leqq\xi$, $t\geqq 0$, (ii) $q(x, t, s)$ where $\|q(x, t, s)\|\leqq\phi(s)k(t)$ for $\|x\|\leqq\xi$, $t\geqq 0$ with $\phi(s)$ continuous at $s=0$ and $\phi(0)=0$, $k(t)$ being a non-negative function with $\int_0^\infty k(t)dt=H<\infty$. Assuming that (*) possesses a Liapunov function $V(x, t)$, it is shown in case (i) that if $V(x, t)$ is a quadratic form in x with uniformly bounded coefficients then every solution of the perturbed equation satisfies $\|x(t, x_0, t_0)\|\leqq M\|x_0\|$ for $t\geqq t_0+T$ where $0<M<1$, $T(M, x_0)>0$, x_0 sufficiently small; and in case (ii) that, given any $\epsilon>0$, there exist constants $\xi(\epsilon, H)>0$, $\delta(\epsilon, H)>0$ such that every solution of the perturbed equation with $\|s\|\leqq\xi$ satisfies $\|x(t, x_0, t_0)\|\leqq\epsilon$ for $t\geqq t_0$ whenever $\|x_0\|\leqq\delta$. (Received June 25, 1954.)

676. Nachman Aronszajn: *Coercive quadratic integro-differential forms.*

A finite number of partial differential operators A_k of orders $\leqq m$ defined in a domain G of the n-dimensional euclidean space, with continuous coefficients, are considered. Denote by $q_k(x;\xi)$ the characteristic polynomial, homogeneous of order m, of A_k ($q_k=0$ if A_k is of order $<m$). For functions u of class $C^{(m)}$ in G and a domain D with $\bar{D}\subset G$ put $\|u\|_D^2=\int_D\sum|A_k u|^2dx$. For $p=0, 1, \cdots, m$ put $\|u\|_{p,D}^2=\sum\int_D|\partial^\alpha u|^2dx$, $\partial^\alpha u$ running through all partial derivatives of u of order $\leqq p$. The quadratic form $\|u\|_D^2$ is called *coercive* in a domain $D'\subset D$ if for some constant $c>0$, $\|u\|_{m,D'}^2\leqq c(\|u\|_D^2+\|u\|_{m-1,D'}^2)$. In what follows, the boundary of D' is of class $C^{(m)}$. *Theorem* 1: If $\overline{D'}\subset D$, $\|u\|_D^2$ is coercive in D' if and only if for each $x\in\overline{D'}$ there is no common real null-vector $\xi\neq 0$ for all $q_k(x, \xi)$ (this is equivalent to ellipticity of $A=\sum A_k^*A_k$ in $\overline{D'}$). *Theorem 2*: If $D'\subset D$ and at every common boundary point y of D and D', the boundary of D is locally of class $C^{(m)}$, $\|u\|_D^2$ is coercive in D' if and only if the condition of Theorem 1 is satisfied and for every common boundary point y there is no common *complex* nullvector $\xi\neq 0$ for all $q_k(y, \xi)$ with imaginary part orthogonal to the tangential hyperplane at y. These theorems allow one to develop a theory of differential problems for the operator $A=\sum A_k^*A_k$ with *general* boundary conditions in a similar way as Gårding's inequality does in the case of Dirichlet boundary conditions (see L. Gårding, C. R. Acad. Sci. Paris vol. 233 (1951) pp. 1554–1556; Math. Scand. vol. 1

(1953) pp. 55–72; also F. Browder, Proc. Nat. Acad. Sci. U.S.A. vol. 38 (1952) pp. 230–235, 741–747). (Received July 19, 1954.)

677*t*. R. G. Bartle: *On compactness in functional analysis.*

S. Kakutani [J. Math. Soc. Japan vol. 1 (1951) pp. 228–231] has introduced the notion of the (*strong*) *pairing* of two abstract sets to obtain short, symmetric proofs of the Arzela-Ascoli compactness criterion in $C(X)$ and Schauder's well-known theorem on compact operators. The author introduces a corresponding notion of *weak pairing* to obtain corresponding weak compactness results. Among the applications are previously announced results of the author [Abstract 59-6-693] giving criteria for pointwise and weak compactness in $C(X)$, and extensions or simplifications of theorems due to Gantmacher, Gelfand, and Sirvint concerning [weakly] compact operators. This paper supersedes the one referred to above. (Received July 15, 1954.)

678. J. S. Bendat (p) and Seymour Sherman: *Monotone and convex operator functions.*

Charles Loewner [*Über monotone Matrixfunctionen*, Math. Zeit. vol. 38 (1934)] showed that analyticity plus the property of mapping the complex open upper half-plane into itself is characteristic for the class of monotone matrix functions of arbitrarily high order. This class coincides with the class of monotone operator functions, the operators being bounded self-adjoint operators in a Hilbert space. For convex operator functions, no characterization or relationship with monotone operator functions was known. The principal result in this paper establishes the proper correspondence between monotone and convex operator functions by showing that a convex operator function $f(x)$, i.e., for each scalar α, $0<\alpha<1$, $\alpha f(X_1)+(1-\alpha)f(X_2)\geq f(\alpha X_1+(1-\alpha)X_2)$, is necessarily analytic, and if $f(x)=\sum_1^\infty a_nx^n$ in $|x|<R<\infty$, where R is the radius of convergence, then $f(x)$ can be uniquely represented by a Stieltjes integral $f(x)=a_1x+\int_{-1/R}^{1/R}x^2(1-tx)^{-1}d\psi(t)$, where $\psi(t)$ is a bounded nondecreasing function, continuous on the left, with $\psi(-1/R)=0$. Further considerations then prove that $f(x)=\sum_1^\infty a_nx^n$ is a convex operator function in $|x|<R$ if and only if $f(x)/x$ is a monotone operator function in $|x|<R$. (Received June 15, 1954.)

679*t*. Stefan Bergman: *On solutions of certain linear partial differential equations in three variables.*

Using analytic functions which are orthonormal, when integrating over the distinguished boundary surface D^3 of a domain M^4, the author derives a formula expressing the values of an analytic function $f(z_1, z_2)$ of two complex variables in terms of its values on the distinguished boundary surface. Since the kernel K of this system is finite, one obtains bounds for $|f|$ inside the domain M^4. The author introduces meromorphic functions which are orthogonal to K in M^4. Using these functions he obtains new representations in M^4 for certain classes of meromorphic functions. Using integral operators which transform analytic functions of two complex variables into solutions $\chi(x, y, z)$ of $L(\chi)=(\partial^2\chi/\partial x^2)+(\partial^2\chi/\partial y^2)+(\partial^2\chi/\partial z^2)+C(r^2)\chi=0$, where C is an entire function of $r^2=x^2+y^2+z^2$ (see Trans. Amer. Math. Soc. vol. 68, 461 ff.), he obtains from the results mentioned above various theorems about the solutions χ of $L(\chi)=0$. (Received September 1, 1954.)

680. Arthur Bernhart: *Note on Fuchsian differential equations.*

For linear differential equations with analytic coefficients Fuchs obtains power series solutions, and shows convergence by comparison with a dominant equation. A modification by Günther simplifies the comparison solution at the expense of its radius of convergence. This paper combines the advantages of both existence proofs. (Received July 19, 1954.)

681. W. W. Bledsoe (p) and A. P. Morse: *Product measures.*

Suppose μ and ν are (outer) measures. Let $\theta \in \text{Fubini}\ \mu\nu$ if and only if θ measures the product space in such a way that: $\int\int f(x, y)\mu dx\nu dy = \int f(z)\theta dz = \int\int f(x, y)\nu dy\mu dx$ whenever f is θ-summable; $A \times B$ is θ-measurable and θ-finite whenever A is μ-measurable and μ-finite, B is ν-measurable and ν-finite. Also let $\theta \in \text{Clin}\ \mathfrak{K}$ if and only if $\mathfrak{K}$ is a topology and θ measures space $\mathfrak{K}$ in such a way that: open sets (members of $\mathfrak{K}$) are θ-measurable; open sets have a certain closed subset property; $\mathfrak{K}$ is θ-almost lindelof (no local compactness is assumed). Suppose $\mathfrak{M}$ and $\mathfrak{N}$ are topologies and $\mathfrak{T}$ is their topological product. A new (topology free) product measure ϕ is defined which allows more freedom of action than the classical product measure and which belongs to Fubini $\mu\nu$. For $\mu \in \text{Clin}\ \mathfrak{M}$, $\nu \in \text{Clin}\ \mathfrak{N}$, it is proved that $\phi \in \text{Clin}\ \mathfrak{T}$, from which it follows (since $\phi \in \text{Fubini}\ \mu\nu$) that $\int\int f(x, y)\mu dx\nu dy = \int f(z)\phi dz = \int\int f(x, y)\nu dy\mu dx$ whenever f is a $\mathfrak{T}$-Borelian function which is dominated by a ϕ-summable function. Other product measures are defined which preserve more restrictive hypotheses on μ and ν. Preliminary to the principal results are discussions of integrable functions and closed subset properties of topological measures. (Received June 2, 1954.)

682*t*. H. D. Block: *Perturbations of nonlinear eigenvalue problems.*

In a paper to appear shortly, P. C. Rosenbloom has considered the eigenvalue problem for a perturbed operator [cf. Bull. Amer. Math. Soc. Abstract 58-4-440]; i.e. let T_0 be a linear operator on a Banach space with $T_0x_0 = \lambda_0x_0$. If U is a "small" operator what can be said about the solutions, (x, λ), of the equation $(T_0 + U - \lambda I)x = 0$? We use the methods introduced by Rosenbloom to deal with this problem when T_0 is not linear, and $U = U(\lambda)$ is not linear. It is proved that, under suitable conditions, the solution (x, λ) is unique and lies close to (x_0, λ_0). An iterative method for obtaining the solution is given as well as numerical estimates of the rate of convergence and the errors of terminating the iteration after a finite number of steps. (Received July 19, 1954.)

683*t*. R. P. Boas, Jr.: *Moments of analytic functions.*

The author proves the following generalization of results of San Juan [C. R. Acad. Sci. Paris vol. 236 (1953) pp. 1941–1943] and Sunyer Balaguer [Revista Mat. Hisp.-Amer. (4) vol. 13 (1953) pp. 241–246]. If $f(z)$ is analytic and of exponential type for $x \geq 0$ and is bounded on the imaginary axis, and if $\int_0^\infty |f(re^{i\theta})| r^n dr < n^n e^{-n\phi(n)}$, $\phi(n) \rightarrow \infty$, for some θ $(-\pi/2 < \theta < \pi/2)$ and for an infinity of n, then $f(z) = 0$. The theorem is deduced from a general result of Ahlfors and Heins [Ann. of Math. (2) vol. 50 (1949) pp. 341–346]. (Received June 1, 1954.)

684. H. D. Brunk, G. M. Ewing (p), and W. T. Reid: *The minimum of a certain definite integral suggested by the maximum likelihood estimate of a distribution function.*

For $a \geq 0$, $b \geq 0$, $0 \leq u \leq 1$, define $L(u; a, b) = -a \ln u - b \ln (1-u)$ if $0 < u < 1$. Let $g(x)$ be bounded nondecreasing and not constant on $[0, X]$, let μ be the Lebesgue-Stieltjes measure generated by g, and let $\alpha(x)$ be μ-measurable on $[0, X]$. After suitably extending $L(u; a, b)$, the function $F(u; x) = L[u; \alpha(x), 1-\alpha(x)]$ is defined finite or ∞ for $0 \leqq u \leqq 1$, $0 \leqq x \leqq X$. Let $\mathfrak{M}$ be the class of nonincreasing functions on $[0, X]$ bounded by 0 and 1 and consider $J[p] = \int_{[0,X]} F[p(x); x] d\mu$. The principal results of this paper are (i) that there exists P minimizing J in $\mathfrak{M}$, P being unique except for sets of μ-measure zero, (ii) that if P minimizes J in $\mathfrak{M}$ then, almost everywhere (μ), P is equal to $\inf_{0 \leqq u \leqq x} \sup_{x \leqq v \leqq X} M(I)$, and to $\sup_{x \leqq v \leqq X} \inf_{0 \leqq u \leqq x} M(I)$, where $x \in I$, an interval with end points u, v, and $M(I)$ is the integral mean of $\alpha(x)$ over I with respect to μ. (Received July 19, 1954.)

685t. Paul Civin: *Abstract Riemann sums.*

A theorem of Jessen [Ann. of Math. vol. 35 (1934) pp. 248–251] asserts for $f(x)$ Lebesgue integrable on $[0, 1]$ and of period one the almost everywhere existence of $\lim_{n\to\infty} 2^{-n} \sum_{k=0}^{2^n-1} f(x+k2^{-n}) = \int_0^1 f(t)dt$. The theorem is shown to be a special case of a theorem analogous to the Birkhoff ergodic theorem asserting the almost everywhere limits of sums $2^{-n} \sum_{k=0}^{2^n-1} f(T^{k/2^n} x)$ on a σ-finite measure space. Analogues are also obtained for the ergodic theorems of Hopf and Hurewicz [Ann. of Math. vol. 45 (1944) pp. 192–196]. (Received May 28, 1954.)

686t. Paul Civin: *Some ergodic theorems involving two operators.*

A version of the Hurewicz ergodic theorem [Ann. of Math. vol. 45 (1944) pp. 192–206] is obtained for two operators T and U on a σ-finite measure space. Each operator is assumed to be a 1-1 transformation of the space onto itself and to satisfy the Birkhoff ergodic theorem. As a corollary the Hopf ergodic theorem is carried over to the form that $\lim_{n\to\infty} \sum_{k=0}^{n} f(T_x^k) / \sum_{k=0}^{n} g(U_x^k)$ exists almost everywhere when T and U are measure preserving, $f(x)$ integrable and $g(x) > 0$. An extension is thus made of a result of Harris and Robbins (Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 860–864) concerning Markov chains admitting an infinite invariant measure. (Received May 28, 1954.)

687. K. L. Cooke: *Forced periodic solutions of a stable nonlinear differential-difference equation.*

In this paper it is proved that the differential-difference equation $u'(t+1) = au(t) + bu(t+1) + f(u(t), u(t+1), \omega t) + kg(\omega t)$ has a continuous, asymptotically stable, periodic solution, of period $1/\omega$, provided $|k|/(1+\omega)$ is sufficiently small, and provided the following conditions are satisfied: (1) all the roots of $se^s - be^s - a = 0$ have negative real parts; (2) a, b, k, and ω are real numbers, $\omega > 0$, and t is a real variable; (3) $g(t)$ is real and continuous for all t, has period one, and the average of $g(t)$ over a period is zero; (4) $f(x, y, t)$ is real and continuous for all (x, y, t), has period one in t, $f(0, 0, t) = 0$, and for any $\epsilon > 0$, $|f(x, y, t) - f(\xi, \eta, t)| \leqq \epsilon(|\xi - x| + |\eta - y|)$ for small enough $|\xi - x|$ and $|\eta - y|$, uniformly in t. The periodic solution is constructed by the method of successive approximations. The method may be extended so as to be applicable to systems of differential-difference equations. (Received June 14, 1954.)

688. Philip Davis and H. O. Pollak (p): *Complex biorthogonality for certain sets of polynomials.*

Let $\{L_n\}$ be a set of linear functionals complete in the set of analytic functions square-integrable over a complex domain B. A basic theorem of Walsh and Davis (*Interpolation and orthonormal systems*, Journal d'Analyse Mathématique vol. 2 (1952) pp 1–28) states that there exist linear combinations L_n^* of L_n, and functions ϕ_n^*, such that

$L_n^*(\phi_m^*) = \delta_{mn}$ and $\iint_B \phi_n^* \overline{\phi_m^*}\, dA = \delta_{mn}$. The present paper extends this theorem and applies it to the development of a unifying theory for different sets of polynomials and rational functions. As examples, an L^2-theory is derived from this unified point of view for Tschebysheff polynomials of the second kind, Takenaka-Walsh interpolation rationals for the unit circle, and Faber-type polynomials for an arbitrary domain B with analytic boundary. (Received July 19, 1954.)

689*t*. M. M. Day: *On smoothing normed spaces*. Preliminary report.

A result of Klee (Trans. Amer. Math. Soc. vol. 74 (1953) pp. 10–43, A1.7) which applies only to reflexive spaces is improved as follows: Let B' be a smooth reflexive space and let T be a linear operator carrying B' into a dense subset of a normed space B; then B is isomorphic to a smooth space B_1 and B_1^* is strictly convex. This lemma proves smoothability of $c_0(\Gamma)$, the space of functions vanishing at infinity on an arbitrary index set Γ, when B' is taken to be $l_2(\Gamma)$, and of $L_1(\quad)$, the space of summable functions on a space of finite measure, when B' is taken to be $L_2(\quad)$. (Received July 19, 1954.)

690. W. F. Donoghue: *A theorem concerning bounded linear operators on reflexive Banach spaces.*

If T is a bounded linear transformation of norm $\leqq 1$ on a reflexive infinite-dimensional Banach space, there exists no element X in the space such that the closed circled convex hull of the sequence $\{T^n X\}$, $n \geqq 0$, contains an interior point. That the hypothesis of reflexivity is needed is shown by an example involving the space l'. (Received September 2, 1954.)

691*t*. R. J. Duffin: *Some properties of discrete analytic functions.*

Complex-valued functions are considered which are defined at the Gaussian integer points of the complex plane. The Gaussian integer points form a lattice which breaks up the plane into squares. Jacqueline Ferrand defined a function to be "analytic" at one of these squares if the difference quotient across one diagonal is equal to the difference quotient across the other diagonal. She found several interesting analogies between such discrete analytic functions and ordinary analytic functions. In this paper analogues of the Cauchy integral formula are developed. The line integral is, of course, replaced by a suitable summation over the lattice points. The explicit evaluation of the Cauchy kernel is found. Asymptotic development of this kernel is furnished by methods previously developed by the writer and D. H. Shaffer. Functions are considered which are bounded and analytic in the upper half-plane; the real and imaginary parts on the real axis are related by an analogue of the Hilbert transform. Expansions in terms of polynomials and other type functions are considered. (Received July 20, 1954.)

692. Paul Erdős, Fritz Herzog, and George Piranian (p): *On polynomials whose zeros lie on the unit circle.*

Let (1) $P(z) = \prod_{j=1}^{n}(z - z_j)$, where $|z_j| = 1$ for $j = 1, 2, \cdots, n$. It is easily seen that there exists a continuous curve Γ, starting at the origin and ending at a point of the unit circle, such that $|P(z)| < 1$ at all points of Γ except the origin; an analogous statement holds for the inequality $|P(z)| > 1$. However, C. Loewner found a complicated example (unpublished) of a polynomial (1) for which $|P(z)| > 1$ on at least one point of every radius of the unit disc. In the present paper is exhibited a simple example of a polynomial (1) with the property that on each radius of the unit disc there exist two points at which $|P(z)|>1$ and $|P(z)|<1$, respectively. It is also shown that, for each polynomial (1) of degree not greater than 4, there exists a radius of the unit disc on which the inequality $|P(z)|<1-|z|^n$ holds everywhere. (Received July 19, 1954.)

693. Paul Erdős, Fritz Herzog (p) and George Piranian: *Sets of divergence of Taylor series and of trigonometric series.*

In two previous papers (Duke Math. J. vol. 16 (1949) pp. 529–534 and vol. 20 (1953) pp. 41–54) Herzog and Piranian have shown that (a) if a set on the unit circle C is of type G_δ, it is the set of divergence of a Taylor series and (b) certain sets on C, not of type G_δ, are sets of divergence of Taylor series whose partial sums are uniformly bounded on C. In the present paper, the authors use certain polynomials, introduced by Fejér, to obtain the following extensions of these results: (i) If a set on C is of type F_σ and of logarithmic measure zero, it is the set of divergence of a Taylor series whose partial sums are uniformly bounded on C and which represents a function continuous on $|z|\leqq 1$; (ii) If a set on C is the union of a set E_1 of type G_δ and a set E_2 of logarithmic measure zero and of type F_σ, it is the set of divergence of a Taylor series. Analogues to (i) and (ii) are proved for Fourier series and trigonometric series, respectively. The analogue to (i) is a generalization of a result due to Tandori (Publ. Math. Debrecen vol. 2 (1952) pp. 191–193). (Received May 28, 1954.)

694t. Evelyn Frank: *On the calculation of the roots of equations.*

The Newton method for the calculation of the solution of equations with n unknowns is here extended to include not only the linear terms of the Taylor series expansions but also terms of higher degree. Remarkably simple formulas are obtained for equations in two variables. These are exceptionally convenient for use in the computation of the complex roots of an equation $F(z)=0$. In this connection, a novel variation of Horner's method is presented for the computation of partial derivatives of rational integral functions. The convergence of the method to the value of the solution is considered, and an estimate of the error involved in the use of only a certain number of terms is indicated. A new method is also presented for the computation of bounds for the roots. (Received February 16, 1954.)

695. W. B. Fulks: *A note on the steady state solutions of the heat equation.*

Let R be a bounded region whose boundary C is a simple closed curve with continuous curvature. Then the unique bounded solution for continuous $\phi(x)$, $\psi(x, t)$ of the boundary value problem $\Delta u(x, t)-\partial u/\partial t=0$, $x\in R$, $t>0$; $u(x, 0)=\phi(x)$, $x\in R$; $u(x, t)=\psi(x, t)$, $x\in C$, $t>0$ is given by $u(x, t)=\int_R G_R(x, y, t)\phi(y)dA_y+\int_0^t\int_C H_R(x, y, t-s)\psi(y, s)dC_y ds$ where G_R and H_R are the Green's functions for the region R. The author uses this formula to show that if $\phi(x)=0$, $\psi(x, t)=\psi(x)\geqq 0$ is independent of time, then the solution $u(x, t)$ is a monotone increasing function of t which converges uniformly to the solution of the boundary value problem $\Delta u(x)=0$, $x\in R$; $u(x)=\psi(x)$, $x\in C$, as $t\rightarrow\infty$. (Received July 19, 1954.)

696. Michael Golomb: *On the mean-square approximation of functions of several variables by products of functions of fewer variables.*

The existence of functions $u(x)$, $v(y)$, $w(z)\in L_2$ which minimize the functional

$\iiint[f(x, y, z)-u(x)v(y)w(z)]^2dxdydz$ for given $f(x, y, z)\in L_2$ is proved and it is shown that u, v, w are the characteristic solutions of a certain triple of nonlinear integral equations belonging to its largest characteristic value. Furthermore, the function $U(x, y)=u(x)v(y)$ is the characteristic solution of a single nonlinear integral equation belonging to its largest characteristic value. Conditions are derived under which the simpler problem of minimizing $\iiint[f(x, y, z)-U(x, y)W(z)]^2dxdydz$ has the solution $U(x, y)=u(x)v(y)$, $W(z)=w(z)$ of the above problem. Similar results are obtained for the functional $\iiint[f(x, y, z)-\sum_{k=1}^{n} u_k(x)v_k(y)w_k(z)]^2dxdydz$ and for the corresponding variational problems in more than three independent variables. (Received July 19, 1954.)

697*t*. E. C. Gras: *Generalized axially symmetric potential theory and the Weber-Schafheitlin discontinuous integrals.*

Let $I=y^{1+\mu}b^{1+\nu}\int_0^\infty e^{-xt}J_{r+\mu}(yt)J_{r+\nu}(bt)t^{-(\mu+\nu+1)}dt$, where $r>0$, $\mu+\nu+1>-1$. Lim I as $x\rightarrow 0^+$ and $x^{-1}(y-b)\rightarrow \cot \beta$ is found explicitly by using the representation of a Bessel function by Sonine's integral (Watson, *Theory of Bessel functions*, p. 373), and by using the formula for the stream function of a "ring of sources" (A. Weinstein, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 342–354). These limits for real values of μ and ν are extensions of previous results which were valid only for integers (E. C. Gras, *On a generalisation of the integrals of Weber and Schafheitlin*, Bull. Amer. Math. Soc. Abstract 60-4-411). (Received May 20, 1954.)

698. W. T. Guy, Jr.: *A generalized Laplace transform.*

A "generalized Laplace" transform is defined using one of Meyer's G-functions as a kernel. The inversion formula is developed and a few examples are given. Varma's transform is obtained as a special case. (Received July 20, 1954.)

699. Guy Johnson, Jr.: *Collective singularities of families of analytic functions.*

Let F denote a family of functions $f(z)=\sum_{n=0}^{\infty} a_n^f(z-z_0)^n$. If there is a neighborhood of z_0 in which each function is holomorphic and F is normal, then z_0 is called a regular point. Otherwise it is called a singular point of F. In particular if there is a neighborhood in which each function is holomorphic but none in which F is normal, then z_0 is called a collective singularity. R is called the radius of regularity of F at z_0 if it is the largest number such that each function is holomorphic and F is normal in $|z-z_0|<R$. In the case where F is a bounded family, Mandelbrojt stated the formula $\limsup_{n\rightarrow\infty} \text{l.u.b.}_f |a_n^f|^{1/n}=1/R$. If F is locally bounded in $|z-z_0|<R$ and for each function there is a sequence of integers $\{\lambda_p^f\}$ such that $n\neq\lambda_p^f$, $p=1, 2, \cdots$, implies $a_n^f=0$ and $\lambda_{p+1}^f/\lambda_p^f\geqq\lambda>1$ where λ is fixed for F, then each point on $|z-z_0|=R$ is a singular point. Let G denote a family of functions $g(w)$ which has radius of regularity $R=\infty$ and is bounded in any finite domain. Let $M_g(r)=\max_{|w|=r}|g(w)|$, $M(r)=\text{l.u.b.}_g M_g(r)$ and $\limsup_{r\rightarrow\infty} (\log M(r))/r=A$. If $A<\pi/e$, then the family F of functions $f(z)=\sum_{n=0}^{\infty} g(n)z^n$ is normal in the complement of $\{z\,|\,|\log z|\leqq Ae, |z|\geqq e^{-\pi}\}$. (Received July 19, 1954.)

700*t*. Guy Johnson, Jr.: *Regions of flatness for analytic functions and their derivatives.*

The following definition and properties of regions of flatness extends those given by Mandelbrojt and Ulrich (Duke Math. J. vol. 18, pp. 549–556) from the special case of circles to simply-connected domains. Let D_α denote a simply-connected domain in the finite z-plane which contains $z=\alpha$. Let $\omega_\alpha(w)$ map the circle $|w|<1$ conformally on D_α such that $\omega_\alpha(0)=\alpha$ and $\omega'(0)>0$. Then $\omega_\alpha(w)$ maps $|w|<\eta<1$ on a domain $D_{\alpha,\eta}$. Let $\mathfrak{S}$ be an infinite set of domains D_α contained in a domain in which $f(z)$ is holomorphic and suppose $f(z)\neq 0$ in $D_{\alpha,\eta}$ except for at most a finite number of domains depending on η. If the family of functions $F_\alpha(w)=f[\omega_\alpha(w)]$ is normal in $|w|<1$, then the domains D_α will be called regions of flatness for $f(z)$. Suppose $f'(z)\neq 0$ in all but a finite number of domains $D_{\alpha,\eta}$. If there exist three positive constants A, B and C such that (1) $\omega'_\alpha(0)\left|[\log_2 f(\alpha)]'\right|>A$ and (2) $\log|f(\alpha)|>\max\{B\omega'_\alpha(0), C\}$ for all but a finite set of numbers α, then the domains D_α are regions of flatness for $f'(z)$. If $\mathfrak{S}$ is denumerable, $f(z)$ tends to infinity in the domains $D_{\alpha,\eta}$ and (2) is replaced by $\log|f(\alpha)|>\max\{B\omega'_\alpha(0), C/\omega'_\alpha(0)\}$ then the domains D_α are regions of flatness for $f^{(k)}(z)$, $k=1, 2, \cdots$. (Received July 19, 1954.)

701. V. L. Klee: *Boundedness and continuity of linear functionals.*

This paper (which will appear in the Duke Math. J.) establishes connections between continuity of linear functionals and their boundedness on certain sets. (A) If E is a metric linear space and C is a convex subset of E, the following statements are equivalent: (i) There is a closed linear subspace L of finite deficiency in E such that $C\subset L$ and $C-C$ has nonempty interior relative to L. (ii) Every additive functional on E which is bounded on C is continuous on E. (B) If, in addition, either E is a Banach space and C is closed or B is complete and C is a closed cone, then (i) and (ii) are equivalent to (iii): Every additive functional on E which is unilaterally bounded on C is continuous on E. The second part of (B) includes a rather general result on continuity of non-negative additive functionals, from which it can be deduced that for $0<r<1$ the space L^r admits no nontrivial non-negative additive functional, and that the algebra L^ω admits no nontrivial multiplicative additive functional. (Received July 6, 1954.)

702*t*. D. C. Kleinecke: *A generalization of complete continuity.*

It remains an unsolved problem whether the completely continuous operators on a general Banach space can be approximated in norm by operators with finite-dimensional ranges. In a sense the approximable operators are a lower "bound." An upper "bound" is found in the set of hyperfinite operators, that is the set of operators which map into the radical of the quotient algebra of the algebra of all operators by the ideal of approximable operators. This set can be characterized as the largest ideal of operators which have spectra of the same type as the completely continuous operators. The weakly completely continuous operators (Dunford and Pettis, Trans. Amer. Math. Soc. vol. 54 (1943) pp. 185–219) show the extension to hyperfinite operators is not trivial. It is shown that hyperfinite operators possess most of the interesting properties of the completely continuous operators. (Received May 14, 1954.)

703*t*. D. C. Kleinecke: *The essential and companion spectra.*

The essential spectrum $\sigma_e T$ of an operator T is now defined as the spectrum of T', the image of T in the quotient algebra by the approximable (or completely continuous) operators. (Compare Schwartz, Bull. Amer. Math. Soc. Abstract 60-2-270.) The companion spectra $\sigma_\nu T$ are the sets of λ for which $\alpha(\lambda I-T)=\nu$ ($\nu=\pm 1, \pm 2, \cdots$) where α is the index (Yood, Duke Math. J. vol. 18 (1951) pp. 599–612). The zero spectrum $\sigma_0 T$ is made up of two parts: $\sigma_e T$, spectral points λ for which $\alpha(\mu I-T)=0$

and $\mu \in \sigma T$ for all μ in some neighborhood of λ; $\sigma_m T$, isolated eigenvalues of finite multiplicity. ΦT is the union of $\sigma_m T$ and the resolvant set ρT. It is shown that $\sigma_\nu T$ $(\nu = \pm 1, \pm 2, \cdots)$, $\sigma_0 T$, ΦT are disjoint open sets whose boundaries lie in closed set $\sigma_\infty T$. If P is hyperfinite (which includes completely continuous) then the essential and companion spectra are preserved, $\sigma_\nu T = \sigma_\nu(T+P)$ $(\nu = \pm 1, \pm 2, \cdots, \infty)$; a component of ΦT may go into a component of $\sigma_0 T$ and conversely. In particular, if $\sigma_\infty T$ does not divide the plane, $\Phi T = \Phi(T+P)$. (Received May 14, 1954.)

704*t*. D. C. Kleinecke: *The spectrum as a function.*

The correspondence $T \to \Phi T$ of elements of a Banach algebra to their spectra may be considered as a function from the algebra to the bounded closed subsets of the complex plane taken with the Hausdorff metric. Newburgh (Duke Math. J. vol. 18 (1951) p. 165–176) investigated the question of continuity for this function. He left open the question when the algebra has the uniform topology. The following example shows σ is sometimes discontinuous in this case: Consider Hilbert space with base e_n $(n = 0, \pm 1, \pm 2, \cdots)$; let $Ae_n = -e_{n-1}$ and $P = (\cdot, e_1)e_0$. Then $A + \delta P \to A + P$ as $\delta \to 1$, but $\sigma(A+\delta P) = \{z \mid |z| = 1\}$ if $\delta \neq 1$, while $\sigma(A+P) = \{z \mid |z| \leqq 1\}$. It is shown that a similar disappearance of a component of the resolvant set occurs in connection with every discontinuity of σ. (Received May 14, 1954.)

705*t*. Jacob Korevaar: *Analytic functions bounded in an angle and nice on a ray.*

The author assumes that $h \in C_\rho(\beta)$ (where ρ is real, $\beta > 0$) if (i) $h(s) \in B(\beta)$, that is, $h(s) = h(re^{i\theta})$ is analytic in $|\theta| < \beta$ while for every $\beta' < \beta$ there exist numbers $A > 0$, a and b such that $|h(re^{i\theta})| < Ar^{-a}(1+r)^b$ in $|\theta| \leqq \beta'$; (ii) $h(x) \in D_\rho$, that is, there exist numbers B, C such that $|h^{(k)}(x)| \leqq BC^k k^{\rho k}$ $(0 < x < 1,\ k = 0, 1, \cdots)$. Various inequalities are derived, showing among other things that if $h \in C_\rho(\beta)$ then $h(xe^{i\theta}) \in D_\rho$ for every θ, $|\theta| < \beta$. The following characterization theorem is proved. $h \in C_\rho(\beta)$ if and only if there exists a function H with the following properties. (i) $H(w)$ is analytic in some half-plane $u = \text{Re } w < \gamma_0$; (ii) for every $\beta' < \beta$ $(\beta' > 0)$ there are b_1, b_2 and $\gamma < \gamma_0$ such that $\log |H(u+iv)| \leqq b_1 + b_2|u| + \rho|u| \log |u| + (\pi/2 - \beta')|v|$ $(u \leqq \gamma,\ -\infty < v < \infty)$; (iii) for every such γ which is not an integer $\leqq 0$ one has $h(s) = P(s) + (2\pi i)^{-1}\int H(w)\Gamma(w)s^{-w}dw$, where the integral is taken along the line $u = \gamma$, and where P is a polynomial of degree $< -\gamma$ $(P = 0$ if $\gamma > 0)$. One consequence is that for $\beta > (1+\rho)\pi/2$ (and $\beta > 0$) there are no functions in $C_\rho(\beta)$ other than polynomials. Applications are made to the case where $h(s)$ is of the form $\sum a_n \exp(-n^\lambda s)$ $(\lambda > 0)$. Then (formally) $H(w) = \sum a_n n^{-\lambda w}$. (Received July 16, 1954.)

706*t*. Jacob Korevaar: *Numerical Tauberian theorems for power series and Dirichlet series.*

Throughout this paper the a_n satisfy the Tauberian condition $a_n \geqq -n^{-1}\phi(n)$ $(n = 1, 2, \cdots)$, where $\phi(n) = A > 0$ or $\phi(n) = An^\alpha$ (or $\phi(n) = n^\alpha L(n)$). The above characterization of $C_\rho(\pi/2)$ is used to show that the following two assertions are equivalent $(\lambda, \rho > 0)$. (a) The series (1) $\sum a_n \exp(-n^\lambda x)$ converges when $x > 0$ and its sum $f_\lambda(x)$ is in D_ρ. (b) The series $\sum a_n \exp(-n^\mu x)$ with $\mu = \lambda/\rho$ converges when $x > 0$ and its sum $f_\mu(x)$ is in D_1. For such functions $f_\mu(x)$ analytic at $x = 0$, or rather, for corresponding integrals, Tauberian theorems may be proved by the methods of previous papers [Nederl. Akad. Wetensch. Proc. Ser. A vol. 57 (1954) pp. 36–45 and 46–56]. Various results for series (1) and series $\sum a_n n^{-s}$ follow. Example: let the series (1) converge when $x>0$ and let $f_\lambda(x)\in D_\rho$ $(\rho>0)$. Then one has the estimate (2) $|s_n-s| \leqq Bn^{-\lambda/\rho}\phi(n)$ $(n=1, 2, \cdots)$, where $s=\lim f_\lambda(x)$ as $x\downarrow 0$, $s_n=\sum_1^n a_k$. (2) is best possible when $\lambda\leqq\rho$. When $\lambda>\rho$ one has the estimate $|s_n-s|\leqq C \exp(-\epsilon n^{\lambda/\rho})$ $(n=1, 2, \cdots; \epsilon>0)$, a result related to gap series results. Several recent Tauberian theorems of the author can be brought within the present framework. Illustration: if the series (1) converges when $x>0$ and if $|f_\lambda(x)-s| \leqq E\exp\{-Fx^{-1/(\rho-1)}\}$ $(0<x<\infty$, $F>0$, $\rho>1)$, then $f_\lambda(x)\in D_\rho$. (Received July 16, 1954.)

707t. Jacob Korevaar: *Numerical Tauberian theorems for Lambert series and the Riemann hypothesis.*

Assuming the truth of the R.H. that $\zeta(s)$ has no zeros for $\sigma=\text{Re } s>1/2$ one has the following Tauberian theorem. Let the Lambert series $\sum a_n nx/(e^{nx}-1)$ be convergent for $x>0$, let its sum $f(x)$ be in $C_\rho(\pi/2)$, and for every $\delta>0$ let $|a_n|\leqq A(\delta)n^{\delta-1}$ $(n=1, 2, \cdots)$. Denote $\sum_1^n a_k$ by s_n, $\lim f(x)$ $(x\downarrow 0)$ by s. Then for every $\epsilon>0$ one has an estimate (*) $|s_n-s|\leqq B(\epsilon)n^{-c+\epsilon}$ $(n=1, 2, \cdots)$ where $c=1/2$ in the case $\rho<1$, $c=1/(\rho+2)$ in the case $\rho\geqq 1$. The estimate (*) with $c=1/2$ holds whenever $\sum_{d|n} da_d = O(n^{-1/2+\eta})$ for every $\eta>0$, and it might well hold in the case $\rho=1$. The above Tauberian theorem conversely implies the truth of the R.H. This follows from the example $a_n=\mu(n)/n$: $f(x)=xe^{-x}\in C_0(\pi/2)$. In this case (*) gives $|\sum_1^n \mu(k)/k| \leqq B(\epsilon)n^{-1/2+\epsilon}$, which implies that the Dirichlet series for $1/\zeta(s)$ converges when $\sigma>1/2$. (Received July 16, 1954.)

708. G. L. Krabbe: *On the logarithm of uniformly bounded operators.* Preliminary report.

Let V be a uniformly bounded operator in a reflexive Banach space, i.e. $\sup_n \|V^n\| < \infty$. In this paper a study is made of the operator solutions of $\exp(iX)=V$. Theorem I: If neither $+1$ nor -1 is an eigenvalue of V, then there is at most one bounded solution A with spectrum on the closed interval $[-\pi, \pi]$; if such an A exists, then any other bounded solution differs from A by a solution Q of $\exp(iQ)=I$ (the identity). In the particular case where V is a unitary operator in Hilbert space, such a solution A exists and is self-adjoint; this is still true if the restriction on the eigenvalues of V is removed, and the uniqueness of A results then from the additional condition that $-\pi$ not be an eigenvalue of A. Any other (possibly unbounded) self-adjoint solution again differs from A by a solution Q of $\exp(iQ)=I$. The explicit forms of the solutions Q are studied in both cases; in particular, these are characterized by the existence of a family of operators P_n such that $Q=2\pi\sum_{n=-\infty}^{\infty} nP_n$, $\sum_{n=-\infty}^{\infty} P_n=I$, and $P_iP_j=0$ when $i\neq j$. Example: The infinitesimal generator of the Titchmarsh semi-group T_α (see Bull. Amer. Math. Soc. Abstract 60-5-611) yields, for a certain class of Banach spaces, a solution A of $\exp(iA)=T_1$ having the required uniqueness properties. The proof of Theorem I is based on the integral representation of V developed by Lorch [Trans. Amer. Math. Soc. vol. 49 (1941) pp. 18–40]. (Received July 6, 1954.)

709. R. B. Leipnik: *Interchange theorems for limits of iterated integrals.*

Let $\Phi=\{\Phi_{m,n}\}$ be a double sequence of functions integrable with respect to $\mu\times\nu$ measure, and let L_1, L_2, L_3, L_4 denote limit w.r.t. m, limit w.r.t. n, integration w.r.t. μ, and integration w.r.t. ν, respectively. There are $4!=24$ possible orders in which L_1, L_2, L_3, and L_4 can be applied to Φ. Conditions are derived for interchange of order.

Applications are made to interchange theorems for improper double integrals of the Fourier type. (Received July 16, 1954.)

710. H. A. Linstone: *Singular perturbations of linear differential equations in the complex plane.*

For large values of the parameter λ, the asymptotic properties of the equation $y^{(n)}+\sum_{j=1}^{n} \alpha_j(x)y^{(n-j)}+\lambda^{n-m}[x^p\tilde{\beta}_0(x)y^{(m)}+\sum_{j=1}^{m} \beta_j(x)y^{(m-j)}]=0$, $n>m$, with n, m, and p positive integers and $\tilde{\beta}_0(0)\neq 0$, are studied. The independent variable x ranges over a bounded region S of the complex plane, in which, for all j, the coefficients $\alpha_j(x)$ and $\beta_j(x)$ are analytic, and which contains no zeros of $x^p\tilde{\beta}_0(x)$. No loss of generality results if $\tilde{\beta}_0(x)\equiv 1$ and $\arg \lambda=0$ are assumed. The use of asymptotic expansions in λ^{-1} to approximate the solutions involves the Stokes phenomenon. Using the techniques of Birkhoff, Turrittin, and Wasow, regions of validity, corresponding to any location of x in S, are found for the $n-m$ dominant-recessive asymptotic approximations and the m balanced asymptotic series. An analysis of an equation of the type $y^{(4)}+\lambda^2(x^2y''+ry) =0$, $r=$constant, by the Laplace contour integral method shows that solutions which are dominant-recessive in the sector obtained by the general theory are balanced in the complementary region, and solutions with balanced behavior in the sector derived previously are dominant in the remaining region. Thus, as for the known cases with coefficients linear in x (e.g., Wasow, Ann. of Math. vol. 52 (1950) pp. 350–361), the minimum sectors of validity predicted by the general theory define maximum regions. (Received July 6, 1954.)

711. A. J. Lohwater (p) and George Piranian: *Sets of radial discontinuity of bounded analytic functions.*

Let E be a set of points on the unit circle, of measure zero and type F_σ (or of type $G_{\delta\sigma}$ and subject to certain metric conditions which cannot be described here); then there exists a function $f(z)$, regular and bounded in $|z|<1$, such that $\lim_{r\to 1} f(re^{i\theta})$ exists if and only if the point $e^{i\theta}$ does not lie in E. If the set E above is closed, there exists a function $f(z)$, regular and bounded in $|z|<1$, and with the following properties: On each radius OP which terminates in E, $\lim\sup |f|=1$ and $\lim\inf |f|=0$; on every other radius of the unit disc, the radial limit of f exists and has modulus 1, except for a denumerable set of radii on which $f\to 0$. (Received July 19, 1954.)

712t. L. L. Philipson: *The asymptotic character of the solutions of a class of ordinary linear differential equations depending on a parameter.*

The asymptotic behavior for large values of a parameter λ of the solutions of $\sum_{k=0}^{n} a_k y^{(n-k)}+\lambda\sum_{k=0}^{m} (b_k+c_kx)y^{(m-k)}=0$ is studied in detail so as to illuminate the Stokes variations in the asymptotic representations about the transition point $-b_0/c_0$ in the complex x-plane. The a's, b's, c's are constants with $c_0\neq 0$; $n>m>0$. Employing the Laplace contour integral, it is found that if $p=n-m\geq 2$, there exist $p+1$ solutions each of which tends, as $\lambda\to\infty$, exponentially to infinity or zero when x is not on a certain ray from the transition point, but when $p=1$, at least in certain cases, an entire sector of width $\pi/2$ is excluded. In the interior of the excluded sector the solution tends to a finite function of x. There also exist in general solutions which converge, as $\lambda\to\infty$, to functions of x either in complete or partial sectors about the transition point. In the latter circumstance, the solution diverges exponentially in the interior of the remaining sector, of width $p\pi/(p+1)$. Expressions for the solutions' "mixed" behavior on the bounding rays of the various sectors are established also. It is proved that a fundamental set of solutions of the above types generally exists. (Received July 6, 1954.)

713. D. H. Potts: *Elementary integrals.*

It is shown that by accepting a restricted concept of elementary function one is able to simplify the treatment of problems on elementary integrals as presented in Ritt, *Integration in finite terms*. The restriction is to narrow the usual idea of "algebraic operation" to one involving only rational operations and the extraction of root. This enables the author to prove a form of the Liouville-Ostrowski theorem without recourse to techniques as advanced as those used in Ritt, e.g. complex variable theory. (Received July 16, 1954.)

714. O. W. Rechard: *Invariant measures for many-one transformations.*

Let X be an abstract space and m a finite measure defined on a completely additive class $\mathfrak{X}$ of measurable subsets of X. Let T be a transformation (not necessarily one-one) of X onto itself such that the complete inverse image of every measurable set is measurable and $m[T^{-1}(A)]=0$ if $m(A)=0$. Then the following theorem is proved: There exists a finite measure m^* defined on $\mathfrak{X}$, absolutely continuous with respect to m, and invariant with respect to T (i.e. $m^*[T^{-1}(A)]=m^*(A)$) if and only if the set functions $m_n(A)=m[T^{-n}(A)]$ are uniformly absolutely continuous with respect to m. The measure m^* can be represented as the indefinite integral of a summable function $m^*(A)=\int_A f dm$. If P is the set of points for which f is positive, then, to within sets of m-measure zero, $P\subset T^{-1}(P)\subset T^{-2}(P)\subset \cdots$ and $X=\bigcup_{n=0}^{\infty} T^{-n}(P)$. This theorem is, in a sense, an extension of a theorem on one-one transformations due to M. Cotlar and R. A. Ricabarra (*Sobre un Teorema de E. Hopf*, Revista Union Mat. Argentina vol. 14 (1949) pp. 49–63). However, the method of proof is entirely different from that used by the above authors. Invariant measures for special transformations of the unit interval are studied. (Received July 15, 1954.)

715. P. V. Reichelderfer: *On the behavior of essential continua in the product of two transformations.*

Given bounded domains D' and D'' in euclidean n-space let T' be a continuous transformation from D' into D'' and let T'' be a continuous transformation from D'' into a bounded portion of n-space. If C' is an essential maximal model continuum for T' and $T'C'$ is an essential maximal model continuum for T'' (see T. Rado and P. V. Reichelderfer, *On n-dimensional concepts of bounded variation, absolute continuity and generalized jacobian*, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) pp. 678–681) it is shown that C' is also an essential maximal model continuum for $T''T'$. Moreover, if C' is essentially isolated for T' and $T'C'$ is essentially isolated for T'', then C' is essentially isolated for $T''T'$ and the product law holds for their essential local indices. Conversely, if C' is an essentially isolated essential maximal model continuum for $T''T'$, then C' is an essentially isolated essential maximal model continuum for T' and $T'C'$ is an essentially isolated maximal model continuum for T''. (Received July 15, 1954.)

716t. Walter Rudin: *Images of radii under analytic mappings of the unit circle.*

Given a sequence $\{a_k\}$ of positive numbers such that $\sum a_k^2$ converges but $\sum a_k$ diverges, it is shown that there exists a sequence of positive integers n_k, tending to infinity sufficiently rapidly, such that the function $f(z) = \sum a_k z^{n_k}$ maps every radius of the unit circle into a non-rectifiable curve, although the power series converges at almost all points of the circumference, and although f is of class H_p for every $p < \infty$. Furthermore, f is the quotient of two bounded analytic functions g and h, and it is shown that there exists a complex number c such that the bounded function $g+ch$ maps almost all radii into non-rectifiable curves. The exceptional set of radii is also of the first category. (Received June 10, 1954.)

717*t*. Walter Rudin: *On mapping properties of Blaschke products.*

If f is analytic in the unit circle, let $V(f; \theta) = \int_0^1 |f'(re^{i\theta})| dr$; i.e. $V(f; \theta)$ is the length of the curve into which f maps the radius which terminates at the point $e^{i\theta}$. It is shown that there exists a Blaschke product B such that $V(B; \theta) = \infty$ for almost all θ. On the other hand, if the zeros z_n of $B(z)$ satisfy the condition $\sum (1-|z_n|)|\log (1-|z_n|)| < \infty$, instead of merely $\sum (1-|z_n|) < \infty$, then $\int_0^{2\pi} V(B; \theta) d\theta < \infty$, so that, in particular, almost all radii are mapped into rectifiable arcs. (Received June 14, 1954.)

718. Walter Rudin: *Radial cluster sets of analytic functions.*

Let E be a set of the first category on the circumference C of the open unit disc U; the linear measure of E may be 2π. Theorem I: For every complex-valued function F, continuous in U, there is a function f, analytic in U, such that $\lim_{r\to 1} \{f(rx) - F(rx)\} = 0$ for every $x \in E$ $(0 < r < 1)$. The proof is based on Mergelyan's theorem on approximation of continuous functions by polynomials. A consequence is Theorem II: Given any continuum K on the Riemann sphere, there exists a function f, analytic in U, whose radial cluster set at every point of E is precisely K. In particular, there exists (1) a harmonic function u in U which has finite radial limits at every point of E and whose conjugate v has no radial limit at any point of E, (2) an analytic function f in U such that $\lim_{r\to 1} |f(rx)| = 1$ for every $x \in E$, whereas $\lim_{r\to 1} f(rx)$ exists for no $x \in E$, (3) an analytic function f in U such that $\lim_{r\to 1} f(rx) = 0$ and $\lim_{r\to 1} \arg f(rx) = 0$ for every $x \in E$. (Received July 6, 1954.)

719. A. L. Shields: *Exceptional sets for uniform distribution.*

The following theorem is proven: Let $a_1 < a_2 < \cdots$ be positive integers such that $\inf (n/a_n) > 0$. Let E be the set of points $x \in [0, 1]$ such that the fractional parts $(a_n x)$ are not uniformly distributed. Then, Hausdorff dimension of E is $\leq 2(1-\alpha)$. This extends a theorem of I. I. Shapiro-Piatetsky, Mat. Sbornik vol. 30 (1952) pp. 669–676. (Received July 19, 1954.)

720*t*. I. M. Singer and John Wermer: *Derivations of commutative Banach algebras.*

A derivation on an algebra is a linear transformation D such that $D(ab) = D(a)b + aD(b)$ for a and b in the algebra. Kaplansky conjectured that the only derivation on a semisimple commutative Banach algebra was the 0 derivation. It is proved that if D is a continuous derivation on an arbitrary commutative Banach algebra, then the range of D is contained in the radical of the algebra. Thus the conjecture is true if D is continuous. If b is an element of any Banach algebra, bracket by b is a continuous derivation and hence the theorem cited above has as corollaries several facts about commuting brackets. (Received July 19, 1954.)

721. Vikramadity Singh and W. J. Thron (p): *A family of best twin convergence regions for continued fractions.*

Combining methods of the theory of linear fractional transformations and their iteration with an application of the Stieltjes-Vitali theorem it has been proved that the continued fraction $1+k(c_n^2/1)$ converges if for all $n \geq 1$, $|c_{2n-1}| \leq \rho$ and $|c_{2n} \pm i| \geq \rho$, where $0<\rho<1$. This result gives a set of best twin convergence regions and is an improvement of an earlier theorem by one of the authors (Duke Math. J. vol. 10 (1943) pp. 667–685). (Received July 6, 1954.)

722. K. T. Smith: *A generalization of an inequality of Hardy-Littlewood and applications.*

In this paper a generalization of a well-known inequality of Hardy and Littlewood is given. Consider a measure μ on a metric space B and make the hypotheses: (a) each closed sphere $S(x, r)$ is measurable and has finite measure; (b) there is a constant k such that for each x and r, $\mu[S(x, 4r)] \leq k\mu[S(x, r)]$; (c) if $\{S_n\}$ is a sequence of closed spheres such that $\mu(S_n) \to 0$, then the diameters $\delta(S_n) \to 0$; (d) if $\{S_n\}$ is a sequence of closed spheres such that $\delta(S_n) \to \infty$, then $\mu(S_n) \to \infty$. The inequality obtained asserts that if $p>1$, then $\int_B \bar{f}(x)^p d\mu \leq A_p \int_B |f(x)|^p d\mu$, when $\bar{f}(x)$ is defined as the supremum of the averages of $|f(x)|$ over the closed spheres centered at x. The applications given concern the case where the metric space B is a smooth surface (bounded curvature suffices) bounding a domain D in n-dimensional Euclidean space and μ is the area measure on B. The results are of the following type. Let $f(x)$ be the boundary function of a function $f(P)$ harmonic in D. For each x on B let $\hat{f}(x)$ be the upper bound of the values of $|f(P)|$ along the normal to B through x. Then for each $p>1$, there is a constant A_p such that $\int_B \hat{f}(x)^p d\mu \leq A_p \int_B |f(x)|^p d\mu$. (The case where D is the unit circle in the plane was treated by Hardy and Littlewood.) There are given also some applications to subharmonic functions. (Received July 19, 1954.)

723. Andrew Sobczyk: *Sections and projections of convex bodies.* Preliminary report.

If $\alpha_1, \cdots, \alpha_n$ are basis vectors for real linear space E_n, define n projections P_j by $P_j(x_1\alpha_1+ \cdots +x_j\alpha_j+ \cdots +x_n\alpha_n)=x_j\alpha_j$, $j=1, \cdots, n$. Denote by L_j the line consisting of the points $\{c\alpha_j\}$ for all real c. A convex region K in E_n is of *standard type* in case there exist a translate $C=K+w$, and a choice of basis $\{\alpha_j\}$, such that each projection P_jC coincides with the section $L_j \cap C$, $j=1, \cdots, n$. This paper partially answers, affirmatively, the following question. Is every convex region K in E_n of standard type? In other words, does every K have a set of n one-dimensional sections, such that K is intermediate between the convex hull and the Cartesian product of the sections? (Received July 15, 1954.)

724. E. W. Titt: *An alternative method of evaluating Hadamard's finite part.*

This paper presents an alternative to Hadamard's Taylor expansion method of evaluating the finite part of a divergent integral. In this paper integration by parts does the work which is accomplished by the Taylor expansion in Hadamard's method. On the other hand the integration by parts technique is more flexible since here neither factor need be expressed as a power function. The equivalence of the two methods is established for the case of single integrals. The new technique is illustrated with the solution of an ordinary equation with a nonintegrable forcing term. The interest in the new method arose out of the problem of reconciling Hadamard's integration formula in terms of finite parts with the type of integration formula to be found in McCulley and Titt (Journal of Rational Mechanics and Analysis vol. 2 (1953) pp. 443–484). (Received July 19, 1954.)

725. F. M. Wright: *On the backward extension of positive definite Hamburger moment sequences.*

Let $\{\mu_n\}$ $(n=0, 1, 2, \cdots)$ be a given positive definite Hamburger moment sequence. Using properties of the polynomials $X_p(z)$ and $Y_p(z)$ $(p=0, 1, 2, \cdots)$, which are the solutions of a certain second order recurrence system $L_p(\eta, z)=0$ $(p=0, 1, 2, \cdots)$, for certain initial conditions, the author, by using what is believed to be a relatively simple and direct method, obtains and enlarges upon the principal results of Hamburger (Math. Ann. vol. 82 (1921) pp. 168–189) relative to the problem of extending the given Hamburger moment sequence backward. A necessary and sufficient condition for a first backward extension is obtained which is first stated in terms of the polynomials $\{X_p(z)\}$ and $\{Y_p(z)\}$ $(p=0, 1, 2, \cdots)$, and then in terms of the given moments. The necessary and sufficient condition for an indeterminate Hamburger moment sequence in terms of the given moments due to Hamburger and reproved by M. Riesz (Arkiv för Mathematik, Astronomi och Fysik vol. 16 (1921) article no. 12) is also proved in this paper in a relatively simple manner. (Received July 9, 1954.)

726*t*. L. C. Young: *Analogues of an inequality due to Aronszajn and Choquet.* Preliminary report.

The area of a surface of finite topological type is not less than the product KLL' where K is an absolute constant and where L, L' are lengths of two curves on the surface. One curve can be taken as any 1-cycle of minimal length in its homology class, the other as a 1-cycle of minimal length not homologous to zero. There is a similar result in terms of homotopy. (Received July 19, 1954.)

727. L. C. Young: *A variational algorithm.*

Parametric surface integrands and curvilinear integrands are not restricted to be continuous. An integrand is exact if, on every closed surface, or on every closed curve, its integral vanishes. Two variational problems with the same boundary conditions are termed equivalent if the difference of their integrands is exact; curves and surfaces are understood to be generalized and the notion of boundary is that of Fleming and Young (Bull. Amer. Math. Soc. Abstract 58-6-615). With these conventions, it is shown that the principle of minimum in a variational problem with given boundary is tantamount to the existence of an equivalent problem with a non-negative integrand which vanishes on some admissible surface, or curve. (Received July 19, 1954.)

728*t*. L. C. Young: *On generalised surfaces of finite topological type.*

A Dirichlet surface of a given topological type is one for which the boundary curve describes a given finite set of simple closed curves C_t together with pairs of Fréchet equivalent arcs with the appropriate characteristic. A generalized surface of finite type is the limit (in the sense of generalized surfaces) of Dirichlet surfaces of bounded topological type and given C_t. It is shown that the track of a generalized surface of finite type is a Dirichlet surface of the same type and a general representation theorem is obtained which is almost completely analogous to the one for generalized curves. In the case in which the generalized surface is the solution of an acceptable variational problem its track is shown to be continuous. If the problem has a convex integrand, its solution coincides with this track. A corollary of these results is the extension to surfaces of given topological type of a recent existence theorem due to Sigalov, Danskin and Cesari. (Received July 19, 1954.)

APPLIED MATHEMATICS

729. W. L. Bade and Herbert Jehle (p): *Specific interactions between identical macromolecules.*

A fundamental biophysical problem, originally stated by the geneticist H. J. Muller, is to explain why nucleic acids and proteins "specifically" attract like molecules, discriminating for an exactly identical partner. The molecules in question present a periodic (presumably helical) structure of great rigidity. They are composed of a backbone of repeating units with a side group attached to each unit. These macromolecules should be represented by an orchestra of electric dipole oscillators vibrating because of the temperature motion. The authors are investigating those vibrations to find out which types of quantum mechanical models of pairs of macromolecules exhibit a specific Van der Waals attraction. Such attraction has been conjectured on the basis of arguments in terms of Bohr's correspondence principle (Proc. Nat. Acad. Sci. U. S. A. vol. 36 (1950) p. 238). Specificity of attraction is analyzed by a study of various "detunings" of one of the macromolecules (slight changes of the identity) of the pair. That analysis was greatly advanced by a complete investigation of a model in which each molecule is represented by only two oscillators (W. L. Bade). It has substantiated the implications of the earlier correspondence arguments for this 2+2 oscillator model. The analysis of an $N+N$ oscillator model of the macromolecule pair (J. M. Yos) is preceded by a group theoretical analysis of vibrations of identical macromolecule pairs and of their detunings. (Research supported by the Research Corporation and National Science Foundation.) (Received July 19, 1954.)

730. R. G. Blake: *A problem in laminated orthotropic material under plane strain.*

Prager and Synge [Quarterly of Applied Mathematics vol. 5 (1947) pp. 241–269] have developed a method of approximation in elasticity based on the concept of function space. This method is used to obtain a first approximation to a problem of plane strain in a rectangle consisting of three layers of orthotropic material cemented together with the two outer layers similar. The line of separation of the layers is taken as $y=\pm c$ and the boundaries of the rectangle as $x=\pm wc$, $y=\pm tc$. The boundary conditions to be satisfied are: at $y=tc$, $\tau_{yy}=Kx^{a}$, $\tau_{xy}=0$; at $y=-tc$, $\tau_{yy}=\tau_{xy}=0$; at $x=\pm wc$, $\int_{-tc}^{tc}\tau_{xy}dy=\mp K(wc)^{a+1}/(a+1)$, $\int_{-tc}^{tc}\tau_{xx}dy=\int_{-tc}^{tc}\tau_{xx}ydy=0$. (Received July 19, 1954.)

731. R. V. Churchill: *A generalized Fourier integral formula.*

The solution of certain types of boundary value problems in partial differential equations, including problems of some importance in applied mathematics, depends on a representation of a function $F(x)$ on the semi-infinite interval $x>0$ in terms of the characteristic functions $y(x, \lambda)$ of the problem $y''+\lambda^2y=0$, $my''(0, \lambda)-hy'(0, \lambda)+ky(0, \lambda)=0$, $y(x, \lambda)$ bounded for $x\geqq 0$. Here m, h, and k are nonnegative constants. Since $y''(0, \lambda)=-\lambda^2y(0, \lambda)$ the boundary condition on y involves the parameter λ

explicitly if $m \neq 0$. The characteristic functions are $y(x, \lambda) = \cos [\lambda x - \alpha(\lambda)]$, where $\tan \alpha(\lambda) = (h - m\lambda^2)/(h\lambda)$, $-\pi/2 \leqq \alpha(\lambda) \leqq \pi/2$ and λ is real and nonnegative. When $F(x)$ satisfies conditions under which it is represented by its Fourier sine and cosine integral formulas, the generalization $\pi F(x) = 2\int_0^\infty y(x, \lambda)d\lambda \int_0^\infty F(\mu)y(\mu, \lambda)d\mu$ of those formulas is established for $x > 0$ with the aid of the Riemann-Lebesgue theorem. This formula includes as a special case ($m = 0$) the generalization arrived at formally by W. Karush [Journal of Applied Physics vol. 23 (1952) pp. 492–494] and by the author [*Modern operational mathematics*, 1944, p. 210]. (Received July 19, 1954.)

732. Jim Douglas, Jr.: *On the integration of $U_{xx} + U_{yy} = U_t$ by implicit methods.*

Three implicit difference equation analogues of the heat flow equation in two space variables are studied. Under proper hypotheses the solution of each is shown to converge to the solution $u(x, y, t)$ of the differential equation for any ratio of Δt to $(\Delta x)^2 = (\Delta y)^2$ as $\Delta t \to 0$ with the error in each case being $O(\Delta t)$. Let $f_{i,j,n} = f(i\Delta x, j\Delta y, n\Delta t)$ and $\Delta_x^2 f_{i,j,n} = (f_{i+1,j,n} - 2f_{i,j,n} + f_{i-1,j,n})/(\Delta x)^2$. Then, let $W_{i,j,0} = U_{i,j,0}$ and $W_{i,j,n} = U_{i,j,n}$ on the boundary of the unit square for $n > 0$. The three difference equations are (I) $\Delta_x^2 W_{i,j,n+1} + \Delta_y^2 W_{i,j,n+1} = (W_{i,j,n+1} - W_{i,j,n})/\Delta t$; (II) $\Delta_x^2 W_{i,j,n+1} + \Delta_y^2 W_{i,j,n+1} + \Delta_x^2 W_{i,j,n} + \Delta_y^2 W_{i,j,n} = 2(W_{i,j,n+1} - W_{i,j,n})/\Delta t$; and (III) $\Delta_x^2 W_{i,j,2n+1} + \Delta_y^2 W_{i,j,2n} = (W_{i,j,2n+1} - W_{i,j,2n})/\Delta t$; $\Delta_x^2 W_{i,j,2n+1} + \Delta_y^2 W_{i,j,2n+2} = (W_{i,j,2n+2} - W_{i,j,2n+1})/\Delta t$. Equation (III), while appearing the most formidable, is in fact much the easiest to evaluate as the linear equation can be solved by well known methods in $9(\Delta x)^{-2}$ calculations per time step. Practical aspects of the methods and applications to Laplace's equation are given in abstract by Peaceman and Rachford (Abstract 735). (Received July 19, 1954.)

733. T. C. Doyle: *Invariantive analytic representation of a finite deformation of a continuous medium.*

The material (undeformed) points are referred alternatively to (a) an absolute (fixed) orthonormal triad, (b) a moving orthonormal material triad, (c) a moving curvilinear material triad, and similarly for the deformed (spatial) points. Any vector may be given by specifying its components in any one of these five reference triads and transformations between these sets of components are accomplished by given matrices of direction cosines and displacement gradients. There exist semi-intrinsic parameterizations (convected coordinates) in which the curvilinear strain tensor components coincide with the metric tensor components in the semi-intrinsic base. (Received June 24, 1954.)

734t. L. E. Payne: *Inequalities for certain eigenvalues of a membrane.*

Let D be a closed domain with boundary C in the xy plane and let $u(x, y)$ satisfy the differential equation $\Delta u + ku = 0$ in D (where k is a positive constant) and either (1) $u = 0$ on C or (2) $\partial u/\partial n = 0$ on C. Denote the eigenvalues in case (1) by $\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \cdots$, and in case (2) by $0 = \mu_1 \leqq \mu_2 \leqq \mu_3 \leqq \cdots$. It has been shown by Szegö (Journal of Rational Mechanics and Analysis vol. 3 (1954) p. 354) that $\mu_2 \leqq p\lambda_1$ where p is the constant giving equality if D is a circle. In this article for a convex domain D the following inequalities are obtained: (3) $\mu_2 < \lambda_1 - 2/(\rho h)_{\max}$ and (4) $\mu_3 < \lambda_1 - (2^{1/2} - 1)/(\rho h)_{\max} \leqq \lambda_1$, where ρ is the radius of curvature of C, and the value of h at any point P on C is the distance from an arbitrary origin inside D to the line tangent to C through P. This research was supported by the Office of Scientific Research, U.S.A.F. (Received June 9, 1954.)

735*t*. D. W. Peaceman and H. H. Rachford, Jr.: *On the numerical integration of two-dimensional parabolic and elliptic partial differential equations.*

I. Stepwise solution of $U_{xx}+U_{yy}=U_t$ through explicit difference equations requires an excessive number of steps owing to stability limitations. Implicit formulations similar to those used in the one-dimensional analogue have been considered. Approximation of the x- and y-derivatives in terms of the unknown values at the end of the step yields a stable solution for all Δt but requires at each step solution of $N^2=(\Delta x)^{-2}$ simultaneous equations, best solved by iterative methods which converge more slowly as Δt increases. A non-iterative implict scheme, stable for all Δt, is achieved by a two-step formulation wherein the x- or y-derivative is alternately expressed in terms of the unknown values. At each step N sets of N simultaneous equations are solved directly, requiring $9N^2$ arithmetic operations. Proof of convergence is given in a paper by Douglas (Abstract 732). II. This alternating direction implicit formulation may be used to solve the elliptic equation $U_{xx}+U_{yy}=0$ by iteration. A trial solution is carried through a succession of "time steps" wherein Δt becomes an iteration parameter. A set of Δt's which optimizes convergence is readily computed. The work to reduce the error by 10^{-3} for large N is $34N^2 \log N$ operations. This is compared with $15N^3$ for optimum extrapolated Liebmann iteration procedures. (Received July 19, 1954.)

736. M. H. Protter: *On some problems in transonic flow.*

Existence and uniqueness theorems are established for certain boundary value problems for the equation $K(y)u_{xx}+u_{yy}=0$ where $K(y)$ is monotone increasing, $K(0)=0$. These problems occur in the two-dimensional transonic flow of an ideal gas between two plane converging walls. The uniqueness result is more general than previous ones obtained in that it contains no restrictions on the domain in the case where $K(y)$ is the particular function occurring in the gas dynamical case. The existence follows from a combination of the uniqueness and a previous result of the author. (Received July 8, 1954.)

737*t*. Domina E. Spencer: *The ordinary differential equations of mathematical physics.*

An important step in the solution of the partial differential equations of field theory is the classification and solution of the ordinary differential equations obtained by separation of variables. A convenient method of classification of these ordinary differential equations stems from the Fuchsian theory, as has been demonstrated by Klein and Bôcher. The present paper extends previous investigations by considering equations obtained by separation of the Laplace and Helmholtz equations in the eleven coordinate systems of Eisenhart and the eleven symmetric cyclidal systems of Bôcher. These twenty-two coordinate systems comprise all the separable systems that are likely to be used in practice. Despite the large number of separable coordinate systems, each of which has three separation equations, there are only twelve distinct differential equations, and four of these are solved by elementary functions. The differential equations are classified with respect to their singularities, and their solutions are indicated. (Received July 19, 1954.)

738. H. F. Weinberger: *A Rayleigh-Ritz procedure for unbounded perturbations.*

Let A be a positive symmetric differential operator with discrete spectrum in an incomplete Hilbert space (functions satisfying differentiability and boundary conditions). Let its eigenvalues μ_n and eigenfunctions v_n be known. Let C be another differential operator which satisfies, for all unit vectors w, $(Cw, Cw)^{1/2} \leqq \phi((Aw, w))$ where $\phi(x)$ is a function satisfying (1) $0 < \phi'(x) < 1$; (2) $\limsup_{x \to \infty} \phi(x)/x \leqq 0.39$; and (3) the quantity $[\phi\phi'/(1-\phi')]' < 0$ for sufficiently small x and changes sign at most once. Then one can calculate the maximum error made in approximating any particular eigenvalue of the perturbed operator $A+C$ with the same boundary conditions by means of the Rayleigh-Ritz upper bound obtained by using the vectors $(v_1, v_2, \cdots, v_N)$. This error bound is given in the terms of the known quantities μ_n, and can be made arbitrarily small by choosing N sufficiently large. The conditions (1), (2), and (3) are satisfied for $\phi = cx^\alpha$ with $\alpha < 1$, so that if A is an elliptic differential operator, C may be any operator of lower order. By a slight modification the method can be applied even when the eigenvalues of $A+C$ are to be found with respect to a new norm which is bounded above and below with respect to the original norm. Application is made to the natural frequencies of a vibrating clamped plate. This research was supported by the United States Air Force through the Office of Scientific Research. (Received July 19, 1954.)

739. J. M. Yos: *Group theoretical analysis of vibrations of macromolecule pairs.*

In the investigation described by H. Jehle and W. L. Bade it is necessary to expand a symmetric function of the characteristic roots of the potential energy matrix of the molecule pair in terms of small "detunings" of the original undetuned pair. This expansion is facilitated by considering the symmetry group of the undetuned system. The transformations of this group produce a representation of the group in the space of detunings, and if the detuning parameters are chosen to reduce this representation the expansion takes on its simplest form. The argument shows that certain coefficients in the expansion vanish identically, and that others are necessarily equal. (Research supported by the National Science Foundation.) (Received July 19, 1954.)

GEOMETRY

740*t*. M. W. Al-Dhahir: *Configurational characterisations of commutativity.* I.

Let S_n, with n odd, be a projective n-space defined over a division ring $\mathcal{F}$. The fundamental theorem on plane complete quadrangles (Veblen and Young, *Projective geometry*, p. 101) is generalized to n dimensions in the following form: Let $012 \cdots n$ and $0'1'2' \cdots n'$ be two complete hyper-plane $(n+1)$-points. If all, but one, of the $n(n+1)/2$ lines of one meet the correspondingly "opposite faces" of the other, then the remaining line of the first meets its corresponding face of the second if, and only if, $\mathcal{F}$ is a field. Through an n-dimensional projection, a "similar" result is obtained in even spaces. Thus, a configurational criteria for the commutativity of the space is obtained. This generalization involves the Möbius proposition for $n=3$, and specializes to the Pappus theorem for $n=2$, and, as such, it may be considered as an extension of both of these propositions also. (Received June 21, 1954.)

741*t*. Rafael Artzy: *Self-dual configurations and their Levi graphs.*

In the Levi graph (Coxeter, Bull. Amer. Math. Soc. vol. 56 (1950) pp. 413–455) of a self-dual configuration there is a one-to-one correspondence between all point-nodes and all line-nodes. Each two corresponding nodes are abstractly identified without altering the joining branches. The *reduced Levi graph* (RLG) thus generated has only half the number of nodes, but it may have loops representing points incident with dually corresponding lines in the configuration. A configuration may have several nonisomorphic RLG's, but no RLG represents more than one configuration. Self-dual specializations of a configuration imply in the RLG certain modifications which are studied. The RLG's of special configurations are dealt with. Thus the RLG of Desargues' configuration is Petersen's graph; several self-dual specializations yield Petersen's graph with loops added. The method is useful for the study of finite geometries: the RLG's of PG(2, 3) and PG(3, 2) contain the RLG's of several well known configurations as subgraphs, thus implying the possibility of constructing these configurations in the respective geometry. (Received July 12, 1954.)

742. P. O. Bell: *On projective tensors of a surface S with respect to a surface S'.*

Let $g_{\alpha\beta}$ and $d_{\alpha\beta}$ denote the first and second fundamental projective tensors of a surface $S(u^1, u^2)$ with respect to a surface $S'(u^1, u^2)$ and let $R_{\alpha\beta}$ and K denote the projective curvature tensor and projective total curvature of S relative to S', respectively. ($g_{\alpha\beta}$ and K were introduced in Trans. Amer. Math. Soc. vol. 60 (1946) pp. 34–39. $R_{\alpha\beta}$ and $d_{\alpha\beta}$ were introduced in Duke Math. J. vol. 18 (1951) pp. 697–705.) Infinitely many pairs of surfaces (S', S'') exist such that the tensor $R_{\alpha\beta}$ of S relative to S' and the tensor $Kg_{\alpha\beta}$ of S relative to S'' are related by the equation $R_{\alpha\beta}+Kg_{\alpha\beta}=0$. Corresponding points x, x', x'' of such surfaces S, S', S'' are collinear. The harmonic conjugate of x with respect to x' and x'' is a point $\sigma(u^1, u^2)$ which generates the *R-associate surface* Σ of S. (See the first paper cited above, p. 38.) A null line of the first fundamental form of S corresponds to a locus of σ whose tangent at σ intersects the tangent to the null line. A null line Γ_1 (Γ_2) of the second fundamental form of S (i.e., an asymptotic curve of S) corresponds to a locus of σ whose tangent at σ intersects the tangent to Γ_2 (Γ_1) at x. This last property completely characterizes Σ. (Received July 6, 1954.)

743*t*. John DeCicco: *Certain generalizations of central and parallel fields of force in Euclidean space E_n.*

An $(m-1)$-dimensional central field of force with an $(m-1)$-dimensional flat L^0_{m-1} as center, in a certain region of E_n, is one such that the lines of the force vectors at all points of the region intersect L^0_{m-1}, and there exist m linearly independent force vectors. The lines of force are all contained in m-dimensional flats L_m passing through L^0_{m-1}, and the dynamical trajectories are in $(m+1)$-dimensional flats L_{m+1}, all of which contain L^0_{m-1}. Similarly in an m-dimensional parallel field of force with the m-dimensional direction μ_m, the lines of force lie in m-dimensional flats L_m and the dynamical trajectories are all contained in $(m+1)$-dimensional flats L_{m+1}, all of which are parallel to the direction μ_m. For the $(m-1)$-dimensional central fields of force, a law which reduces to that of Kepler's second law of planetary motion when $m=1$ is developed. These are the only positional fields of force for which every dynamical trajectory is contained in some $(m+1)$-dimensional flat L_{m+1}, and m is the smallest such integer. (Received July 12, 1954.)

744. J. W. Gaddum: *Distance and angle sums on a sphere.*

The angle between two planes in E_n can be expressed as a distance between two points in $S_{n-1,1}$ and the angle at a point formed by n hyperplanes can be treated as the "area" of a simplex in $S_{n-1,1}$. With the motivation of seeking bounds on the angle sums in a simplex in E_n, bounds are found for the sum of the distances between points of a set in $S_{n-1,1}$, and for the content of a spherical simplex in terms of the sum of the angles between its faces. If A is the content of a simplex in $S_{n-1,1}$ and $\sum a_{ij}$ the sum of the angles between its faces, it is proved: for n odd $\sum a_{ij} \geqq ((n-1)^2/4)\pi + ((n^2-1)/2e_n)A$, for n even $\sum a_{ij} \geqq (n(n-2)/4)\pi + (n^2/2e_n)A$, and $\sum a_{ij} \leqq ((n-1)(n-2)/2)\pi + (2(n-1)\pi/e)A$ for all n, where $e_n = 2\pi^{n/2}/\Gamma(n/2)$. If $p_1, \cdots, p_k$ is a global set in $S_{n-1,1}$, $\sum p_ip_j \geqq (K-1)\pi$, and for any set $p_1, \cdots, p_k$ in $S_{n-1,1}$, $\sum p_ip_j \leqq [k^2/4]\pi$. All bounds stated are best possible. The methods are elementary. (Received July 19, 1954.)

745. Michael Goldberg: *Basic rotors in spherical polygons.*

In an earlier paper by the author [Journal of Mathematics and Physics vol. 30 (1952) pp. 235–244], a kinematic method was given for generating a special class of spherical rotors which are convex spherical curves rotatable through all orientations while keeping contact with all the sides of a given regular spherical polygon. This class corresponds to the circular-arc rotors in plane polygons. The present paper exhibits another class of spherical rotors generated by kinematic methods. These correspond to the basic rotors in plane polygons, namely, the convex involutes of the hypocycloids. Their analytic expressions are derived. (Received June 28, 1954.)

746. P. C. Hammer: *Convex bodies in linear spaces.*

Let C be a convex set in a linear space W. Let A_s be the minimal set covered by the union of translations of all line segments in C with translated centers on the origin θ. Let C_s be the set obtained by closing A_s on every line through θ. Then C_s is the *symmetroid* of C. If $x \in C$, then $2C_s + x \supset C$ and C_s is the smallest relatively closed convex set centered on θ with this property. If the intersection of C_s with each line through θ is a proper finite line segment and if C is closed and contains interior points in the topology with C_s as a unit sphere, C is a *convex body*. From this intrinsic definition of a convex body generalizations of results of Klee, Sobczyk, and the author for finite-dimensional spaces are obtained. Two convex sets with the same symmetroid have equivalent breadths in a generalized sense. (Received July 20, 1954.)

747. W. R. Hutcherson (p) and N. A. Childress: *Surfaces obtained from involutions generated by homographies of periods three, five, and thirteen.*

The image of a planar cyclic involution of period three is represented as a surface [Lucien Godeaux, *Étude élémentaire sur l'homographie plane de période trois et sur une surface cubique*, Nouvelles Annales de Mathématiques vol. 16 (1916) pp. 49–61] of order three in S_3. Also, the image of a planar cyclic involution of period five is represented as a surface [W. R. Hutcherson and N. A. Childress, *Étude d'une involution cyclique de période cinq*, Bulletin de l'Académie royale de Belgique (Classe des Sciences) (6) vol. 40 (1954) pp. 103–108] of order five in S_4. The image of a planar cyclic involution of period thirteen is represented as a surface of order thirteen in S_8. The plane is in (13, 1) correspondence with this surface. A series of five successive projections is performed on this surface, with the point of projection being on the surface which is being projected each time. A rational surface of order one hundred and sixty nine in S_9 is shown. This surface is in (13, 1) correspondence with the image surface. (Received June 3, 1954.)

748t. W. R. Hutcherson and J. C. Morelock: *Concerning a pattern for perfect points.*

Using a certain surface [W. R. Hutcherson, *Voisage du cinquieme ordre d'une involution de periode* 13, Bulletin Société Royale des Sciences de Liège No. 11 (1952) pp. 483–487; Bull. Amer. Math. Soc. Abstract 59-6-556] invariant under an involution of order p (prime number), it is shown that a simple point P_1 has perfect points within certain order neighborhoods, depending upon p. A table, exhibited for the first six neighborhoods when $p \leqq 47$, is followed by several theorems. The largest prime number found in the cataloging of the first twenty neighborhoods was 14251. (Received June 1, 1954.)

749t. Herbert Knothe: *On polyhedrons in multi-dimensional curved spaces.*

The well known theorem that the area of a triangle bounded by great circles on the unit sphere is equal to the sum of the angles minus π can be generalized in spaces of constant curvature of an even number of dimensions. By a simple method of combination a formula is derived expressing the volume of an $(n+1)$-hedron in the n-dimensional space by means of the solid angles at the vertices and edges. In the same manner the volume of a polyhedron can be expressed by angles. It is supposed that the volume of a regular curved convex body can be expressed by the $g_{\lambda\mu}$ of the surface. (Received August 30, 1954.)

750. T. G. Ostrom: *Conics in finite projective planes.*

Baer (Bull. Amer. Math. Soc. vol. 52 (1946)) has shown that a polarity in a finite projective plane with $n+1$ points on a line has $n+1$ absolute points unless n is a square and that, if n is odd, no three absolute points lie on the same line. Assuming n odd and not a square, the author defines a conic as the set of absolute points of a polarity. The author investigates the conditions under which such a conic acts, in certain respects, like the classical conics in a Desarguesian plane. These conditions turn out to be Desargues configurations for pairs of triangles appropriately located with respect to the conic. (Received July 19, 1954.)

751t. Valdemars Punga: *Note on covariant differentiation by means of measuring vectors.*

Professor Schouten in his Ricci-Kalkül derives the formulas for the covariant differential of co- and contravariant vectors using measuring vectors $e^{\kappa}_{(\lambda)}$ and $e^{(\kappa)}_{\lambda}$, but in deriving the formula for the covariant differential of affinors of valence higher than 1 he uses ideal vectors. In this paper it is shown how the process of covariant differentiation of affinors of any valence can be developed by means of the measuring vectors. Similarly to $v^{\kappa}=v^{(\lambda)}e^{\kappa}_{(\lambda)}$, $w_{\lambda}=w_{(\kappa)}e^{(\kappa)}_{\lambda}$, $v^{(\kappa)}=v^{\lambda}e^{(\kappa)}_{\lambda}$, $w_{(\lambda)}=w_{\kappa}e^{\kappa}_{(\lambda)}$, the author writes for affinors of valence higher than 1 (taking as example affinor $v^{\kappa\lambda}_{\nu}$): $v^{\kappa\lambda}_{\nu}=v^{(\rho)(\sigma)}{}_{(\tau)}e^{\kappa}_{(\rho)}e^{\lambda}_{(\sigma)}e^{(\tau)}_{\nu}$, $v^{(\kappa)(\lambda)}_{(\nu)}=v^{\rho\sigma}{}_{\tau}e^{(\kappa)}_{\rho}e^{(\lambda)}_{\sigma}e^{\tau}_{(\nu)}$ and for example $v^{(\kappa)\lambda}_{(\nu)}=v^{\rho(\sigma)}_{\tau}e^{(\kappa)}_{\rho}e^{\lambda}_{(\sigma)}e_{(\nu)}$. Assume that $\nabla_{\mu}e^{\kappa}_{(\lambda)}=\Gamma^{\kappa}_{\mu(\lambda)}$, $\delta e^{\kappa}_{(\lambda)}=\Gamma^{\kappa}_{\mu(\lambda)}d\xi^{\mu}$ and $\nabla_{\mu}e^{(\kappa)}_{\lambda}=-\Gamma^{(\kappa)}_{\mu\lambda}$, $\delta e^{(\kappa)}_{\lambda}=-\Gamma^{(\kappa)}_{\mu\lambda}d\xi^{\mu}$. Since $e^{\kappa}_{(\sigma)}$ and $e^{(\kappa)}_{\sigma}$ act as index-substitution operators, define $\Gamma^{\kappa}_{\mu\lambda}=\Gamma^{\kappa}_{\mu(\sigma)}e^{(\sigma)}_{\lambda}$, $\Gamma^{\kappa}_{\mu\lambda}=\Gamma^{(\sigma)}_{\mu\lambda}e^{\kappa}_{(\sigma)}$. Using the introduced formalism, the author derives the formulas for the covariant differentiation for affinors of any valence, affinor-densities, etc. (Received July 12, 1954.)

752*t*. Andrew Sobczyk: *Families of planes.*

The family of all lines in n-dimensional vector space E_n is equivalent to the fiber bundle of the tangent hyperplanes to the sphere S_{n-1}. In E_4, there exist three types of covering families of planes, U, V, W. If P_1, P_2 are any two planes of a family U, then either P_1 and P_2 intersect in a line, or there is a third plane Q of U which intersects both P_1 and P_2, in lines parallel to a fixed direction. The fibering of the sphere S_3 with S_2 as base determines a family V of planes, all passing through the origin and having no other common points. If P_1, P_2 are any two planes of a family W, then P_1 and P_2 have no common point (they lie in parallel hyperplanes). By means of considerations involving fiber bundles, projective duality, hyperplanes at infinity, and results of the author on families of lines, these and numerous further results are established concerning covering, intersectional, and directional properties of families of planes, especially in E_3 and in E_4. (Received July 15, 1954.)

753*t*. Andrew Sobczyk: *Simple families of lines.*

Let "non-horizontal" lines in $(n+1)$-dimensional real vector space E_{n+1} be coordinatized by their intersections x, y with parallel hyperplanes H, K. Then the equation for any pencil of lines with center $(w; w_{n+1})$ is $(y-w)=cI(x-w)$, where I is the identity, and $c=(b-w_{n+1})/(a-w_{n+1})$, $z_{n+1}=a$, $z_{n+1}=b$ being respectively the equations of H, K. In case $(n+1)$ is odd, the family $(y-w)=T(x-w)$, where T is any linear transformation having only complex eigenvalues, has the property that it contains exactly one line in every "non-horizontal" direction, and that each point $(z; z_{n+1})$ of E_{n+1} is covered by exactly one line of the family. In general if T has real eigenvalues, if any, all different from 1, then any family $y=Tx+u$ is included under the previous form, and the family simply covers all of E_{n+1} except for one hyperplane corresponding to each distinct real eigenvalue. If one or several eigenvalues are equal to 1, the family is a "stack" of lower dimensional families in a sheaf of parallel flats, and therefore does not contain a line in every "non-horizontal" direction. Generalization is made to families of lines in Banach spaces. (Received July 15, 1954.)

LOGIC AND FOUNDATIONS

754*t*. M. W. Al-Dhahir: *The Möbius class of configurations.*

A configuration $\mathfrak{A}(a_1, a_2, a_3)$ is a set of subsets a_1, a_2, a_3 in which a relation R is defined. A configuration $\mathfrak{A}^*(a_1, a_2, a_3; x)$ is a "natural extension" of $\mathfrak{A}$, $\mathfrak{A}\leqq\mathfrak{A}^*$, if each element of x is determined by R among some elements of $\mathfrak{A}$. Two configurations $\mathfrak{A}$ and $\mathfrak{B}$ are "geometrically equivalent," $\mathfrak{A}\sim\mathfrak{B}$, if $\exists$ a configuration $\mathfrak{C}\ni\mathfrak{A}\leqq\mathfrak{C}\&\mathfrak{B}\leqq\mathfrak{C}$. Some theorems are proved; in particular $\sim$ is an equivalence relation. Then these notions are applied to configurations corresponding to some configurational propositions equivalent to Möbius theorem. It is shown that these configurations are $\sim$ to the Möbius configuration; and hence they are all "unified" into one class—the Möbius class—of configurations "leveling" with the Möbius class of propositions. Commutativity of multiplication may be characterized by any member of the class. (Received June 3, 1954.)

755*t*. A. R. Schweitzer: *Outline of a form of general analysis.* Preliminary report.

The content of E. H. Moore's form of general analysis (New Haven, 1910) is classified first with regard to concepts relevant to Peano-Russell logic and Boole-

Grassman-Schroeder logic such as property, relation, class, operation, relation between classes, composition; and second with regard to mathematical subjects such as set, semi-group, number system, function, dominance of function, convergence, continuity. Moore's analysis is interpreted as the theory of systems of classes of functions based on some or all of the categories symbolized by $ALCDK_1K_2\Delta$. The first four categories are associated with convergence; the last six, with continuity. The author's outline proceeds as follows. I. Concerning classes (loc. cit., pp. 55–76). II. Concerning convergence (pp. 13–55; 76–88). III. Concerning continuity (pp. 89–149). IV. Moore's "relative uniformity" of convergence is held to be a key to the unification of certain theories (pp. 13–14) expressed by a formal principle (p. 1). In connection with this principle, following Moore the author traces the genesis of Moore's memoir and its relation to integral equation theory (p. 2, note). Reference is made to the author's article in Rev. de Mét. et de Mor., Paris, 1914. (Received June 22, 1954.)

756*t*. A. C. Sugar: *Axiomatic foundation of thermodynamics*. Preliminary report.

As a first task the author considers a set of axioms for the thermodynamics of the reversible phenomena of the so-called perfect, ideal or polytropic gases. By a differentiable reversible polytropic intensive system of thermodynamics, briefly called a hyperclassical system of thermodynamics, is meant an ordered sextuple, $\Sigma=\langle S, T, q, t_1, c_v, R\rangle$, which satisfies axioms A1–A5. Axiom A1. T is a closed interval of real numbers. A2. t_1 is in T. A3. R and c_v are positive real numbers. A4. S is the set of ordered pairs, $(\langle p, v\rangle)$, of all real differentiable functions (whose domains are T and whose values are positive real numbers) such that for each $\langle p, v\rangle$ and every t in T, the Riemann integral $\int_{t_1}^{t} pDv$ exists. A5. q is a differentiable function, whose domain of definition is $S\times T$ (cartesian product) and whose value is a real number, such that for every $\langle p, v\rangle$ in S and every t in T, $q(\langle p, v\rangle, t)-\int_{t_1}^{t} pDv=(c_v/R)[p(t)v(t)-p(t_1)v(t_1)]$. The second law is a consequence of axioms A1–A5. The intriguing aspect of this axiomatization is that these axioms are really all definitions. Consequently one is confronted with the surprising result that the operation of clarification has established that this portion of thermodynamics is a science of no assumptions only definitions. Temperature is defined using the equation of state $\theta=pv/R$. Reversibility is taken as a continuity condition on the set of admissible state paths, S. T is a set of real numbers measuring elapsed periods of time and $q(t)$ the amount of heat supplied to the "perfect gas" in t units of time. It is planned to include isolated and more complicated systems, to axiomatize statistical thermodynamics and to reduce, in a strictly logical fashion, classical to statistical thermodynamics. (Received June 28, 1954.)

STATISTICS AND PROBABILITY

757*t*. A. G. Anderson: *The prediction of quantitative characteristics in polygenic systems*. II.

A method devised by the author (Bull. Amer. Math. Soc. Abstract 59-3-315) is employed in a specific example for the purpose of predicting yield in hybrid corn crosses. The resulting analysis shows that the improvement over present methods of prediction, while not statistically significant, is comparable to that made by present methods over chance. (Received July 19, 1954.)

758*t*. G. E. Baxter: *An analogue of the law of the iterated logarithm*.

Let $\{X_{nk}\}$ $(k=1, 2, \cdots, n; n=1, 2, \cdots)$ be a sequence of sequences of Bernoulli variables, mutually independent within each row, with $P\{X_{nk}=1\}=\lambda/n$ and $P\{X_{nk}=0\}=1-\lambda/n$ $(k=1, 2, \cdots, n)$. The classical Poisson theorem states that the distributions of the row sums $S_n=X_{n1}+X_{n2}+\cdots+X_{nn}$ converge to the Poisson distribution with parameter λ. It is now shown that $P\{\limsup_{n\to\infty}(\log\log n/\log n)S_n\leqq 1\}=1$. Under the additional hypothesis that the family $\{X_{nk}\}$ consists entirely of mutually independent Bernoulli variables it is shown that $P\{\limsup_{n\to\infty}(\log\log n/\log n)S_n=1\}=1$ and that for any positive normalizing factor $f(n)$, $P\{\liminf_{n\to\infty}S_n/f(n)=0\}=1$. (Received May 24, 1954.)

759*t*. R. K. Getoor: *On stochastic processes*. I.

Let $X(t)$, $0\leqq t<\infty$, be a stochastic process with finite variance and let $L(X)$ be the smallest closed linear manifold of random variables containing all $X(t)$. It is well known that $L(X)$ is a Hilbert space. Necessary and sufficient conditions in terms of the covariance function $R(t, s)$ of $X(t)$ are given that the semi-group of translation operators V_t: $X(s)\to X(t+s)$ be well defined on a dense subset of $L(X)$. These conditions are then strengthened to insure that each V_t is bounded. If $R(t, s)$ admits a representation in the form $R(t, s)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f(t;\lambda_1, \cdots, \lambda_n)f(s; \lambda_1, \cdots, \lambda_n)\,d\alpha(\lambda_1, \cdots, \lambda_n)$, where α is a bounded measure on E^n, it is shown that there exist n self-adjoint operators $A_1, \cdots, A_n$ densely defined, but not necessarily bounded, on an appropriate extension of $L(X)$ such that $X(t)=f(t; A_1, \cdots, A_n)X(0)$. This representation is closely related to a result of Karhunen. (Received July 6, 1954.)

760*t*. R. K. Getoor: *On stochastic processes*. II.

Using the notation of the preceding abstract, stochastic processes whose translation semi-group $\{V_t\}$ consists of either symmetric or normal operators are investigated. A necessary and sufficient condition that $\{V_t\}$ consist of symmetric operators is that $R(t, s)=r(t+s)$ and thus are the processes of "convex exponential character" of Loève (*Sur les fonctions aléatoires du seconde ordre*, Note in P. Lévy, *Processus stochastiques et mouvement Brownien*, Paris, 1948). For both symmetric and normal operators first the unbounded and then the bounded case is investigated. Under certain mild restrictions various integral representations and mean ergodic theorems are obtained for both these classes of stochastic processes. The relationship between these processes and stationary processes is also considered. (Received July 6, 1954.)

761*t*. H. T. McAdams: *Factorial experiments as determinants*. Preliminary report.

A relation is established between the theory of higher dimensional determinants and analysis of variance in a factorial experiment. Let L be a set of p elements $i, j, \cdots, m, n$ and let $A=(a_{ij\cdots mn})$ be a p-dimensional matrix of order 2 in which $a_{ij\cdots m1}=1$ and $a_{ij\cdots m2}$ are observations from a 2^{p-1} factorial experiment. Consider all subsets S_r $(r=1, 2, \cdots, 2^{p-1})$ of L containing an even number of elements, and let D_r denote the p-dimensional determinant of A with the elements of S_r as the signant indices (R. Oldenburger, *Higher dimensional determinants*, Amer. Math. Monthly vol. 47 (1940) pp. 25–33). If $S_{\bar{r}}$ is that subset of S_r which contains all the elements of S_r except n, then $(1/2)^{p-1}D_r^2$ is the mean square obtained (in the usual analysis of variance) for the component identified by the elements of $S_{\bar{r}}$. The D_r determine the coefficients for expressing the experimental data as a linear combination of orthogonal polynomials constructed from the elements of $S_{\bar{r}}$ by considering these elements as continuous variables in the interval -1 to 1, and $(1/2)^{p-1}D_r^2$ is a measure of residual variance accumulated when the rth term is rejected by a statistical test of significance. More general applications of higher dimensional determinants to experiment designs is anticipated. (Received July 19, 1954.)

762. Brockway McMillan: *Information rates and the information lattice.* Preliminary report.

Let P be a given probability measure on a Borel field (BF) I. All BF below will be subfields of I. Using P, this paper defines a non-negative numerical function $H\{A; B|C\}$, where A, B, C are BF. Theorem (i): Let X_n (X_∞) be the smallest BF with respect to which the variables $\{x_t, t\leqq n\}$ (all the variables) of a stationary stochastic process $\{x_t, t=0, \pm 1, \pm 2, \cdots\}$ are measurable, then $H\{I; X_n|X_{n-1}\}$ is C. E. Shannon's $H(x)$, the entropy per symbol for this process (Bell System Technical Journal vol. 27 (1948)). Theorem (ii): Define Y_n, Y_∞ similarly for $\{y_t, t=0, \pm 1, \cdots\}$; if all X_n and Y_n are closed by a simple closure operation, $H\{Y_\infty; X_n|X_{n-1}\} = H\{X_\infty; Y_n|Y_{n-1}\} = H\{I; X_n \cap Y_n|X_{n-1} \cap Y_{n-1}\}$ and the common value is Shannon's $H(x)+H(y)-H(x, y)$ (loc. cit.). This last permits defining a metric lattice whose elements are certain monotone sequences of closed BF (cf. C. E. Shannon, Proceedings of the International Congress of Mathematicians, 1950, Vol. II, p. 262). From its modularity and the formula (ii) follow several theorems of information theory. (Received July 16, 1954.)

763. F. L. Spitzer: *On a class of random variables.*

Let Z be a complex valued random variable, E its expectation operator, and $f(z)$ an analytic function. It is noted that the mean value theorems of analysis can be represented in the form $E[f(Z)]=f[E(Z)]$. This relation is used to define regular random variables (r.r.v.'s): Z is a r.r.v. if its absolute moments exist and the defining relation holds for every polynomial f. It is shown that there exists a class of r.r.v.'s with independent real and imaginary parts which may, but do not have to be normally distributed. The generalization of the defining relation is $E[u(\mathbf{X})]=u[E(\mathbf{X})]$, where $\mathbf{X}$ is an n-dimensional vector random variable $\mathbf{X}=\{X_1, \cdots, X_n\}$, $E(\mathbf{X})=\{E(X_1), \cdots, E(X_n)\}$. It is to hold for all harmonic polynomials $u(x_1, \cdots, x_n)$, and the components X_i are to be mutually independent and all their moments are to exist. If $n\geqq 3$ it is shown that the X_i have to be normal random variables. (Received July 12, 1954.)

TOPOLOGY

764. F. W. Anderson: *A lattice characterisation of completely regular C-spaces.*

A topological space is called a *C-space* in case each of its points is a G_δ. Spaces of this type were first mentioned by E. W. Chittenden [Trans. Amer. Math. Soc. vol. 31 (1929)] and later were studied by C. W. Burrill [Doctoral Thesis, State University of Iowa, 1951]. It is shown that the lattice of all real-valued continuous functions on a completely regular C-space characterizes the space. The proof employs techniques similar to those used by Shirota [Osaka J. Math. vol. 4 (1952)]. It is easily seen that locally compact Hausdorff C-spaces satisfy the first axiom of countability, but the converse is clearly false. An example is given of a paracompact C-space in which the first axiom of countability fails. Finally, it is noted that the space T_Ω [Hewitt, Trans. Amer. Math. Soc. vol. 64 (1948)] provides an example of a completely regular C-space which is not a Q-space in the sense of Hewitt. (Received May 17, 1954.)

765. R. D. Anderson: *Upper semi-continuous collections of arcs in one-dimensional continuous curves.*

Let M be a one-dimensional compact metric locally connected continuum (regarded as space) with the property that no point of M separates an open subset of M. Let ϵ be any positive number. Then there exists an upper semi-continuous collection G of mutually exclusive arcs filling up M such that G with respect to its elements as points satisfies the above conditions for M and such that for each point P of M and element g of G, $\rho(g, P) < \epsilon$. If, furthermore, M is assumed to satisfy a special, rather strong, almost-homogeneity property, then it can be required additionally that G be homeomorphic to M. The universal curve and a certain subset of the plane, among other sets, satisfy this homogeneity property. The demonstration of the existence of the collection G is accomplished by a construction and argument in the general spirit of that used by the author in his paper *Monotone interior mappings of manifolds* to appear in Trans. Amer. Math. Soc. (Received July 19, 1954.)

766. R. H. Bing: *Another homogeneous plane continuum.*

An example is given of a homogeneous bounded plane continuum which is neither a point, a simple closed curve, nor a pseudo arc. Like a circle, it separates the plane and is decomposable. It may be called a circle of pseudo arcs since it has a continuous decomposition such that the decomposition space is a circle and the elements of the decomposition are pseudo arcs. It is circular in that for each positive member ϵ, it is covered by a circular ϵ-chain. This precisely prescribes the continuum in that if M_1, M_2 are any two circular continua such that there is a monotone interior map of each onto a circle such that the inverses of points are pseudo arcs, M_1 and M_2 are homeomorphic and each is homogeneous. (Received July 19, 1954.)

767. Eldon Dyer: *Certain transformations which reduce dimension.*

In this paper it is shown that if G is a continuous collection of mutually exclusive arcs which is homeomorphic to a closed n-cell with respect to its elements as points, the dimension of the union of the elements of G is $n+1$. It is also shown that if G is a continuous collection of mutually exclusive $(n-1)$-dimensional continuous curves in the n-sphere S^n, then G with respect to its elements as points does not contain any compact two-dimensional subspace. With the aid of the first of these theorems, it is shown that there does not exist a continuous collection of mutually exclusive arcs filling up either S^3 or E^3. (Received July 19, 1954.)

768. J. B. Giever: *Cohomology groups for a ring of continuous functions.*

Let X be a locally compact Hausdorff space, R the Banach algebra of real-valued continuous functions on X which vanish at infinity, and R_* the set of nonzero elements of R. Define $\Phi^n(R, G) = [\phi \mid \phi: R_*^{n+1} \to G]$ and define the coboundary $\delta\phi \in \Phi^{n+1}$ as usual. The function $\phi(r_0, \cdots, r_n)$ $(r_i \in R_*)$ has empty support if and only if there is a family $\mathfrak{A} = \{I_\alpha\}$ of closed ideals which generate R such that $\phi(r_0, \cdots, r_n) = 0$ whenever $r_i \in I_\alpha \in \mathfrak{A}$. Denoting the group of functions with empty support by Φ_0^n, the group of cochains is Φ^n/Φ_0^n and cohomology groups are defined as usual. $H^n(X, G) \approx H^n(R, G)$ as follows. The correspondence between open sets, $O \subset X$, and the closed ideals of functions which are zero on the complement of O can be used to build a Čech type cohomology theory on R. Dowker's paper on the *Homology groups of relations* (Ann. of Math. 1952) provides the transition to the above defined Alexander type cohomology groups. The above groups serve to supplement the homology groups defined by Hu (Bull. Calcutta Math. Soc. 1950). (Received July 13, 1954.)

769. S. T. Hu: *On fiber spaces with cross-sections.*

Let $p. E\rightarrow B$ be a fiber space in the sense of Serre and assume that there is a cross-section $q: B\rightarrow E$. Then B can be identified with the subset $q(B)$ of E and q becomes the inclusion map. Let b_0 be a given point in B and F the fiber $p^{-1}(b_0)$. Then it is well known that $\pi_n(E, b_0)\approx\pi_n(B, b_0)\times\pi_n(F, b_0)$. The main purpose of this paper is to generalize this formula for the relative homotopy groups. Let $A\subset B$ and $D=p^{-1}(A)$. Then $B\cap D=A$ and, for each $n\geq 2$, $\pi_n(E, A)\approx\pi_n(B, A)\times\pi_n(D, A)\approx\pi_n(B, A)\times\pi_n(F, b_0)$. This is an easy consequence of the following exact sequence proved in this paper $\cdots\rightarrow\pi_{n+1}(E, A_0)\xrightarrow{\Delta}\pi_n(A, A_0)\xrightarrow{I^*}\pi_n(B, A_0)\times\pi_n(D, A_0)\xrightarrow{J^*}\pi_n(E, A_0)\rightarrow\cdots\rightarrow\pi_1(A, A_0)\xrightarrow{I^*}\pi_1(B, A_0)\times\pi_1(D, A_0)\xrightarrow{J^*}\pi_1(E, A_0)$, where A_0 is any subset of A. Further, if $\pi_1(A, A_0)=0$, then J^* maps $\pi_1(B, A_0)\times\pi_1(D, A_0)$ onto $\pi_1(E, A_0)$. Here $\pi_1(A, A_0)$ is not a group but a set of homotopy classes of paths defined in an obvious way. (Received July 12, 1954.)

770. F. B. Jones: *On a certain type of homogeneous plane continuum.*

An example is given to show the existence of a class of compact, homogeneous plane continua each of which is decomposable and separates the plane but is not a simple closed curve. Some of the properties of these continua are developed. For example, each continuum in this class may be decomposed by an upper semicontinuous collection (of continua) into a simple closed curve. (Received July 19, 1954.)

771t. Jean-Pierre Meyer: *Classification of mappings of a 3-complex into the projective plane.*

This paper is concerned with the homotopy classification of mappings of a 3-dimensional complex into a 2-dimensional projective space. The method is patterned after that of Steenrod, and uses the obstruction theory with local coefficients (Olum, Ann. of Math. vol. 52). Certain functional operations, introduced by Massey (Bull. Amer. Math. Soc. vol. 55, p. 303), and variants thereof, are also used. If (K, L) is a simplicial pair with dim $K\leq 3$ and $H^3(K, L; Z)$ has no element of order 2, then the mappings $K\rightarrow P^2$ which agree on L are completely classified. If (K, L) satisfies a further condition, effectively computable invariants are defined which enable us to determine whether two such mappings are homotopic rel L. As an application of the above, a complete and effective classification of mappings of compact orientable 3-manifolds with finite Abelian fundamental groups into 2-dimensional projective space is obtained. (Received April 22, 1954.)

772t. J. T. Mohat: *Concerning spirals in the plane.*

It is shown that if, in the plane, M is an inner limiting set such that (1) for every point P and positive number ϵ there exists a simple closed curve of diameter less than ϵ enclosing P and containing no point of M and (2) there is a line l such that the projection of M onto l is a connected open subset of l and no perpendicular to l contains two points of M, then if $ABCD$ is a rectangle there exist an equicontinuous collection G of mutually exclusive arcs and a point set K such that (1) each arc of G has one end point in the segment BC and the other in the segment AD, lies, except for its end points, in the interior of $ABCD$, and spirals down on only one point, (2) every point of the interior of $ABCD$ belongs to some arc of G, (3) a point belongs to K if, and only if, some arc of G spirals down on it, and (4) M is topologically equivalent to K. (Received August 10, 1954.)

773. P. S. Mostert: *On locally connected homogeneous spaces.*

THEOREM. *Let X be a locally connected, locally compact Hausdorff space of finite dimension which admits a locally compact, transitive transformation group satisfying the second axiom of countability. Then X is homeomorphic to an analytic manifold.* COROLLARY. *Let X be a locally compact, locally connected, connected Hausdorff space of finite dimension. Then X is homeomorphic with a metrically homogeneous space if and only if it is homeomorphic with the space of cosets of a Lie group modulo a compact subgroup.* The proof of the theorem involves, principally, the use of a generalization of the author's local cross section theorem for locally compact groups (Proc. Amer. Math. Soc. vol. 4 (1953) pp. 645–649). (Received July 15, 1954.)

774. H. M. Trent: *A note on the enumeration and listing of all possible trees in a connected linear graph.*

This paper presents theorems which provide efficient means for determining the number of distinct trees in any connected linear graph and for the specification of each of these trees. In each case the results depend only on the expansion of a single determinant. Dual theorems are presented for the enumeration and listing of all complements to trees. These theorems were discovered independently by Okada and Onodera but were published in a not readily available source (Bull. Yamagata University, Natural Science, Vol. 2, No. 2). The proofs of the theorems have been materially shortened. (Received July 15, 1954.)

775. P. M. Swingle: *n-indecomposable and related connexes.*

A set W is an essential sum of the elements of a class (C) of subconnexes if $\bigcup(C) = W$ and no C of (C) is contained in the closure of the sum of the rest. A connexe W is n-indecomposable if it is the essential sum of a class of n, but not more than n, subconnexes. For an integer $n > 0$, the compact connexe W is n-indecomposable if and only if it is the essential sum of n indecomposable subconnexes and is an irreducible joining connexe closure about a finite set. (See C. E. Burgess, Proc. Amer. Math. Soc. vol. 4 (1953) Theorem 5, p. 237; for definitions see P. M. Swingle, Bull. Amer. Math. Soc. vol. 47 (1941) p. 797). If the compact connexe W is n-indecomposable, n as above, $x \in$ (closure of W), then there exist integers $n_x \leq n$ and $2 \leq m_x \leq n$ such that W is locally n_x-indecomposable at x and locally m_x-irreducible at x; also then the set of points where W is locally 1-indecomposable are dense in W. If W is a finitely-indecomposable connexe in a completely separable space then there exists a set P of power aleph-zero such that W is an irreducible joining connexe closure about P. (Received March 11, 1954.)

776. W. R. Utz: *Equicontinuity and the limits of orbits.*

Pointwise periodicity, pointwise almost periodicity, and pointwise recurrence under equicontinuous and equi-uniformly continuous homeomorphisms of a metric space onto itself are studied to compare an equicontinuity hypothesis on the homeomorphism with the hypothesis that the metric space be compact. These two hypotheses are compared in several theorems on extensions from sets dense in the metric space to the entire space and in theorems concerning the limits of sequences of orbits under a given homeomorphism. For example, if X is a metric space, $T(X) = X$ is an equicontinuous homeomorphism and if α_i is a sequence of point orbits under T such that $\lim \alpha_i = \beta$, then β is the closure of the orbit of each of its points and hence is a minimal set. From this it follows that if T is pointwise periodic, then β is a (periodic) point orbit. This is similar to part of a theorem due to G. E. Schweigert [*A note on the limit of orbits*, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 963–969; cf. also D. W. Hall and G. E. Schweigert, *Properties of invariant sets under pointwise periodic homeomorphisms*, Duke Math. J. vol. 4 (1938) pp. 719–724, and G. T. Whyburn, *Analytic topology*, New York, 1942, p. 259] in which X is assumed compact but T is only assumed to be a homeomorphism. The remainder of Schweigert's theorem (pertaining to component orbits) does not hold if compactness of X is replaced by equicontinuity of T. (Received July 15, 1954.)

W. T. Puckett,
Acting Associate Secretary

REPORT OF THE TREASURER

To the Board of Trustees of the
American Mathematical Society

Gentlemen:

I have the honor to submit herewith the report of the Treasurer for the fiscal year ended May 31, 1954. This is the first report covering a full fiscal year since the report presented on November 30, 1952. An interim report covering the six months' period December 1, 1952 to May 31, 1953 was submitted as of the latter date.

Attention is called to the large sum held in cash on May 31, 1954. This includes grants for special purposes, and some $65,000 was held in savings banks until expenditures could be made, thereby securing a small amount of additional income for the Society.

While the Society's finances are healthy, and its assets begin to approach a half million dollars, it should not be assumed that the Society is in any sense an affluent organization. It is still necessary to exercise all reasonable economies, and to conduct the affairs of the Society frugally.

The Society operates on a budget approved in advance by the Board of Trustees, and expenditures are rigidly maintained within the approved budgetary allotments.

During the fiscal year reported upon the Society's accountants, Christiansen and Company of Providence, have completed a thorough study of the Society's accounts and accounting methods, and have presented a voluminous report, together with numerous suggestions. These are being reviewed by the appropriate officers of the Society and doubtless many of the suggestions will be accepted.

Amortization of the cost of moving to Providence continues as originally scheduled, but the Trustees have authorized the complete repayment of the indebtedness incurred in moving the headquarters to Providence; this will be reported in the accounts of the next fiscal year.

Following is a summary of the changes in security holdings made during the year:

Acquired— 3 shares Standard Oil Company of N.J.
60 shares American Cyanamid Company
100 shares Phelps Dodge Corp.
Sold— 150 shares R. J. Reynolds Tobacco Co. Class B
$1000 Texas & Pacific Railway General and Refunding Series E 3-7/8 1985 (called)

Our return on invested funds during the year under review has been at the rate of 4.64 per cent.

The surplus for the year is approximately equal to the surplus in the preceding six months and about one-quarter of the surplus earned during the last twelve month fiscal year. This reflects the increasing cost of the Society's activities, and confirms the statement made above that the cash situation should be viewed with caution.

Respectfully submitted,
ALBERT E. MEDER, JR.
Treasurer

EXHIBIT A

BALANCE SHEET

Assets

	May 31, 1954	May 31, 1953
CURRENT FUNDS:		
Cash	$121,072.24	$ 68,660.23
Accounts Receivable:		
Members and Non-Members	7,734.46	
Navy	3,280.31	5,665.10
Investments	67,279.54	67,208.26
Due from Invested Funds	558.05	
Insurance Claims Receivable	183.05	
Prepaid Postage	651.53	
	$200,759.18	$141,533.59
INVESTED FUNDS:		
Cash	$	$ 81.53
Investments	293,210.75	291,892.57
	$293,210.75	$291,974.10
TOTAL ASSETS	$493,969.93	$433,507.69

Liabilities

	May 31, 1954	May 31, 1953
CURRENT FUNDS:		
Accounts Receivable—Credit Balances	$ 2,328.19	$ 2,496.03
Special Publication Funds	26,119.53	22,257.16
Other Special Funds	54,117.27	26,610.80
General Surplus	118,194.19	90,169.60
	$200,759.18	$141,533.59
INVESTED FUNDS:		
Endowment Fund	$ 94,000 00	$ 94,000.00
Library Proceeds Fund	52,800.00	52,800.00
Other Expendable Funds	28,551.47	28,551.47
Prize Funds	6,575.00	6,575.00
Mathematical Reviews	80,000.00	80,000.00
Reserves	30,726.23	30,047.63
Due to Current Funds	558.05	
	$293,210.75	$291,974.10
TOTAL LIABILITIES	$493,969.93	$433,507.69

EXHIBIT B

GENERAL FUNDS INCOME AND EXPENSE

GENERAL RECEIPTS:			
Dues from Ordinary Memberships		$54,241.93	
Dues from Contributing Memberships		653.00	
Dues from Institutional Memberships		21,725.00	
Publication Charges from Non-Member Institutions		787.50	
Initiation Fees		5.00	
Income From			
Henderson Estate	$ 7,450.00		
Current Funds Investments	1,619.26		
Invested Funds	9,600.24		
Transferred from Invested Funds	342.76	19,012.26	
Regular Publications		25,505.25	
Other Publications		1,273.19	
Meeting Fees		611.00	
Overhead on Contracts		12,378.67	
TOTAL GENERAL RECEIPTS			$136,192.80
GENERAL DISBURSEMENTS:			
Regular Publications		$53,775.42	
Other Publications		317.52	
General Activities			
Accounting Department	$14,803.40		
Executive Director	20,089.90		
New Haven Office	4,713.02		
Officers' Expense	1,748.36		
Meetings	5,729.50		
Membership Records	5,272.72		
Directory	1,698.63		
Combined List of Members	2,966.65		
Policy Committee	206.90		
Custodian Expense	166.53		
Accounts Receivable Adjustment	466.11		
Foreign Exchange	.97	57,862.69	
Subsidies			
American Journal of Mathematics	$ 420.94		
Canadian Journal of Mathematics	1,000.00		
Pacific Journal of Mathematics	3,000.00		
U.S. National Committee on Theoretical and Applied Mechanics	50.00	4,470.94	
TOTAL GENERAL DISBURSEMENTS			$116,426.57
EXCESS OF GENERAL RECEIPTS OVER GENERAL DISBURSEMENTS			$ 19,766.23
SURPLUS, May 31, 1953			$ 90,169.60
			109,935.83
Adjustments			8,258.36
SURPLUS, May 31, 1954			$118,194.19

SUMMARY OF CURRENT FUNDS

	Balance 6/1/53	Additions	Deductions	Balance 5/31/54
Mathematical Reviews	$ 2,958.25	$ 71,922.10	$ 74,521 19	$ 359.16
Colloquium	20,473.54	10,467.18	4,990.26	25,950.46
Mathematical Surveys	(3,590.48)	2,512.93	1,148.96	(2,226.51)
Birkhoff's Collected Papers	(448.02)	382.00	72.10	(138.12)
Symposia on Applied Mathematics	(407.39)	500.09	229.28	(136.58)
Memoirs	984.18	1,073.36	1,458.60	598.94
Russian Vocabulary	(75.62)	568.43	1,075.53	(582.72)
Translations	(137.30)	802.20	870.00	(205.10)
Transactions Reprinting	2,500.00			2,500.00
TOTAL SPECIAL PUBLICATION FUNDS	$22,257.16	$ 88,228.29	$ 84,365.92	$26,119.53
Office Transfer Fund	$ 1,815.83	$ 3,500.00	$	$ 5,315.83
Policy Committee	688.78	206.90	385.48	510.20
Combined List of Members	(7,861.96)	6,354.16	3,460.59	(4,968.39)
Prize Funds Accumulated Income	2,561.05	308.31	530.11	2,339.25
Profit/Loss on Sale of Securities	1,998.45			1,998.45
Summer Institute (1953)	19,961.99	430.11	20,392.10	
Summer Institute (1954)		25,000.00	11.29	24,988.71
Publication Project	6,292.76		4,089.33	2,203.43
Navy Overhead Accrual	1,153.90	2,315.37	3,469.27	
Congress Proceedings		1,459.54	1,459.54	
National Science Foundation Translation Project		14,500.00	842.83	13,657.17
Office of Scientific Research			1,719.24	(1,719.24)
Register of Mathematicians		14,450.00	5,158.14	9,291.86
Journal d'Analyse de Mathématique		1,200.00	700.00	500.00
Euler		37.50	37.50	
TOTAL OTHER FUNDS	$26,610.80	$ 69,761.89	$ 42,255.42	$54,117.27
Directory	$ (533.88)	$ 533.88	$	$
Bulletin	(5,505.96)	11,331.32	15,217.44	(9,392.08)
Proceedings	(8,521.21)	14,615.14	20,731.31	(14,637.38)
Transactions	(2,798.71)	16,946.39	18,388.39	(4,240.71)
TOTAL GENERAL PUBLICATIONS	($17,359.76)	$ 43 ,426.73	$ 54,337.14	($28,270.17)

SUMMARY OF INVESTED FUNDS

	PRINCIPAL				INCOME			
	Balance 6/1/53	Additions	Deductions	Balance 5/31/54	Balance 6/1/53	Additions	Deductions	Balance 5/31/54
Permanent Endowment..	$ 94,000.00			$ 94,000.00		$ 4,387.43	$ 4,387.43*	
Expendable Funds								
Boyle...	$ 250.00			$ 250.00		$ 12.28	$ 12.28	
Brown.....	1,000.00			1,000.00		46.39	46.39	
Henderson	1,000.00			1,000.00		46.39	46.39	
Hutchinson	1,000.00			1,000.00		46.39	46.39	
Merrill..	650.00			650.00		30.01	30 01	
Reilly........	23,651.47			23,651.47		1,103.68	1,103.68	
Whittemore.....	1,000.00			1,000.00		46.38	46.38	
Library Proceeds.	52,800.00			52,800.00		2,465.20	2,465.20	
	$ 81,351.47			$ 81,351.47		$ 3,796.72	$ 3,796.72	
Prize Funds......								
Bôcher	$ 1,450.00			$ 1,450.00	$ 903.01	$ 68.21	$ 100.00	$ 871.22
Cole..	2,550.00			2,550.00	1,246.56	120.05	430.11	936.50
Moore........ ..	2,575.00			2,575.00	411.48	120.05		531.53
	$ 6,575.00			$ 6,575.00	$2,561.05	$ 308.31	$ 530.11	$2,339.25
Mathematical Reviews .	$ 80,000.00			$ 80,000 00		$ 3,733.95	$ 3,733.95†	
Reserves								
Life Membership...	$ 3,726.96	$	$340.94	$ 3,386.02		$ 165.07	$ 165.07	
Transactions Life Sub..	95.52		1.82	93.70		4.09	4.09	
Against Invest. Loss..	5,550.00			5,550.00		259.21	259.21	
Securities Profit or Loss.	20,675.15	1,021.36		21,696.51		987.72	987.72	
	$ 30,047.63	$1,021.36	$342.76	$ 30,726.23		$ 1,416.09	$ 1,416.09	
TOTAL...	$291,974.10	$1,021.36	$342.76	$292,652.70	$2,561.05	$13,642.50	$13,864.30	$2,339.25

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* Transferred to General Receipts.

† Transferred to Mathematical Reviews Current Fund.

BOOK REVIEWS

Darstellungstheorie unendlicher Gruppen und fastperiodische Funktionen. By W. Maak. Enzyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Vol. I_1, No. 7, Part I. 2d ed. Leipzig, Teubner, 1953. 1–26 pp.

This is primarily a résumé of the theory of almost periodic functions on groups. Group representations are treated only in so far as they relate to almost periodic functions (for the most part), so that infinite-dimensional or unbounded representations get practically no discussion. The booklet is eclectic in style and moderately comprehensive though necessarily sketchy in detail.

More than half of it, the first chapter, is devoted to a general account of the theory of almost periodic functions on groups. The approach is essentially that of von Neumann and Weyl. Next there is a chapter on generalized almost periodic functions on groups, etc., necessarily a short chapter in view of the difficulty of treating such matters on general groups. As shown by Følner, the Weyl functions can to some extent be carried over from the real line, but the interesting Besicovitch functions have yet to be so extended, the key difficulty being that of handling discrete groups. Finally there is a chapter on miscellaneous aspects. It mentions among some other topics the Hilbert fifth problem for compact groups, the Freudenthal structure theorem for connected maximally almost periodic groups, the compact-discrete case of the Pontryagin duality theorem, and the original form of the Tannaka duality theorem.

In an encyclopedia article such as this one expects more comprehensiveness and up-to-dateness than in an independent introduction to the subject. Therefore one is disappointed to find that while it is a very readable account of the theory, it deals mainly with the theory as it existed a few years ago and has some notable omissions. To be more specific: (1) little mention is made of recent pertinent topological and algebraic developments. For example, it is stated that it has not yet been possible to set up the basic theory without the use of the theory of integral equations. While this may be true in spirit, it may well be misleading to have no indication whatsoever of how Banach algebras or partially ordered linear spaces can be used effectively to treat almost periodic functions. (2) Although the author discusses spherical harmonics and the decomposition of the regular representation of a compact group, the essentially equal simple structure of an arbitrary representation of a compact group is not described.

(3) No mention is made of Chevalley's version and application of the Tannaka theorem, nor of the work of Krein in related connections. (4) It is mentioned that the unimodular group in $n(>1)$ dimensions is minimally almost periodic; notably more generally, the same is true for any noncompact simple Lie group.

There is some occasional vagueness that may bother the non-expert reader. Thus, in the discussion of Hilbert's fifth problem, "finite-dimensional" is used as if it meant "finite-dimensional and locally euclidean," or at least "finite-dimensional and locally connected." And in quoting van der Waerden's result on the continuity of representations of semi-simple groups, the author neglects to make explicit the qualification that the representation be finite-dimensional.

This booklet conveys the scope of the theory of almost periodic functions on groups more rapidly and pleasantly than any other treatment with which the reviewer is familiar, and as such performs a valuable and important service.

I. E. SEGAL

Undecidable theories. By A. Tarski in collaboration with A. Mostowski and R. M. Robinson. Amsterdam, North Holland Publishing Company, 1953. 9+98 pp. $2.50.

The material of this book is based on results which were obtained by the authors, in particular by Tarski, between 1938–1952, and were originally intended to be published as a series of papers in some mathematical journal. Fortunately they are combined here in one volume.

A given axiomatic theory T is called decidable or undecidable according as we can or cannot find a mechanical procedure which permits us to decide (in a finite number of steps), for each particular sentence formulated in the symbolism of T, whether this sentence is provable by means of the devices available in T. (The number of steps necessary will, however, in general depend on the structure of the sentence under test, just as in applications of the Euclidean algorithm the number of divisions necessary depends on the two numbers whose g.c.d. we wish to find.) The decision problem of a theory T is the problem of determining whether T is decidable or undecidable. The problem of deciding which axiomatic theories have decision procedures and which do not is one of the central problems of modern symbolic logic, and should be of interest to all mathematicians, not just logicians. Here the authors describe all the principal methods and results in this field. The exposition throughout is excellent, so that the book is valuable not only to the expert, but also to students.

The first of the three papers is by Tarski and it outlines the general methods in proofs of undecidability and gives the necessary definitions and background. It forms both an introduction and outline of the whole work. Two general methods have been developed to show that a theory T (formalized in the first order predicate calculus) is undecidable. The first, direct, method can be used only in case T allows us to build up a sufficiently large part of number theory so that we can carry through arguments based on Gödel numbering and use diagonal procedures. The second method is indirect: The decision problem for a theory T_1 is reduced to that for some theory T_2 which has already been shown to be undecidable. In many applications, T_2 is taken to be arithmetic based on Peano's axioms, for Rosser has shown that every consistent extension of this is undecidable. This can be expressed by saying that Peano's arithmetic is essentially undecidable. If now T_1 is obtained from an undecidable theory T_2 by deleting a finite number of axioms (without, however, removing any constant from the symbolism of T_2), then T_1 is undecidable. The same holds if some essentially undecidable theory T_2 is interpretable in T_1. Moreover, T_1 is undecidable if some finitely axiomatizable and essentially undecidable theory T_2 is interpretable in some consistent extension of T_1. Some fragments of arithmetic are both finitely axiomatizable and essentially undecidable and applications of the last result showed the undecidability of many formal theories, such as the elementary theory of groups, rings, fields and lattices.

Paper II is by A. Mostowski, R. M. Robinson, and A. Tarski. Here the first, direct, method is used to give a very elegant proof of the essential undecidability of a subtheory of the arithmetic of natural numbers which is based on only 7 axioms. The first such theory was constructed by Mostowski and Tarski in 1939 and the present simplified version is due to Robinson. It is also shown that every subtheory of arithmetic (having the same constants) is also undecidable. The results are extended to systems of arithmetic with different sets of constants; and, using the indirect method, the elementary theories of various kinds of rings can be shown to be undecidable, and in some cases essentially undecidable.

The third and last paper is by A. Tarski and shows that the elementary theory of groups is undecidable. More precisely, if we consider the class of all sentences which can be deduced from the axioms $xo(yoz)=(xoy)oz$, $\bigvee z(x=yoz)$, $\bigvee y(x=yoz)$ by the first order functional calculus, then there is no general algorithm for deciding whether a given sentence is in this class or not. [This is not yet the same as the word-problem for groups which deals with sentences of a particular form. The word-problem for groups is still unsolved.] The proof is by the indirect method by interpreting a translation of the finitely axiomatized essentially undecidable arithmetic theory in an extension of group theory.

Much credit is also due to the editors of the series on "Studies in logic and the foundations of mathematics" for their part in the publication of this and many other excellent volumes.

ILSE NOVAK GÁL

Pages choisies d'analyse générale. By M. Fréchet. Paris, Gauthier-Villars; Louvain, Nauwelaerts, 1953. 216 pp. 2.000 fr. ($5.94).

Jubilees or seventieth anniversary celebrations of savants usually carry with them the publication of an issue of a journal dedicated to the celebrant, or a special volume consisting of papers in his main field of interest, contributed by his friends, pupils and admirers. The present volume, which is not in either category, was not planned for in connection with the Jubilee of M. Fréchet. It is, however, an outgrowth of this celebration in that friends and pupils on that occasion urged on him the desirability of issuing a volume which might contain some of the material planned for in a second volume of his *Espaces abstraits*, had not a change of positions and consequent change of fields of interest and research intervened in 1928. The volume under discussion might fall into the category of "excerpts from collected works" which would be insufficiently inclusive in that it is limited to re-publication of some papers connected with the topic of general spaces and not intended to give a systematic insight into the development of these researches. We have then before us a group of papers published in various periodicals, a half dozen of which may be somewhat inaccessible at present. The selection was made by the author and can be considered an indication of what phases of his researches in this field he considers worthy of reproduction and important.

The papers are roughly grouped into chapters headed as follows: 1. Survey of sets (1 paper); 2. Functional spaces (13 papers); 3. Functional analysis (13 papers); 4. Abstract spaces (4 papers); 5. General analysis (6 papers). However, there is no sharp distinction between the topics included in the various chapters. For instance, papers pertaining to the differential are found in the third and fifth chapters, matters pertaining to the generalization of the Weierstrass theorem of approximation in the third and fifth chapters, considerations pertaining to curves, surfaces, and surface area in the second, third, and fifth chapters. While the reprinted papers are chronologically arranged in each subheading of the chapters, one is aware of much jumping back and forth in point of time, so that papers from various periods rub elbows with each other. There are reprints of items appearing in the very early researches, i.e. 1904, including two excerpts from his famous thesis of 1906, and one paper which appeared in 1950. As a consequence, consecutive reading of the articles in this collection gives one a rather blurred image of the historical development of ideas contributing to general analysis theory on the part of the author.

As is to be expected, the volume contains a number of papers which have been basic and stimulated further mathematical developments. There is the paper of 1915 on the integral of a function over an abstract set generalizing the Lebesgue integral, which incidentally is one place where the points of view of Fréchet and E. H. Moore on general analysis met in the consideration of a functional analysis where the basic class of elements is unconditioned, and one manipulates the real-valued functions on the class. There are two papers on the notion of differential which has proved valuable for instance in implicit function theory on linear normed spaces. There is the paper on bilinear functionals on the space $C\times C$, where C is the space of continuous functions on a linear interval, which introduced an interesting notion of bounded variation for functions of two variables and which has recently been extended and generalized by M. Morse. There are a number of papers on surfaces which have influenced the development of theory of surfaces. There is an excerpt from the 1910 paper on dimension theory proposing the existence of a homomorphism as a basis for an ordering of dimensions.

One is a little puzzled by the insertion of a note headed "malentendu" in the chapter on abstract spaces. It seems to be an apology or defense for having tied the topology of abstract spaces initially to the notion of convergent sequences, i.e. the so-called L(imit) spaces, and later abandoned this type of space for topologies based on vicinity notions. It is not clear why any apology is necessary, since every openminded mathematician knows that initial basic concepts which open up new fields of investigation frequently give way to others which are either better suited to the purpose or receive vociferous support from a prominent member of a mathematical group and his supporters or pupils. Interesting in this same chapter also is the paper "De l'écart numérique à l'écart abstrait," a reaction to the spaces with uniform structure of A. Weil.

There are supplementary remarks after some papers, especially the more important ones, indicating briefly further developments and some of the individuals who have built upon and expanded the material of the original publication.

Although an offset process has been used in the publication of this volume, the papers included have all been retyped, leading to a uniform typography, but introducing occasional unimportant misprints.

T. H. HILDEBRANDT

Trattato di analisi matematica. Vol. 1. By M. Picone and G. Fichera. Rome, Tuminelli. 5700 Lire ($9.20).

This volume is the first of three which will be devoted to introducing the young student to classical analysis. Into it, Professor Picone, with the very able assistance of Professor Fichera, has poured all the knowledge and experience derived from forty-five years of teaching and some twenty-five as director of Italy's chief computing laboratory. And "poured" is but a meager word to indicate the enormous enthusiasm and energy which have been lavished on the undertaking. The discussion is very detailed and the material elaborate. The book will serve ably not only the budding mathematician but the future physicist and engineer as well. Indeed, being written by men who are outspoken adherents of the "applied school" of mathematics, this treatise is especially recommended to those beginning scientists who frequently complain that most mathematicians ignore their needs.

The present volume treats in its five chapters: matrices and determinants; sets, functions, and continuity; differentiation; integration; and the beginnings of complex analytic functions. The treatment varies from what in the United States would be considered elementary to the level, roughly, of our advanced calculus courses. A principal difference is that the amount of material is more extensive than that which we usually cover. According to the American method, this material is taught twice, once in a general and intuitive fashion to undergraduates, and again in a strictly rigorous fashion for serious or graduate students. The European system (if there is one such) seems to steer an intermediate course of teaching the material once, more thoroughly than we do it the first time and lacking the rigor which we deploy in the second. The superiority of one method over another, we leave for discussion to some more relaxed moment and less public place. National systems of education being what they are, no suggestion is being made that we should adopt this or another similar treatise for teaching our students. However, we would advise the inquiring mind to turn to a book such as this in order to derive some ideas as to the training of students in another land and to learn to appreciate the Latin mind. Here the lesson is rich and the knowledge it imparts adds to the stature of a human being. One might add that the lesson is never learned, being too extensive. We look forward to the appearance of volumes 2 and 3, which we are certain will reveal once more all the personal warmth and all the pedagogical cunning of these two authors.

E. R. LORCH

Mathematical aspects of the quantum theory of fields. By K. O. Friedrichs. New York, Interscience, 1953. 8+272 pp. $5.00.

Attempting a mathematical treatment of quantum fields may be a bit like trying to run a cross-country mile in 4 minutes. One of the main obstacles is the psychological one arising from the prevailing opinion that it can't really be done. This book is one of the first serious attempts to get in training for a successful run, and is the only one of such scope. A number of shorter stretches are run in good enough time to be encouraging, and some of the longer practice runs show interesting technique and suggest novel possibilities. If at times the pace slows down almost to a fast walk and if not all of the mile is covered, the degree of success is still greater than one had a right to expect at this stage of the enterprise.

The subject does not at present admit a systematic or rigorous presentation. To a considerable extent it is not even so much a subject in the mathematical sense as a set of techniques for dealing with specific problems with various elements in common. The most incisive of these techniques, and in particular the renormalization approach developed within the past ten years, look just as firmly mathematically unrigorizable now as did the theory of *interacting* quantum fields when it was initiated over 25 years ago. It is even a significant accomplishment to present, as this book does, a quasi-rigorous treatment of selected parts of the theory of an individual quantum field.

The material is presented in a relatively mathematical language and style. It is thereby more readable for most mathematicians than most of the corresponding articles in the physical literature, even if definitions and statements are not always mathematically precise and if proofs of some of the more technical statements are omitted. The author apparently intends to give a more mathematical description to some of the simpler parts of the existing theory, and the only substantial really novel feature is the treatment of quantum statistics. The basic annihilation and creation operators of conventional physics exist only in a formal sense, but smoothed-out versions of them can be identified with perfectly good mathematical operators (unbounded, but closed and densely defined). This development was independently arrived at by J. M. Cook, whose approach was perhaps less concrete but more invariant, using tensors over a Hilbert space rather than square-integrable functions of several variables. The large number of applications and comments that are made in the book may be judged from the fact that the mathematical part of Cook's paper is 10 pages long while the corresponding material in the book is scattered through some fifty odd pages, without including proofs of such results as the self-adjointness of the canonical P's and Q's.

There is no doubt that the author has succeeded in his aim "to present basic sections of field quantum theory in a consistent mathematical language without carrying out all mathematical deductions with complete rigor." It is regrettable that the standard of rigor and the mode of presentation are not such as to make the mathematical consistency of many of the sections wholly manifest and explicit, but one must be thankful that the ice has been broken with this first large-scale attempt to deal mathematically with quantum fields. While the book is not easy reading in detail and is not closely integrated in the large (the relation between statistics and dynamics seemed blurred to the reviewer), a mathematician interested in a more rigorous and searching approach to the specific problems of quantum field theory may well find this a stimulating and useful book.

I. E. SEGAL

Collected mathematical works. By H. Bohr. Ed. by E. Følner and B. Jessen. Copenhagen, Dansk Matematisk Forening, 1952. Vol. I, 34+771 pp., 1 plate; vol. II, 9+852 pp., 1 plate; vol. III, 10+985 pp., 1 plate. 110 kr.

These volumes are an impressive reminder that Harald Bohr was one of the great mathematicians of the first half of this century. Most of his work is now generally regarded as "classical" and much of it has been superseded or subsumed in more general results, but this fact is in itself a tribute to the importance and fundamental character of his work and to the stimulation which it gave to other mathematicians. His papers are sure to be admired by generations of future mathematicians and the publication of his collected works is accordingly most welcome.

Aside from the intrinsic interest of Bohr's papers, they are models of mathematical exposition. Their eminent readability makes them excellent reading for students of analysis at practically all levels. While this readability is largely due just to good mathematical writing, it stems to some extent from the use of a more leisurely style and the inclusion of more expository material than seems fashionable or permissible nowadays. Evidently Bohr was not a believer in the dictum that "authors must present their results in as compact a form as possible and should avoid space-consuming exposition and discussion" —with the happy result that almost any one of his papers can be read as an entity in itself.

This collection contains all of Bohr's mathematical writings with the exception of obituaries, elementary articles in Danish, textbooks in Danish, and the address he delivered at the 1950 International Congress in connection with the presentation of the Fields medals. In particular it includes his joint article with Cramér *Die neuere Entwicklung der analytische Zahlentheorie* in the Enzyklopädie der mathematischen Wissenschaften (1923) and his monograph *Fastperiodische Funktionen* in the series Ergebnisse der Mathematik und ihrer Grenzgebiete (1932). His doctoral dissertation *Bidrag til de Dirichlet'ske Rækkers Theori* (1910) is given both in the original Danish and in English translation, and English summaries are provided for the other research papers in Danish. An English translation of an interesting lecture *Looking Backward*, given by Bohr on his 60th birthday, appears as a preface. Aside from the translations, the papers have been reproduced by a photographic method. The last volume contains a fifteen-page general index and a dozen pages of notes giving cross-references among the papers and the most important references to later work by other authors.

Bohr's work is described rather aptly by the following quotation from an introductory note by the editors of these volumes, Erling Følner and Børge Jessen: "Bohr's contribution to mathematics is one of great unity. His investigations in his youth on Dirichlet series and the Riemann zeta-function established his reputation as one of the leading analysts of his time. The problem as to which functions may be represented by Dirichlet series led to his main achievement, the theory of almost periodic functions, on which the greater part of his later work is concentrated. With the exception of a small number of papers, dealing mostly with problems in the general theory of analytic functions, his papers on other subjects are all closely connected with, or are even auxiliaries of, his investigations on these main subjects." A fuller discussion of Bohr's life and the significance of his work may be found in Jessen's memorial address, which is given (in English) as a fitting postlude to this collection.

Mathematicians everywhere will be grateful to Bohr's countrymen for publishing these collected works, especially in view of their low price and handsome physical appearance. They are a "must" on the analyst's bookshelf.

PAUL T. BATEMAN

Methods of mathematical physics. By R. Courant and D. Hilbert. Vol. I. 1st English ed. New York, Interscience, 1953. 16+561 pp. $9.50.

This book is mainly a translation of the second German edition of the celebrated textbook written by Courant and his collaborators at the University of Göttingen. While Hilbert did not take any active part in its preparation his name was put as a co-author to indicate the tremendous influence which he exerted on the mathematical thinking of his surroundings and indeed of the whole mathematical world. Since the content of the book is well known to every worker in the field, let us recall only that the text covers the following subjects: linear transformations and quadratic forms, development of arbitrary functions in series of orthogonal functions, linear integral equations, calculus of variations, eigenvalue and vibration problems, application of variational calculus to eigenvalue problems, and special functions.

This work presents a cross-section of the subject matter as it appeared to Courant's school in Göttingen in 1931. It is, of course, not a valid criticism that the translation contains no newer developments, inasmuch as the author states in his preface that the pressure for publication of an English "Courant-Hilbert" became irresistible. However, regardless of its contents as seen today, one may reasonably ask what will be the reaction of some students who will miss the possibility of using the work as a reference book in which each theorem is stated and numbered in a precise way. Instead of a catalogue of theorems, the reader will find an artistic exposition of the profound meaning of mathematical thinking. The author is greatly aided in his exposition by his natural inclination to somewhat fluid statements which greatly stimulate the imagination of the reader. The reviewer gratefully acknowledges being one of the large community of scientists outside of Göttingen who were influenced by Courant's book.

Very few additions and alterations have been included in the English edition; of these, only the interesting appendix by W. Magnus has been mentioned in the preface. This appendix deals with the question of how a set of linearly independent spherical harmonics in three variables is transformed if the coordinate system is rotated.

The main addition not mentioned in the introduction is a paragraph entitled: Reciprocal quadratic variational problems (chapter 4, §11, pp. 252–257), which complements a preceding section (§9) devoted to the discussion of upper and lower bounds of quadratic functionals and to Friedrichs' analysis of Trefftz's method. In the new addition, the author emphasizes a geometrical approach given by Prager and Synge (Quarterly of Applied Mathematics vol. 5 (1947) pp. 241–269). Similar questions were widely discussed by various methods in recent publications, and a part of the literature is mentioned by Courant in one of the longest footnotes of the book. In view of the great and deserved influence of the book, such a footnote cannot be passed over without comment. The footnote states that the theories indicated in §9 have been recently rediscovered and advanced by several authors. Let us emphasize that the word rediscovered is incorrect, since all papers refer either to Trefftz's method or to its analysis by Friedrichs. The purpose of the recent investigations was to simplify the method of Trefftz and to obtain stronger results. Incidentally, the footnote omits a reference to the work of Diaz and Weinstein (see e.g. *Schwarz' inequality and the methods of Rayleigh-Ritz and Trefftz*, Journal of Mathematics and Physics vol. 26 (1947) pp. 133–136) which uses an analytic approach. The various methods were recently analyzed by Diaz, Collectanea Mathematica vol. 4 (1951) pp. 1–47, specially pp. 41–46, who pointed out the advantages of the analytic method, which yields simpler formulas and more correct results than the geometric procedure.

The tendency of understatement of the work done outside the author's circle reappears in the footnote on page 175, where Lord Rayleigh's contributions to variational methods are evaluated as follows: "Even before Ritz, such ideas were successfully employed by Lord Rayleigh."

The book preserves not only all the high points of the original, but also some of its misprints, which, being by now classical, only add to the pleasure of the reader. It is with great expectations that the reviewer is looking forward to the translation and additions to the second volume.

A. WEINSTEIN

Forced oscillations in non-linear systems. By C. Hayashi. Osaka, Nippon Printing and Publishing Co., 1953. 14+164 pp. $4.50.

This book was written "to provide engineers and physicists with practical knowledge concerning the important subject of non-linear oscillations," in particular forced oscillations governed by non-linear equations of the second order.

The text is divided into two parts; the first concerns the stability of steady state oscillations, whereas the second is devoted to a discussion of the geometry of solutions of a slightly perturbed system of two first order equations. The first chapter considers the equation $v''+f(v)v'+g(v)=e(t)$ $('=d/dt)$, where e is periodic. The existence of a periodic solution of the same (or integral multiple of the) period of e is assumed, and the stability of this solution is defined in terms of the characteristic exponents of the corresponding equation of the first variation. A discussion of the Mathieu equation and Hill's equation follows in which "approximate" characteristic exponents are obtained, and corresponding stability regions are plotted. The question of the degree of approximation is not considered. In the next three chapters special cases are worked out in detail, a typical equation being $v''+2\delta v'+(c_1v+c_3v^3)=B \cos \nu t$, where δ, c_1, c_3, B, ν, are constants, with a subharmonic of the form $v=k_1 \sin t+k_2 \cos t+k_3 \cos \nu t$ assumed. Various experiments with electrical oscillatory circuits are described and the results compared with the approximate stability regions determined. In the last two chapters (Part II) the integral curves of some special cases of the system $x'=X(x, y)$, $y'=Y(x, y)$ are depicted in the vicinity of the equilibrium points. The solutions of the van der Pol equation are also sketched.

The book is replete with excellent illustrations.

Although from the mathematical viewpoint the equations and solutions considered are rather special, this work should serve its purpose well, and in addition provide mathematicians with a supply of nicely illustrated examples of forced oscillations.

E. A. CODDINGTON

Hypergeometric and Legendre functions with applications to integral equations of potential theory. By Chester Snow. (National Bureau of Standards, Applied Mathematics Series, no. 19.) Washington, Government Printing Office, 1952. 11+427 pp. $3.25.

This is the second edition of a monograph of which the first edition appeared in 1942 (as MT15 in the Mathematical Tables Series of the National Bureau of Standards) and was exhausted within a year, a second printing being sold out soon afterwards. Continuing, and indeed increasing, demand for the monograph and the impending retirement of the author from active service with the National Bureau of Standards were the motivations for the present second edition. Known misprints have been corrected, some chapters have been rewritten and expanded, and a new chapter (on confluent hypergeometric functions) has been added.

A brief indication of the contents, chapter by chapter, follows.

I. Definitions and preliminary formulas relating to the gamma function and Gauss' hypergeometric series.

II. Homographic substitutions (linear transformations) of the hypergeometric differential equation.

III. Non-linear substitutions (quadratic transformations) of the hypergeometric differential equation.

IV. Integral representations of the hypergeometric function.

V. A few relations of contiguity.

VI. Associated Legendre functions. It should be noted that in this book $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ stand for functions which are numerical multiples of the associated Legendre functions defined by Hobson, and commonly used, in the z-plane cut along the real axis from $-\infty$ to 1. The functions defined in the complex plane cut from $-\infty$ to -1 and again from 1 to ∞ are denoted by $T_\nu^\mu(z)$ and $q_\nu^\mu(z)$, and the first of these differs again by a constant factor from MacRobert's $T^{-\mu}_\nu(z)$.

VII. Heun's function, which is a standard solution of an ordinary linear differential equation of the second order with four regular singular points and prescribed exponents at these singular points.

VIII. Some integral representations. This is one of the longest chapters of the book, it is one of its most original portions, and has been expanded considerably in the new edition. The general problem is the representation of "arbitrary" functions as integrals involving solutions of a given differential equation (or rather of a differential boundary value problem with a continuous spectrum), in a sense the problem of the generalized Fourier integral. After a general formulation of the problem, integral representations in terms of hypergeometric, Legendre, Bessel, and related functions are considered and illustrated by a considerable number of interesting examples.

IX. Some integral equations of potential theory with $Q_{m-1/2}$ as nucleus; canonical expansions of $Q_{m-1/2}$. Another long and highly original chapter. The problem of determining a potential whose only sources are simple layers on surfaces of revolution leads to integral equations of the first kind whose nuclei are Legendre functions, m being an integer. Such integral equations are discussed in various systems of curvilinear coordinates in the meridian planes, and the bilinear expansions of the nuclei in series of characteristic functions of the integral equations are the canonical expansions of $Q_{m-1/2}$. Of particular importance are those curvilinear coordinate systems in which Laplace's equation is separable, and a determination of all such coordinate systems is given in this chapter.

X. Applications. The results of the last two chapters are used to construct potentials determined by certain data on given surfaces of revolution. The answers are given in (circular) cylindrical, (spherical) polar, dipolar, toroidal, prolate and oblate spheroidal, paraboloidal, and annular (cyclidical) coordinates. This is the longest chapter of the book, and is full of useful information. The thoroughness in laying the groundwork in chapters VIII and IX simplifies the work of this chapter considerably.

XI. Integral representations with integrals of confluent hypergeometric functions. This chapter has been added in the second edition.

As J. H. Curtiss remarks in the Foreword, this work is "a labor of love on the part of the author" with whom "to derive new formulas pertinent to the hypergeometric function was, quite literally, his hobby as well as his profession." The resulting book has a flavor all of its own which sets it apart from all other books on the subject. The monograph is written by a specialist for mathematicians seeking highly specialized information; it does not attempt to replace, or to compete with, standard texts, and offers much that will be new even to experts in this field. The continuing demand for the book is due in a large measure to the increasing number of mathematicians who have "discovered" Snow's monograph and found it so helpful that they would like to own a copy. In the course of the last ten years the present reviewer loaned his copy of the first edition to numerous colleagues, and on the book being returned (somewhat reluctantly in many cases), the borrower almost invariably asked how he could buy a copy. It is a pleasant thought that in the future it will be possible to give a simple answer to this question.

A. ERDÉLYI

Introduction to the theory of stochastic processes depending on a continuous parameter. By H. B. Mann. (National Bureau of Standards Applied Mathematics Series, no. 24.) Washington, Government Printing Office, 1953. 45 pp. 30 cents.

To have written a serious textbook on the theory of stochastic processes in the small compass of forty-five pages is an astonishing *tour de force* and the reviewer is full of admiration for the ingenuity with which so much has been packed into so small a space. But brevity can be an enemy of clarity and even the "educated mathematician" to whom (in the foreword by Dr. J. H. Curtiss) the argument is addressed may find some difficulty in discovering from the evidence presented what a stochastic process *is* (unless he already knows). The following notes, supplementing Chapter I, may help in pinpointing the author's point of view in relation to other surveys.

Professor Mann starts by defining an (indexed) family of random variables $\{x_t: t\in T\}$, and he calls it a stochastic process if the index-set T is a set of real numbers. If we ignore this distinction (and it is not a useful one) then we can say with the author that a stochastic process has been defined when we are given:

(i) an index-set T;

(ii) for each $n \geqq 1$ and for each $(t_1, \cdots, t_n) \in T^n$ an n-dimensional distribution-function $F_{t_1,\cdots,t_n}$;

(iii) consistency relations of the two types,

$$F_{t_1,\cdots,t_n,t}(\xi_1, \cdots, \xi_n, \infty) = F_{t_1,\cdots,t_n}(\xi_1, \cdots, \xi_n)$$

and

$$F_{s_1,\cdots,s_n}(\eta_1, \cdots, \eta_n) = F_{t_1,\cdots,t_n}(\xi_1, \cdots, \xi_n)$$

where $(s_1, \cdots, s_n)$ is any permutation of $(t_1, \cdots, t_n)$ and $(\eta_1, \cdots, \eta_n)$ is the same permutation of $(\xi_1, \cdots, \xi_n)$.

Abstract objects $\{x_t: t \in T\}$ are then introduced so that (whenever A is an n-dimensional Borel set) we can write

$$P\{[x_{t_1}, \cdots, x_{t_n}] \in A\}$$

instead of

$$\int_A dF_{t_1,\cdots,t_n};$$

these objects x_t are called the "random variables" composing the family, but the customary language of probability theory is employed only as a colourful way of making statements about the distribution-functions F.

The justification for this procedure is not explained in the book but it is, of course, the theorem of Daniell and Kolmogorov. Let Ω be the set of all mappings ω of T into the real line. Then $\omega(t)$ can be regarded either as the value of the real-valued function $\omega(\cdot)$ at t or as the value of the real-valued function $\cdot(t)$ at ω. A Borel field $\mathcal{J}$ of subsets of Ω is defined by requiring it to be the smallest Borel field with regard to which every one of the functions $\cdot(t)$ is measurable. Their theorem then asserts, given (i) to (iii) above, that there exists a unique probability-measure P on $(\Omega, \mathcal{J})$ such that, for each $n \geqq 1$, for each $(t_1, \cdots, t_n) \in T^n$, and for each n-dimensional Borel set A,

$$P\{[\omega(t_1), \cdots, \omega(t_n)] \in A\} = \int_A dF_{t_1,\cdots,t_n}.$$

Thus the indexed family of abstract random variables $\{x_t: t \in T\}$ can be identified with the indexed family $\{\cdot(t): t \in T\}$ of real-valued

$\mathcal{F}$-measurable functions with domain Ω. It is usual to complete $\mathcal{F}$ under P and the resulting triple $(\Omega, \mathcal{F}, P)$ is then a fundamental probability space (f.p.s.) which could have been taken as the starting point of the discussion (and is so taken, for example, in the recent book by Doob).

The more usual procedure is in fact to start with a f.p.s. $(\Omega, \mathcal{F}, P)$ and then to say that a stochastic process is any (indexed) family of random variables (measurable functions). The insistence on indexing is interesting. It is trivial that an indexing can always be supplied (e.g., by well-ordering) and thus it is sometimes useful to speak of the "universal" stochastic process consisting of all measurable functions on the f.p.s.

But in most cases the index-set serves an end more important than a mere labelling, and is itself endowed with a significant structure. For example, T may be ordered and the process Markovian, or T may be an abelian group and the process stationary, or T may be a topological space and the process continuous in some sense or other.

If the process is to be continuous then reference must be made to one of the stochastic topologies. Of these the most important are those associated with the following modes of convergence:

(a) convergence in probability (in measure);

(b) convergence in mean square;

(c) convergence with probability one (almost everywhere).

Now the statement "$x_{t_n} \to x_t$ as $n \to \infty$" can be expressed in terms of the finite-dimensional distributions F in cases (a) and (b) only, and this is why, in the work under review, no reference is made to the third mode of convergence. Chapter I begins with a detailed study of modes (a) and (b) and a completeness theorem is proved for each. Here the author's approach, specially constructed to avoid measure-theoretic difficulties, leads him at once into difficulties of another sort; the trouble is that the limiting random variable (whose existence is to be proved in the completeness theorem) will not normally be a member of the family of random variables with which he starts. He is not allowed to say that it is some measurable function on the f.p.s. (and so a member of the "universal" stochastic process mentioned above) and the following complicated evolutions are necessary. A new abstract object τ is introduced and adjoined to T to give an enlarged index-set T^+. A new stochastic process $\{y_u : u \in T^+\}$ with index-set T^+ is then defined in such a way that its natural contraction to T is identifiable with the original process, so that any Cauchy sequence $\{x_{t_n}\}$ can be identified with a Cauchy sequence $\{y_{t_n}\}$ from the new process. Finally it is shown that when the extension is suitably carried out, we shall have $y_{t_n} \to y_\tau$, and this is the form taken by the completeness theorem. The disadvantage of this procedure is that a new family of random variables has to be introduced with every limit-operation, a difficulty which could have been avoided by introducing a family of random variables closed under all the modes of stochastic convergence to be considered. This could have been done in several ways and would not seem to present any difficulties of principle, but it is the price which must be paid for the avoidance of measure theory, and the reader may find Chapter I quite mystifying until this fact is realized.

The last five words in the title to the work mean that attention is to be focused on those stochastic processes $\{x_t : t \in T\}$ for which T is a finite or infinite real interval. Thus it is natural to expect an analogue to the classical theory of functions of a real variable for the "function" $x_.$, and its continuity, differentiation, and integration with regard to t are discussed in the remainder of Chapter I. There then follow four chapters devoted to processes having independent increments, and their close relatives. Here the essential idea is that of a family of random variables with an ordered index-set T and such that the differences $x_{t_2} - x_{t_1}, \cdots, x_{t_n} - x_{t_{n-1}}$ are mutually independent whenever the intervals $(t_1, t_2), \cdots, (t_{n-1}, t_n)$ are non-overlapping. This branch of the subject has a long history starting with remarkable work by Bachelier and Wiener and developed extensively by Paul Lévy and others. The general theory is presented here in Chapter IV together with an account of infinitely divisible distribution-functions. The Wiener process (in which the independent increments have Gaussian distributions) is dealt with in Chapter II and called the fundamental random process (f.r.p.); a more realistic model for the Brownian motion (associated with the names of Ornstein and Uhlenbeck and so here called the o.u.p.) is also discussed at some length. Chapter III is devoted to the problems of estimating the parameters in a f.r.p. or an o.u.p. when a single realization of the process is available, and of estimating a deterministic motion masked by the superposition of a f.r.p. In Chapter V the author discusses two ways in which a process having independent increments may be modified by a registering device. In the first of these the original process is replaced by a moving average of itself; in the second a Geiger counter transforms a process initially of the Poisson type.

The last chapter contains a brief selection of results concerning the harmonic analysis of a stochastic process, when this is of the second order (i.e., when $\int \xi^2 dF_t(\xi)$ is finite for every t). The reader of this part of the book may easily not realize how much more is known and for further information he should be referred to the longer book by Doob (which is now available but was not when Professor Mann's book was published); there he will find these problems treated in their Hilbert space versions. The particular topics covered by the work under review are (i) the expansion of x_t as a Fourier series in t (with random coefficients) over a finite interval, the Paley-Wiener expansion for the f.r.p. being worked out as an example, and (ii) the mean ergodic theorem.

To summarize: this is a very useful little book and everyone concerned with the subject will want to possess it for the light which a study of it throws on the larger works by Lévy and Doob. But for all its brevity it does not make easy reading; the approach adopted is a difficult one to follow conscientiously; and the reader should be warned that many of the best things now known are not referred to.

DAVID G. KENDALL

Differential operators and differential equations of infinite order with constant coefficients. Researches in connection with integral functions of finite order. By P. C. Sikkema. Groningen, Noordhoff, 1953. 4+223 pp. 11.50 florins; cloth 13.50 florins.

This monograph is essentially the author's Groningen thesis and describes the results of his researches into the following three general problems. Let $F(D)=\sum_{n=0}^{\infty} a_n D^n$, $D=d/dx$, designate a differential operator of infinite order with constant coefficients, and let $F(D)\rightarrow y(x)$ designate the result of applying $F(D)$ to y, i.e., $F(D)\rightarrow y(x)=\sum_{n=0}^{\infty} a_n y^{(n)}(x)$. (1) Under what conditions does the expression $F(D)\rightarrow y(x)$ possess more than formal meaning? (2) What are the properties of the function $h(x)=F(D)\rightarrow y(x)$? (3) What can be said about the solutions $y(x)$ of the differential equation of infinite order $F(D)\rightarrow y(x)=h(x)$? The functions y and h are restricted in this work to be entire and of finite order, and the theorems obtained relate the properties of F with the order and type properties of y and h. Some of the results yield generalizations of theorems of H. Muggli, I. M. Sheffer, and J. M. Whittaker. The exposition is detailed, and workers in the field will find useful the exactitude with which the author has worked out the statements of his theorems as well as the numerous summaries of previously known results.

PHILIP DAVIS

JOURNAL

The Michigan Mathematical Journal. Vol. 2, no. 1. University of Michigan Press, Ann Arbor, 1954. 94 pp. $2.00 per volume to individuals ordering directly, $4.00 to others.

Vol. 1 was reviewed in this Bulletin, vol. 59, p. 483. It is pleasant to report that, while the journal is still produced by photo-offset, its appearance is vastly better than that of Vol. 1. The 13 papers in this issue (which was received in July, 1954 and is cautiously dated 1953–54) bear dates from March to December, 1953; several of them emanate from far beyond the immediate vicinity of Ann Arbor.

R. P. BOAS, JR.

BRIEF MENTION

Introduction to elliptic functions with applications. By F. Bowman. New York, Wiley, 1953. 115 pp. $2.50.

The purpose of this little book would perhaps be more clearly indicated if the title were *Applications of elliptic functions*. In fact, only about 44 of the 108 pages of text deal with the theory of the Jacobian elliptic functions (the Weierstrassian functions are not even mentioned). Hence the larger part of the book is devoted to applications of elliptic functions, principally to certain conformal mappings which are very important for electrical engineering, aerodynamics and so on. This part is sure to be useful to the physicist or engineer who encounters the corresponding problems.

As to the theoretical part of the book, there is some question whether such a short treatment of a mathematical theory can have any genuine use. In particular, I wonder whether the indications given in the text would suffice for the reduction to one of the three standard forms of an elliptic integral of a slightly malicious nature. Moreover, even if this reduction were to be carried out somehow, it would generally be of little use, since the book says nothing about the evaluation of elliptic integrals of the third kind. I realize that to have dealt also with ϑ-functions and so on would have involved a considerable increase in the size of the book; but, considering that Landen's transformation is briefly considered, it should have been possible to indicate at least its usefulness in the numerical calculation of integrals of the third kind.

F. G. TRICOMI

Colloque sur les fonctions de plusieurs variables. Tenu à Bruxelles du 11 au 14 mars 1953. Liège, Thone; Paris, Masson. 161 pp. 250 Belgian fr., 1800 French fr.

This volume contains lectures by F. Severi, P. Lelong, H. Cartan, J.-P. Serre, P. Roquette, H. Behnke, K. Stein, E. Martinelli, W. Saxer, and S. Bergman.

Premier colloque sur les équations aux dérivées partielles. Tenu à Louvain du 17 au 19 décembre 1953. Liège, Thone; Paris, Masson. 129 pp. 200 Belgian fr., 1400 French fr.

This volume contains lectures by A. Lichnerowicz, Y. Fourès, J. Delsarte, G. Doetsch, T. Lepage, P. Gillis and R. Sauer.

Theory of functions of a complex variable. By C. Carathéodory. Trans. by F. Steinhardt. Vol. 1. New York, Chelsea, 1954. 14+301 pp.

The German edition appeared in 1950 and was reviewed in this Bulletin, vol. 57, pp. 190–192.

Wave motion and vibration theory. (Proceedings of Symposia in Applied Mathematics, vol. 5.) New York, London, Toronto, McGraw-Hill, 1954. 6+169 pp. $7.00.

The volume contains 14 papers and one abstract; the symposium was held in June, 1952.

Logarithmetica Britannica, being a standard table of logarithms to twenty decimal places of the numbers 10,000 *to* 100,000. By A. J. Thompson. 2 vols., unpaged. Cambridge University Press, 1952. $28.50.

These tables, which have been appearing in parts since 1924, have finally been collected in two massive volumes. They are issued by the Department of Statistics, University College, London, to commemorate the tercentenary of Henry Briggs' publication of the *Arithmetica Logarithmica*, 1624.

RESEARCH PROBLEMS

25. Allen Shields: *A convergence problem.*

If $a_1 < a_2 < \cdots$ are positive integers, if C is compact, and if $\sin a_n x \to 0$ for all x in C, prove that the convergence must actually be uniform. (Received July 18, 1954.)

26. Allen Shields: *Measurability in groups.*

If G and H are locally compact groups and f is a continuous function from G to H, and if A is a Borel set in G, prove that $f(A)$ is measurable in the sense of Haar. (By Borel sets we mean the σ-ring generated by the compact sets.) The solution probably involves analytic sets; see, for example, Sierpinski, *General topology*, Toronto, 1934. (Received July 18, 1954.)

27. Walter Rudin: *The cluster set of a meromorphic function.*

If f is meromorphic in the open unit disc, then $C(f)$, the cluster set of f in the large, is the set of all w (including ∞) for which there is a sequence $\{z_n\}$ such that $|z_n| < 1$, $|z_n| \to 1$, and $f(z_n) \to w$ as $n \to \infty$. Every $C(f)$ is a continuum, but not every continuum is a $C(f)$. A counter-example is the set consisting of (a) a spiral, say $r = \theta/(\pi + \theta)$, $0 \leqq \theta < \infty$, (b) the unit circumference, (c) the interval $1 \leqq x \leqq 2$, $y = 0$. The problem is to characterize, by means of intrinsic properties, those continua which are cluster sets in the above sense. Reference: Collingwood and Cartwright, Acta Math. vol. 87 (1952). (Received August 26, 1954.)

28. Walter Rudin: *Approximation of continuous functions by analytic functions.*

Let E be a set of the first category (and arbitrary measure) on the circumference C of the open unit disc U, and let ϕ be a continuous complex-valued function in U. The proposer has recently proved that there exists an analytic function f in U such that, for every x in E, (*) $\lim_{r \to 1} [f(rx) - \phi(rx)] = 0$. Without further conditions on ϕ, the category condition cannot be dropped. The problem is to characterize those continuous functions ϕ in U for which (*) may hold (i) for every x on C, (ii) uniformly on C, (iii) in some other norm. (Received August 26, 1954.)

ERRATA, VOLUME 59

The Sixth Symposium in Applied Mathematics, p. 513. In the write-up of the Applied Mathematics Symposium in Santa Monica, California, in August, 1953, the name of the Consolidated Engineering Corporation of Pasadena, California, was inadvertently omitted from the list of computing facilities visited by members of the Symposium.

ERRATA, VOLUME 60

David Nelson, *Contraposition with strong negation.* Preliminary report. Abstract 60-1-141. Line 5 of the abstract: for the second and third "$\mathcal{P}$" read "$\mathcal{N}$."

REPRESENTATIVES AND COMMITTEES OF THE SOCIETY

Representatives of the Society in the Division of Mathematics of the National Research Council:

1954–1955—W. L. Duren, J. R. Kline, Norman Levinson, E. J. McShane, D. C. Spencer, R. L. Wilder.
1955–1956—W. L. Duren, H. J. Ettlinger, Norman Levinson, C. C. MacDuffee, E. J. McShane, R. L. Wilder.
1956–1957—W. L. Duren, H. J. Ettlinger, D. H. Lehmer, Norman Levinson, C. C. MacDuffee, N. E. Steenrod.

Representatives on the Council of the American Association for the Advancement of Science:

1954–1955—G. A. Hedlund, E. J. McShane.

Representatives on the Editorial Board of the Annals of Mathematics:

1955–1956—A. M. Gleason, J. L. Kelley.

Representatives on the Editorial Board of the Duke Mathematical Journal:

1953–1955—R. H. Cameron, S. S. Chern.

Representative on Advisory Board of Applied Mechanics Reviews:

K. O. Friedrichs.

Representative on U. S. Committee on Theoretical and Applied Mechanics:

1953–1956—M. H. Martin.

Liaison Officer in Connection with Editorial Management of Quarterly of Applied Mathematics:

1953–1955—A. H. Taub.

Colloquium Lecturers:

1896—James Pierpont.
    Maxime Bôcher.
1898—W. F. Osgood.
    A. G. Webster.
1901—Oskar Bolza.
    E. W. Brown.
1903—H. S. White.
    F. S. Woods.
    E. B. Van Vleck.
1906—E. H. Moore.
    E. J. Wilczynski.
    Max Mason.
1909—G. A. Bliss.
    Edward Kasner.
1913—L. E. Dickson.
    W. F. Osgood.
1916—G. C. Evans.
    Oswald Veblen.
1920—G. D. Birkhoff.
    F. R. Moulton.
1925—L. P. Eisenhart.
    Dunham Jackson.
1927—E. T. Bell.
    AnnaPell-Wheeler.
1928—A. B. Coble.
1929—R. L. Moore.
1930—Solomon Lefschetz.
1931—Marston Morse.
1932—J. F. Ritt.
1934—R. E. A. C. Paley.
    Norbert Wiener.
1935—H. S. Vandiver.
1936—E. W. Chittenden.
1937—John vonNeumann.
1939—A. A. Albert.
    M. H. Stone.
1940—G. T. Whyburn.
1941—Oystein Ore.
1942—R. L. Wilder.
1943—E. J. McShane.
1944—Einar Hille.
1945—Tibor Radó.
1946—Hassler Whitney.
1947—Oscar Zariski.
1948—Richard Brauer.
1949—G. A. Hedlund.
1951—Deane Montgomery
1952—Alfred Tarski.
1953—Antoni Zygmund.

Committee to Select Gibbs Lecturers for 1954 and 1955:

A. A. Albert (Chairman), J. L. Doob, C. B. Morrey.

Gibbs Lecturers:

1923—M. I. Pupin.
1924—Robert Henderson.
1925—James Pierpont.
1926—H. B. Williams.
1927—E. W. Brown.
1928—G. H. Hardy.
1929—Irving Fisher.
1930—E. B. Wilson.
1931—P. W. Bridgman.
1932—R. C. Tolman.
1934—Albert Einstein.
1935—Vannevar Bush.
1936—H. N. Russell.
1937—C. A. Kraus.
1939—Theodore von Kármán.
1941—Sewall Wright.
1943—Harry Bateman.
1944—John von Neumann.
1945—J. C. Slater.
1946—Subrahmanyan Chandrasekhar.
1947—P. M. Morse.
1948—Hermann Weyl.
1949—Norbert Wiener.
1950—G. E. Uhlenbeck.
1951—Kurt Gödel.
1952—Marston Morse.
1953—Wassily Leontief.
1954—K. O. Friedrichs.

Committee on Society Visiting Lectureship:

Garrett Birkhoff (Chairman), Einar Hille, P. R. Halmos.

Visiting Lecturers:

1927–1928—Constantin Carathéodory, University of Munich.
1928–1929—Hermann Weyl, Zurich Technical School.
1929–1930—Enrico Bompiani, University of Rome.
1930–1931—Wilhelm Blaschke, University of Hamburg.
1931–1932—R. L. Moore, University of Texas.
1936–1937—Thirukkannapuram Vijayaraghavan, University of Dacca.
1948–1949—Casimir Kuratowski, University of Warsaw.
1951–1952—Marshall H. Stone, University of Chicago.

Auditors for 1954:

B. P. Gill, C. H. W. Sedgewick.

Committee on Publication of Moore Manuscript:

R. W. Barnard, E. W. Chittenden, L. M. Graves, T. H. Hildebrandt, M. H. Ingraham.

Committees to Select Hour Speakers:

For Annual and Summer Meetings: E. G. Begle (Chairman), Wilfred Kaplan, Oscar Zariski.
For Eastern Sectional Meetings: L. W. Cohen (Chairman), A. M. Gleason, Nelson Dunford.
For Western Sectional Meetings: J. W. T. Youngs (Chairman), E. E. Moise, M. E. Shanks.
For Far Western Sectional Meetings: J. W. Green (Chairman), R. D. James, J. L. Kelley.
For Southeastern Sectional Meetings: J. H. Roberts (Chairman), J. J. Gergen, W. V. Parker.

Committee on Places of Meetings:

M. F. Smiley (Chairman), E. R. Lorch, W. M. Whyburn.

Policy Committee for Mathematics:

Representatives of American Mathematical Society: A. A. Albert, S. S. Cairns (Chairman), J. R. Kline, Leo Zippin.
Representative of Association for Symbolic Logic: Alfred Tarski.
Representative of Institute of Mathematical Statistics: J. F. Daly.
Representatives of Mathematical Association of America: J. S. Frame, H. M. Gehman, E. J. McShane.
Representative of National Council of Teachers of Mathematics: E. H. C. Hildebrandt.
Secretary: E. G. Begle.

Committee on Applied Mathematics:

F. J. Murray (Chairman), R. E. Bellman, Garrett Birkhoff, R. S. Burington, Shizuo Kakutani, Eric Reissner.

Editorial Committee for Applied Mathematics Symposium Proceedings:

R. V. Churchill (Chairman), A. E. Heins, F. J. Murray.

Committee on Translations from Russian and Other Foreign Languages:

J. L. Doob (Chairman), R. E. Bellman, R. P. Boas, Irving Kaplansky, Hans Samelson.

FORMER PRESIDENTS

J. H. Van Amringe, 1889–1890.
Emory McClintock, 1891–1894.
G. W. Hill, 1895–1896.
Simon Newcomb, 1897–1898.
R. S. Woodward, 1899–1900.
E. H. Moore, 1901–1902.
T. S. Fiske, 1903–1904.
W. F. Osgood, 1905–1906.
H. S. White, 1907–1908.
Maxime Bôcher, 1909–1910.
H. B. Fine, 1911–1912.
E. B. Van Vleck, 1913–1914.
E. W. Brown, 1915–1916.
L. E. Dickson, 1917–1918.
Frank Morley, 1919–1920.
G. A. Bliss, 1921–1922.
Oswald Veblen, 1923–1924.
G. D. Birkhoff, 1925–1926.
Virgil Snyder, 1927–1928.
E. R. Hedrick, 1929–1930.
L. P. Eisenhart, 1931–1932.
A. B. Coble, 1933–1934.
Solomon Lefschetz, 1935–1936.
R. L. Moore, 1937–1938.
G. C. Evans, 1939–1940.
Marston Morse, 1941–1942.
M. H. Stone, 1943–1944.
T. H. Hildebrandt, 1945–1946.
Einar Hille, 1947–1948.
J. L. Walsh, 1949–1950.
John von Neumann, 1951–1952.

ENDOWMENT FUND

In 1923 an Endowment Fund was collected to meet the greater demands on the Society's publication program caused by the ever increasing number of important mathematical memoirs. Of this fund, which now amounts to some $94,000, a considerable proportion was contributed by members of the Society. Under the terms of the will of the late Robert Henderson, for many years a Trustee of the Society, the Society receives approximately $4,000 yearly. Upon the death of the other legatees, the Society will receive the entire principal of the estate for its Endowment Fund.

SPECIAL FUNDS

($500 or more)

The Bôcher Memorial Prize.

This prize was founded in memory of Professor Maxime Bôcher. It is awarded every five years for a notable research memoir in analysis which has appeared during the preceding five years in a recognized journal published in the United States or Canada; the recipient must be a member of the Society, and not more than fifty years old at the time of publication of his memoir.

First (Preliminary) Award, 1923: To G. D. Birkhoff, for his memoir *Dynamical systems with two degrees of freedom.*

Second Award, 1924: To E. T. Bell, for his memoir *Arithmetical paraphrases,* and to Solomon Lefschetz, for his memoir *On certain numerical invariants with applications to abelian varieties.*

Third Award, 1928: To J. W. Alexander, for his memoir *Combinatorial analysis situs.*

Fourth Award, 1933: To Marston Morse, for his memoir *The foundations of a theory of the calculus of variations in the large in m-space,* and to Norbert Wiener, for his memoir *Tauberian theorems.*

Fifth Award, 1938: To John von Neumann, for his memoir *Almost periodic functions and groups.*

Sixth Award, 1943: To Jesse Douglas, for his memoirs *Green's functions and the problem of Plateau, The most general form of the problem of Plateau,* and *Solution of the inverse problem of the calculus of variations.*

Seventh Award, 1948: To A. C. Schaeffer and D. C. Spencer for their memoir *Coefficients of schlicht functions.*

Eighth Award, 1953: To Norman Levinson for his contributions to the theory of linear, nonlinear, ordinary and partial differential equations contained in his papers of recent years.

The Frank Nelson Cole Prize in Algebra.
The Frank Nelson Cole Prize in the Theory of Numbers.

These prizes were founded in honor of Professor Frank Nelson Cole, on the occasion of his retirement as Secretary of the American Mathematical Society and Editor of the Bulletin, after twenty-five years of service; the fund was later doubled by his son, Charles A. Cole. They are awarded at five-year intervals for contributions to algebra and the theory of numbers, respectively, under restrictions similar to those for the Bôcher prize.

First Award, 1928: To L. E. Dickson, for his book *Algebra und ihre Zahlentheorie,* Zurich, 1927.

Second Award, 1931: To H. S. Vandiver, for his several papers on Fermat's last theorem published in the Transactions of the American Mathematical Society and the Annals of Mathematics during the last five years, with special reference to a paper entitled *On Fermat's last theorem*, which appeared in volume 31 of the Transactions.

Third Award, 1939: To A. A. Albert, for his papers on the construction of Riemann matrices, published in volumes 35 and 36 of the Annals of Mathematics.

Fourth Award, 1941: To Claude Chevalley, for his paper entitled *La théorie du corps de classes*, published in volume 41 of the Annals of Mathematics.

Fifth Award, 1944: To Oscar Zariski, for four papers on algebraic varieties, published in volumes 61 and 62 of the American Journal of Mathematics and volumes 40 and 41 of the Annals of Mathematics.

Sixth Award, 1946: To H. B. Mann for his paper entitled *A proof of the fundamental theorem on the density of sums of sets of positive integers*, published in volume 43 of the Annals of Mathematics.

Seventh Award, 1949: To Richard Brauer, for his memoir *On Artin's L-series with general group characters*, published in volume 48 of the Annals of Mathematics.

Eighth Award, 1951: to Paul Erdös, for his many papers in the Theory of Numbers, and in particular for his paper, *On a new method in elementary number theory which leads to an elementary proof of the prime number theorem*, published in the Proceedings of the National Academy, vol. 35, pp. 374–385, July 1949.

The Eliakim Hastings Moore Fund.

This fund was founded in 1922 in honor of Professor Eliakim Hastings Moore, on the occasion of the twenty-fifth anniversary meeting of the Chicago Section of the American Mathematical Society. The income from the fund of $2,575 is to be used at the discretion of the Council of the Society, for the publication of important mathematical books or memoirs, or the award of prizes.

The Marion Reilly Fund.

Dean Marion Reilly willed to the American Mathematical Society a portion of her estate to be used for the advancement of research in pure mathematics. The principal of this fund is $23,650.

The Ernest William Brown Fund.

From the estate of Professor Ernest William Brown, a fund of $1,000 is available, the interest on which, at the discretion of the Council, can be used for the furtherance of such mathematical interests as (a) the publication of important mathematical books, memoirs, and periodicals, and (b) lectures to be delivered on special occasions by invited guests of the Society.

The Robert Henderson Fund.

Upon his retirement from the Board of Trustees on December 31, 1940, Dr. Robert Henderson presented to the Society an unrestricted gift of $1,000 which the Trustees have set aside as the Robert Henderson Fund.

The James K. Whittemore Fund.

From the estate of Professor James K. Whittemore, a fund of $1,000 is available for use in the interest of mathematical research.

The Generva B. Hutchinson Fund.

A fund of $1,000 from the estate of the late Professor Generva B. Hutchinson for use, at the discretion of the Council and Trustees, in the advancement of mathematical research.

The Helen A. Merrill Fund.

From the estate of the late Professor Helen A. Merrill, a fund of $650 is available for use in the interest of mathematical research.

Library Proceeds Fund.

A fund of $66,000 established by the Trustees in 1951 from the proceeds of the sale of the Society's Library.

BY-LAWS OF THE AMERICAN MATHEMATICAL SOCIETY

ARTICLE I

OFFICERS

Section 1. There shall be a President, a President Elect (during the even-numbered years only), an Ex-president (during the odd-numbered years only), three Vice Presidents, a Secretary, four Associate Secretaries, and a Treasurer.

Section 2. It shall be a duty of the President to deliver an address before the Society at the close of his term of office or within one year thereafter.

ARTICLE II

BOARD OF TRUSTEES

Section 1. There shall be a Board of Trustees consisting of seven trustees, five trustees elected by the Society in accordance with Article VII, together with the President and Treasurer of the Society ex officio. The Board of Trustees shall designate its own presiding officer and secretary.

Section 2. The function of the Board of Trustees shall be to receive and administer the funds of the Society, to have full legal control of its investments and properties, to make contracts and, in general, to conduct all business affairs of the Society.

Section 3. The Board of Trustees shall have the power to appoint a manager and such assistants and agents as may be necessary or convenient to facilitate the conduct of the affairs of the Society, and to fix the terms and conditions of their employment. The Board may delegate to the officers of the Society duties and powers normally inhering in their respective corporative offices, subject to supervision by the Board. The Board of Trustees may appoint committees to facilitate the conduct of the financial business of the Society and delegate to such committees such powers as may be necessary or convenient for the proper exercise of those powers. Agents appointed, or members of committees designated, by the Board of Trustees need not be members of the Board.

Nothing herein contained shall be construed to empower the Board of Trustees to divest itself of responsibility for, or legal control of, the investments, properties and contracts of the Society.

ARTICLE III

PUBLICATIONS COMMITTEES

Section 1. There shall be eight Publications Committees, which shall be the seven Editorial Committees specified in Section 2 of this Article and the Committee on Printing and Publishing specified in Section 3 of this Article.

Section 2. There shall be seven Editorial Committees, as follows: a committee of two members for the *Bulletin;* committees of three members each of the *Proceedings*, the *Colloquium Publications*, *Mathematical Surveys* and *Mathematical Reviews;* a committee of four members jointly for the *Transactions* and the *Memoirs*, and one consisting of two representatives of the Society on the Board of Editors of the *American Journal of Mathematics.*

Section 3. There shall be a Committee on Printing and Publishing consisting of three members.

ARTICLE IV

COUNCIL

Section 1. The Council shall consist of fifteen members at large and the following ex officio members: the officers of the Society specified in Article I, the members of the Publications Committees specified in Article III, any former Secretary for a period of two years following his term of office, and members of the Executive Committee who remain on the Council by the operation of Article VII, Section 4.

Section 2. The Council shall formulate and administer the scientific policies of the Society and shall act in an advisory capacity to the Board of Trustees.

Section 3. In the absence of the Secretary from any meeting of the Council, one of the Associate Secretaries present may be designated as Acting Secretary for the meeting, either by written authorization of the Secretary, or, failing that, by majority agreement among the Associate Secretaries present.

Section 4. All members of the Council shall be voting members. The method for settling matters before the Council at any meeting shall be by majority vote of the members present. If the result of a vote is challenged, it shall be the duty of the presiding officer to determine the true vote by a roll call. In a roll call vote, each Council member shall vote only once (although he may be a member of the Council in several capacities), and he shall state before the vote in which capacity he votes. The group consisting of the four Associate Secretaries shall have one vote, and it shall be divided equally among those who vote as Associate Secretaries. Each of the eight Publications Committees shall have one vote, and it shall be divided equally among those who vote as members of the respective Publications Committees. All other members of the Council shall have one vote each. Fractional votes shall be counted.

Section 5. Any group of members of the Council who have a total of five votes as defined in Section 4 of this Article shall constitute a quorum for the transaction of business at any meeting of the Council.

Section 6. Between meetings of the Council, business may be transacted by a mail vote. Votes shall be counted as in the case of a roll call vote as specified in Section 4 of this Article, "members present" being replaced by "members voting." An affirmative vote by mail on any proposal shall be declared if and only if (a) more than half of the total number of possible votes is received by the time announced for the closing of the polls, and (b) at least three-quarters of the votes received by then are affirmative. If members who have a total of five or more votes request postponement at the time of voting, action on the matter at issue shall be postponed until the next meeting of the Council, unless either (1) at the discretion of the Secretary, the question is made the subject of a second vote by mail, in connection with which brief statements of reasons, for and against, are circulated or (2) the Council places the matter at issue before the Executive Committee for action.

Section 7. The Council may delegate to the Executive Committee (Article V) certain of its duties and powers. Between meetings of the Council the Executive Committee shall act for the Council on such matters and in such ways as the Council may specify. Nothing herein contained shall be construed as empowering the Council to divest itself of responsibility for formulating and administering the scientific policies of the Society.

Section 8. The Council shall also have power to speak in the name of the Society with respect to matters affecting the status of mathematics or mathematicians, such as proposed or enacted Federal or State Legislation, conditions of employment in universities, colleges, or business, research or industrial organizations, regulations, policies or acts of governmental agencies or instrumentalities, and other items which tend to affect the dignity and effective position of mathematics.

With the exception noted in the next paragraph, a favorable vote of two-thirds of the entire membership of the Council, taken in accordance with the provisions of Section 4 of this Article, shall be necessary to authorize any statement in the name of the Society with respect to such matters. With the exception noted in the next paragraph, such a vote may be taken only if written notice shall have been given to the Secretary by the proposer of any such resolution not later than one month prior to the Council meeting at which the matter is to be presented, and the vote shall be taken not earlier than one month after the resolution has been discussed by the Council.

If, at a meeting of the Council, there are present members having a total of twelve votes, as defined in Section 4 of this Article, then the prior notification to the Secretary may be waived by unanimous consent. In such a case, a unanimous favorable vote by those present shall empower the Council to speak in the name of the Society.

The Council may also refer the matter to a referendum mail vote of the entire membership of the Society, and shall make such reference if a referendum is requested, prior to final action by the Council, by two hundred or more members. The taking of a referendum shall act as a stay upon Council action until the votes have been canvassed, and thereafter no action may be taken by the Council except in accordance with a plurality of the votes cast in the referendum.

ARTICLE V

EXECUTIVE COMMITTEE

Section 1. There shall be an Executive Committee of the Council, consisting of four elected members and the following ex officio members: the President, the Secretary, the President Elect (during even-numbered years), and the Ex-president (during odd-numbered years).

Section 2. The Executive Committee of the Council shall be empowered to act for the Council on matters which have been delegated to the Executive Committee by the Council. If three members of the Executive Committee request that any matter be referred to the Council, the matter shall be so referred. The Executive Committee shall be responsible to the Council and shall report its actions to the Council. It may consider the agenda for meetings of the Council and may make recommendations to the Council.

Section 3. Each member of the Executive Committee shall have one vote. An affirmative vote on any proposal before the Executive Committee shall be declared if and only if at least four affirmative votes are cast for the proposal. A vote on any proposal may be determined at a meeting of the Executive Committee, but it shall not be necessary to hold a meeting to determine a vote.

ARTICLE VI

EXECUTIVE DIRECTOR

Section 1. There shall be an Executive Director who shall be a paid employee of the Society. He shall have charge of the central office of the Society, and he shall be responsible for the general administration of the affairs of the Society in accordance with the policies that are set by the Board of Trustees and by the Council.

Section 2. The Executive Director shall be elected by the Council for a specified term and approved by the Board of Trustees. The terms and conditions of his employment shall be fixed by the Board of Trustees.

Section 3. The Executive Director shall work under the immediate direction of the Board of Trustees and of the Council, and in cooperation with the Secretary. He shall attend meetings of the Board of Trustees, of the Council, and of the Executive Committee, but he shall not be a member of any of these bodies.

ARTICLE VII

ELECTION AND TERMS OF OFFICERS

Section 1. The term of office shall be one year in the case of the President Elect and the Ex-president; two years in the case of the Trustees, the President, the Vice Presidents, the Secretary, the Associate Secretaries, and the Treasurer; three years in the case of the members of the Publications Committees. The term of office for members at large of the Council shall be three years, five of the members at large retiring annually. The term of office for elected members of the Executive Committee shall be two years, two of the elected members retiring annually. In every case, however, the officials specified in Articles I, II, III, IV, and V, with the exception of the President Elect and the Ex-president, shall continue to serve until their successors have been duly elected and qualified.

Section 2. Election of officers specified in Article I (with the exception of the President and the Ex-president), Trustees, members of the Publications Committees, and members at large of the Council shall be by ballot at the Annual Meeting. An official ballot shall be sent to each member of the Society by the Secretary at least one month before the Annual Meeting, and such ballots, if returned to the Secretary in envelopes bearing the name of the voter and received prior to the closing of the polls, shall be counted at the Annual Meeting. Each ballot shall contain one or more names proposed by the Council for each office to be filled, with blank spaces in which the voter may substitute other names. A plurality of all votes cast, whether cast in person or by mail, shall be necessary for election. In case of failure to secure a plurality for any office, the members present at the Annual Meeting shall choose by ballot among the members having the highest number of votes.

Section 3. At the end of his term of office, the President Elect shall become the President. At the end of his term of office, the President shall become the Ex-president.

Section 4. Within fourteen days after the election of officers at the Annual Meeting the Secretary shall send to all members of the Council for a mail vote a ballot containing two names for each place to be filled on the Executive Committee. The nominees shall be chosen by a committee appointed by the President. Members of the Council may vote for persons not nominated. Any member of the Council who is not an ex officio member of the Executive Committee (see Article V, Section 1) shall be eligible for election to the Executive Committee. In case a member is elected to the Executive Committee for a term extending beyond his regular term on the Council, he shall automatically continue as a member of the Council during his term on the Executive Committee.

Section 5. The President and Vice Presidents shall not be eligible for immediate re-election to their respective offices. A member at large or an ex officio member of the Council shall not be eligible for immediate election (or re-election) as a member at large of the Council.

Section 6. If the President of the Society die or resign while a President Elect is in office, the President Elect shall serve as President for the remainder of the year and thereafter shall serve his regular two-year term. If the President of the Society die or resign when no President Elect is in office, the Council, with the approval of the Board of Trustees, shall designate one of the Vice Presidents to serve as President for the balance of the regular presidential term. If the President Elect of the Society die or resign before becoming President, his office shall remain vacant until the next regular election of a President Elect, and the Society shall, at the next Annual Meeting, elect a President for a two-year term. If the Ex-president die or resign before the expiration of his term of office, the Council, with the approval of the Board of Trustees, shall designate a former President of the Society to serve as Ex-president during the remainder of the regular term of the Ex-presidency. Such vacancies as may occur at any time in the group consisting of the Vice Presidents, the Secretary, the Associate Secretaries, the Treasurer, and the members of the Publications Committees shall be filled by the Council with the approval of the Board of Trustees. If a member of a Publications Committee takes temporary leave from his duties, that Committee shall nominate a substitute for consideration by the Council. The Council shall then elect a substitute. The Council shall fill from its own membership any vacancy in the elected membership of the Executive Committee.

Section 7. If any Trustee die or resign during his tenure of office, the vacancy thus created shall be filled for his unexpired term by the Board of Trustees.

Section 8. If any member at large of the Council die or resign more than one year before the expiration of his term, the vacancy for the unexpired term shall be filled by the Society at the next Annual Meeting.

ARTICLE VIII

Members and Their Election

Section 1. Election of members shall be by vote of the Council or of its Executive Committee.

Section 2. There shall be four classes of members, namely ordinary, contributing, corporate, and institutional.

Section 3. Application for admission to ordinary membership shall be made by the applicant on a blank provided by the Secretary, and shall be approved either by two ordinary members of the Society or by the institutional member whose nominee he is (see Article IX, Section 6). Such applications shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail), except in the case of members of other societies entering under special action of the Council approved by the Board of Trustees.

Section 4. An ordinary member may become a contributing member by paying the dues for such membership. (See Article IX, Section 3.)

Section 5. A university or college, or a firm, corporation, or association interested in the support of mathematics may be elected a corporate or an institutional member.

ARTICLE IX

Dues and Privileges of Members

Section 1. Any applicant shall be admitted to ordinary membership immediately upon election by the Council (Article VIII) and the discharge within sixty days of election of his first annual dues. Dues may be discharged by payment or by remission when the provision of Section 7 of this Article is applicable. The first annual dues shall apply to the year of election, except that if any applicant is elected after August 15 of any year he may elect to have his first annual dues apply to the following year.

Section 2. The annual dues of an ordinary member of the Society shall be fourteen dollars, except that the Council, with the approval of the Board of Trustees, may establish special rates in exceptional cases and for members of an organization with which the Society has a reciprocity agreement.

Section 3. The minimum dues for a contributing member shall be twenty dollars per year. Members may, upon their own initiative, pay larger dues.

Section 4. The minimum dues of an institutional member shall depend on the amount of published material credited to that member in certain journals during a specific period. The formula for computing these dues shall be established from time to time by the Council subject to approval by the Board of Trustees. Institutions may pay larger dues than the computed minimum.

Section 5. The minimum dues of a corporate member shall be one hundred dollars.

Section 6. The privileges of a corporate or an institutional member shall depend on its dues in a manner to be determined by the Council, subject to approval by the Board of Trustees. These privileges shall be in terms of Society publications to be received by the institution and of the number of persons it may nominate for ordinary membership in the Society.

Section 7. The dues of an ordinary member of the Society shall be remitted for any years during which he is the nominee of an institutional member.

Section 8. After retirement from active service on account of age, any ordinary or contributing member who is not in arrears of dues and with membership extending over at least twenty years may, by giving proper notification to the Secretary, have his dues remitted, on the understanding that he will thereafter receive the programs of the meetings but not the *Bulletin* and *Proceedings*.

Section 9. An ordinary or contributing member shall receive the *Bulletin* and *Proceedings* as privileges of membership during each year for which his dues have been discharged.

Section 10. Nine dollars of the annual dues of those who receive the *Bulletin* and *Proceedings* under the provisions of this Article shall be allocated in payment therefor.

Section 11. The annual dues of ordinary, contributing, and corporate members shall be payable after October first of the preceding year and due by January first of the year to which they apply. The Society shall submit bills for dues. If the annual dues of any member remain undischarged beyond what the Board of Trustees deem to be a reasonable time, his name shall, after due notice, be removed from the list of members. If a member wishes to discontinue his membership at any time, he shall submit his resignation in writing to the Society.

Section 12. Any member who became a life member before October 25, 1941, by the payment of a sum determined in accordance with actuarial principles shall have for life the status and privileges of an ordinary member without further payment of dues. No additional applications for life memberships will be accepted.

ARTICLE X

MEETINGS

Section 1. The Annual Meeting of the Society shall be held between the fifteenth of December and the fifteenth of January next following. Notice of the time and place of this meeting shall be mailed by the Secretary or an Associate Secretary to the last known post office address of each member of the Society. The times and places of the Annual and other meetings of the Society shall be designated by the Council. There shall be a business meeting of the Society at the Annual Meeting and at the Summer Meeting. A business meeting of the Society shall take final action only on business accepted by unanimous consent, or business notified to the full membership of the Society in the call for the meeting. Such notification shall be made only when so directed by a previous business meeting of the Society or by the Council.

Section 2. Meetings of the Executive Committee may be called by the President; he shall call a meeting at any time upon the written request of two of its members.

Section 3. The Council shall meet at the Annual Meeting of the Society. Special meetings of the Council may be called by the President; he shall call a special meeting at any time upon the written request of five of its members. No special meeting of the Council shall be held unless written notice of it shall have been sent to all members of the Council at least ten days before the day set for the meeting.

Section 4. The Board of Trustees shall hold its regular Annual Meeting each year between December 15 and January 20, but not before the Annual Meeting of the Society. Special meetings of the Board of Trustees may be called by the Chairman of the Board upon three days' notice of such meeting mailed to the last known post office address of each Trustee. He should call a meeting upon the receipt of a written request of two of the Trustees. Meetings for the transaction of business may also be held by common consent of all the Trustees.

Section 5. Papers intended for presentation at any meeting of the Society shall be passed upon in advance by a program committee appointed by or under the authority of the Council; and only such papers shall be presented as shall have been approved by such committee. Papers in form unsuitable for publication, if accepted for presentation, shall be referred to on the program as preliminary communications or reports.

ARTICLE XI

PUBLICATIONS

Section 1. The Society shall publish an official organ called the *Bulletin of the American Mathematical Society*. It shall publish two journals, known as the *Transactions of the American Mathematical Society* and the *Proceedings of the American Mathematical Society*. It shall publish a series of mathematical papers known as the *Memoirs of the American Mathematical Society*. The object of the *Transactions*, *Proceedings*, and *Memoirs* is to make known important mathematical researches. It shall publish a periodical called *Mathematical Reviews*, containing abstracts or reviews of current mathematical literature. It shall publish a series of volumes called *Colloquium Publications* which shall embody in book form new mathematical developments. It shall publish a series of monographs called *Mathematical Surveys* which shall furnish expositions of the principal methods and results of particular fields of mathematical research. It shall also cooperate in the conduct of the *American Journal of Mathematics*.

Section 2. The editorial management of the publications of the Society listed in Section 1 of this article, and the participation of the Society in the editorial management of the *American Journal of Mathematics* shall be in charge of the respective Editorial Committees as provided in Article III, Section 2.

Section 3. The Committee on Printing and Publishing shall advise the Council and Board of Trustees on business and non-editorial matters concerning the publications of the Society, and shall perform such other functions and shall exercise such powers as are properly assigned or delegated to it by the Council or the Board of Trustees.

ARTICLE XII

AMENDMENTS

These By-Laws may be amended or suspended at any meeting of the Society on recommendation of the Council and by a two-thirds vote of the members present, provided notice of such proposed action and of its general nature shall have been given in the call for such meeting.

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